We have $k < i_1$ and $m_{i_1} > t_k$. For any choice of $a_{i_1}^{ji} \in h_1^{-1}(a_k^i)$, $i = 1, \ldots, s_k$ (such choice clearly exists) there exists $b_{i_1}^j$ so that $a_{i_1}^{ji} \sigma_{i_1} b_{i_1}^j$. Therefore $a_k^i \sigma_k h_1(b_{i_1}^j)$, which gives $h_1(b_{i_1}^j) = a_k$ or b_k. This is a contradiction to (7). Therefore (6) is valid and hence $A_k \subset h_1(B_{i_1})$. By induction we get (5). (5) together with $h_r(B_{i_r}) \subset B_k$ implies $A_k \subset B_k$ which is a contradiction to the definitions of A_k and B_k.

From Lemma 3 we can deduce

LEMMA 4. $P_1 \neq P_2 \Rightarrow F_{P_1} \neq F_{P_2}$.

The proof is immediate, as by Lemma 3 $s_k \in P_2 - P_1$ $(s_k \in P_1 - P_2)$ implies $\varrho_{P_2}(C_k, \sigma_k) \neq \varrho_{P_1}(C_k, \sigma_k)$. $\operatorname{card} \boldsymbol{T}^* = 2^{\aleph_0}$ follows obviously from Lemma 4, the definition of P and from the evident upper bound $\operatorname{card} \boldsymbol{T}^* \leqslant 2^{\aleph_0}$.

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HOMOMORPHISMS OF GROUP RINGS

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Introduction

Homomorphisms of group rings with the ring of integers as the coefficient ring and torsion-free group were first investigated by Higman [1]. These investigations were continued among others, by Smirnov [13]. Parmenter and Sehgal considered automorphisms of the group ring $A[G]$ for infinite cyclic group G and arbitrary ring of coefficients A ([7], [8]). Lantz [4] described automorphisms of group rings of free abelian groups of finite rank with commuting coefficients.

The aim of this paper is to present a new method of investigation of a group of units and homomorphisms of group rings. For this purpose we shall investigate in § 1 properties of some subgroups of a group $U(C[G])$ of units of a group ring $C[G]$, where C is a commutative ring. In § 2 a structure of the group $U(C[G])$ is described in the case where G is a u.p. group. In § 3 we introduce 4 classes of homomorphisms of group rings related to subgroups defined in § 1. They are called G_i-homomorphisms $(i = 0, 1, 2, 3)$ and it is shown that in the case of u.p.-groups every homomorphism is a G_3-homomorphism. In § 4 structure of G_0- and G_1-homomorphisms is described. In § 5 we investigate properties of G_2- and G_3-homomorphisms using in the essential way results concerning G_1-homomorphisms. In § 4 and § 5 some criteria for a homomorphisms to be an injection, a surjection or an automorphism are given. In § 6 our results are applied to the description of the structure of group of automorphisms and hopficity and cohopficity of group rings of u.p.-groups.

The paper is written in such a way that it is possible to extend all the results on u.p.-groups to the arbitrary torsion-free group after showing the triviality of the group of units of group algebras of such groups over fields.

In the paper, standard notation of the theory of group rings is used ([9], [14]). All the necessary information on Boolean algebras and categories can be found in [11] and [4].

1. Subgroups of units

In the sequel, C will denote a commutative ring with unity, N its nil-radical and X will denote a group.

Let $D(C)$ denote the set of decompositions of the unity element of C into finite sets of orthogonal idempotents. For any $E \in D(C)$ let $X_E = \{\sum_{e\in E} ex_e\colon x_e \in X\}$.

We shall define

$$G_0(C[X]) = X \quad \text{and} \quad G_1(C[X]) = \bigcup_{E\in D(C)} X_E.$$

Let $G_2(C[X])$ be a set of products uv, where $u \in U(C)$, $v \in G_1(C[X])$. Moreover, let $G_3(C[X])$ be a set of elements of the form $u+t$, where $u \in G_2(C[X])$, $t \in N[X]$.

THEOREM 1.1. *In the foregoing notations, the sets $G_i(C[X])$ $(i = 1, 2, 3)$ are subgroups of $U(C[X])$ and*

$$X \subseteq G_1(C[X]) \subseteq G_2(C[X]) \subseteq G_3(C[X]).$$

Proof. Since $\left(\sum_{e\in E} ex_e\right)\left(\sum_{e\in E} ex_e^{-1}\right) = 1$, X_E is a group for any $E \in D(C)$. If $E, F \in D(C)$, then we shall say that $E \leqslant F$ whenever for any $f \in F$ there exists an $e \in E$ such that $ef = f$. The set $D(C)$ is ordered by the relation "$\leqslant$" and for $E, F \in D(C)$ a set of all non-zero elements of the form $e\cdot f$ where $e \in E$, $f \in F$ is their upper bound. Therefore the set $D(C)$ is directed. If $E, F \in D(C)$ and $E \leqslant F$, then $X_E \leqslant X_F$. By the above arguments it easily follows that $G_1(C[X])$ is a group. Now it is apparent that $G_2(C[X])$ is also a group. Taking into consideration this and the fact that $N[X]$ is a nil-ideal of $C[X]$, one can easily check that $G_3(C[X])$ is a group. The inclusions

$$X \subseteq G_1(C[X]) \subseteq G_2(C[X]) \subseteq G_3(C[X])$$

are obvious.

Henceforth if I is an ideal of a (not necessarily commutative) ring, then a_I will denote a natural homomorphism of A onto A/I. Moreover, $a_I\otimes 1$ will denote a homomorphism of a ring $A[X]$ onto $(A/I)[X]$ induced by a_I.

The following result describes the behaviour of just introduced groups under the change of coefficients.

THEOREM 1.2. *Let I be an ideal of a ring C. Then*

(1) *$(a_I\otimes 1)\big(G_k(C[X])\big) \subseteq G_k(C/I)[X]$ for $k = 0, 1, 2, 3$;*

(2) *if $I \subseteq N$, then $a_I\otimes 1$ is a homomorphism of $G_k(C[X])$ onto $G_k(C/I)[X]$ for $k = 0, 1, 2, 3$;*

(3) *if I contains no non-zero idempotents, then $a_I\otimes 1$ is an injection of $G_1(C[X])$ into $G_1(C/I)[X]$;*

(4) *if $I \supseteq N$, then $(a_I\otimes 1)G_3(C[X]) \subseteq G_2(C/I)[X]$.*

Proof. (1) Since homomorphisms preserve invertibility, idempotence and nilpotence of elements and orthogonality of idempotents, (1) is directly verified.

(2) The statement is obvious for $k = 0$. Assume that $I \subseteq N$. Let

$$v = \sum_{i=1}^{n} \bar{e}_i x_i \in G_1(C/I)[X], \quad \text{where} \quad 1 = \sum_{i=1}^{n} \bar{e}_i$$

be a decomposition of the unity element of the ring C/I into a sum of orthogonal idempotents. Since I is a nil-ideal, the decomposition $1 = \sum_{i=1}^{n} \bar{e}_i$ can be lifted, as it is known [9], to the decomposition $1 = \sum_{i=1}^{n} e_i$ such that $\bar{e}_i = a_I(e_i)$.

It is clear that $\bar{v} = (a_I\otimes 1)\left(\sum_{i=1}^{n} e_i x_i\right)$. Now it is easy to complete the proof of (2) for $k = 2, 3$.

(3) Let I contains no non-zero idempotents and suppose that $v = \sum_{i=1}^{n} e_i x_i \in G_1(C[X])$ is such that $(a_I\otimes 1)(v) = 1$. We may assume that $e_i \neq 0$ for any $1 \leqslant i \leqslant n$ and that $x_i \neq x_j$ for $i \neq j$. Then $1 = (a_I\otimes 1)(v) = \sum_{i=1}^{n} a_I(e_i)x_i$. Hence it follows that $n = 1$ and $v = 1$.

Proof of (4) follows directly from the definitions.

The following result characterizes elements of the investigated subgroups:

THEOREM 1.3. *Let $p \in C[X]$. Then*

(1) *$p \in G_1(C[X])$ iff $\sum_{x\in X} p_x = 1$ and $p_x p_y = 0$ for $x \neq y$;*

(2) *$p \in G_2(C[X])$ iff $\sum_{x\in X} p_x \in U(C)$ and $p_x p_y = 0$ for $x \neq y$;*

(3) *the following statements are equivalent:*

(a) *$p \in G_3(C[X])$,*

(b) *$(a_N\otimes 1)(p) \in G_2(C/N)[X]$,*

(c) *$\sum_{x\in X} p_x \in U(C)$ and $p_x p_y \in N$ for $x \neq y$.*

Proof. (1) If $p \in G_1(C[X])$, then there exists $E \in D(C)$ such that $p \in X_E$, thus the coefficients of p satisfy our conditions. On the contrary, if $p \in C[X]$ satisfies the conditions $\sum_{x \in X} p_x = a$ and $p_x p_y = 0$ for $x \neq y$, then it is easy to see that non-zero elements p_x give an orthogonal decomposition of the unity element into idempotents and thus $p \in G_1(C[X])$.

(2) If $p \in G_2(C[X])$, then from the definition we get $p = uv$, where $u \in U(C)$, $v \in G_1(C[X])$. It follows immediately that the required conditions are satisfied. On the contrary, if $u = \sum_{x \in X} p_x \in U(C)$ and $p_x p_y = 0$ for $x \neq y$, then an element $u^{-1}p$ fulfils the conditions of (1) and therefore it belongs to $G_1(C[X])$. This implies that $p \in G_2(C[X])$.

(3) Assume that $p \in G_3(C[X])$. By Theorem 1.2 it follows that $(a_N \otimes 1)(p) \in G_2(C/N)[X]$. Now suppose that $(a_N \otimes 1)(p) \in G_2(C/N)[X]$. Applying (2), we get that $\sum_{x \in X} a_N(p_x) \in U(C/N)$ and $a_N(p_x) a_N(p_y) = 0$ for $x \neq y$. Thus $\sum_{x \in X} p_x \in U(C)$ and $p_x p_y \in N$ for $x \neq y$. On the other hand, if $\sum_{x \in X} p_x \in U(C)$ and $p_x p_y \in N$ for $x \neq y$, then by (2) we obtain that $(a_N \otimes 1)(p) \in G_2(C/N)[X]$. By Theorem 1.2 it follows that there exists $u \in G_2 \times (C[X])$ such that $(a_N \otimes 1)(u) = (a_N \otimes 1)(p)$. Hence $t = p - u \in \ker(a_N \otimes 1) = N[X]$ and $p = u + t \in G_3(C[X])$.

If $p \in G_1(C[X])$, then it has a unique presentation by means of idempotents e_x and a support of p. If $p \in G_2(C[X])$, then it has a unique presentation of a form $p = uv$ where $u \in U(C)$, $v \in G_1(C[X])$. On the contrary, if $p \in G_3(C[X])$, then the presentation $p = uv + t$, where $u \in U(C)$, $v \in G_1(C[X])$, $t \in N[X]$, is not unique in general. If $p \in G_3(C[X])$ and $p = uv + t$ where $u \in U(C)$, $v \in G_1(C[X])$, $t \in N[X]$, and $v_x t_x = 0$ for any $x \in X$, then we shall say that p is *presented in a canonical form.*

LEMMA 1.4. *Let $p = uv + t \in G_3([X])$. Then an element $v \in G_1([X])$ is uniquely determined. Moreover, there is exactly one presentation of p in a canonical form.*

Proof. Let $p = uv + t = u_1 v_1 + t_1$. Then $(a_N \otimes 1)(p) = (a_N \otimes 1)(u)(a_N \otimes 1)(v) = (a_N \otimes 1)(u_1)(a_N \otimes 1)(v_1) \in G_2(C/N)[X]$. By the foregoing remarks it follows that $(a_N \otimes 1)(v) = (a_N \otimes 1)(v_1)$. Theorem 1.2 (3) yields that $v = v_1$, and the first part of the lemma is proved.

Now let $t' = \sum_{x \in X} (1 - v_x) t_x x$. Since $t \in N[X]$, we have that $t' \in N[X]$. Put $u' = u + \sum_{x \in X} v_x t_x$. Then $u' \in U(C)$, as $t_x \in N$. Since for any $x \in X$ holds an equality $(u'v + t')_x = p_x$, we have $p = u'v + t'$ and $v_x t'_x = v_x(1 - v_x) t_x = 0$. Thus we have shown that there exists a presentation $p = u'v + t'$ of p in a canonical form. Now let $u''v + t''$ be the second canonical presentation of an element p and let $x \in X$. Then multiplying the equality $u'v_x + t'_x = u''v_x + t''_x$ by v_x and applying the hypothesis we get $u'v_x = u''v_x$. Since $\sum_{x \in X} v_x = 1$, $u' = u''$ and consequently $t' = t''$, which ends the proof.

If $p \in G_3(C[X])$ and $p = uv + t$ is a presentation of p in a canonical form, then v will also be denoted by (p).

By Lemma 1.4 it follows that δ is a well defined mapping of $G_3(C[X])$ into $G_1(C[X])$. Similarly if $I \subseteq C$ is an ideal, then we have a mapping $\delta_I\colon G_3(C/I)[X] \to G_1(C/I)[X]$. Obviously, $\delta_{(0)} = \delta$.

THEOREM 1.5. *Let I be an ideal of C. Then the above defined mapping δ_I is a homomorphism of a group $G_3(C/I)[X]$ onto a group $G_1(C/I)[X]$ such that $\delta_I = \delta_I^2$. Moreover, the following diagram is commutative:*

$$\begin{array}{ccc} G_3(C[X]) & \xrightarrow{\delta_0} & G_1(C[X]) \\ \downarrow{\scriptstyle a_I \otimes 1} & & \downarrow{\scriptstyle a_I \otimes 1} \\ G_3(C/I)[X] & \xrightarrow[\delta_I]{} & G_1(G/I)[X] \end{array}$$

Proof. A routine verification shows that δ_I is a homomorphism of groups. Now if $v \in G_1(C/I)[X] \subseteq G_3(C/I)[X]$, then $\delta_I(v) = v$ which implies that $\delta_I = \delta_I^2$ and $\delta_I(G_3(C/I)[X]) = G_1(C/I)[X]$.

Theorem 1.2 and Lemma 1.4 yield the rest of the proof.

In the sequel we shall need ideals of C generated by idempotents. Let $B(C)$ denote a set of idempotents of the ring C. Then $B(C)$, together with operations $\vee$ and $\wedge$ defined by:

$$e \vee f = e + f - ef, \qquad e \wedge f = ef,$$

is a Boolean algebra [12]. If $e_1, \ldots, e_n \in B(C)$, then the ideal $e_1 C + \ldots + e_n C$ is generated by $e_1 \vee \ldots \vee e_n$.

This implies the following:

LEMMA 1.6. *Let I be an ideal of the Boolean algebra $B(C)$. Then*

(1) *if S is a finite subset of IC, then there exists an element $e \in I$ such that $es = s$ for any $s \in S$;*

(2) $IC \cap B(C) = I$.

Let $P(C)$ be the Stone space of the Boolean algebra $B(C)$, i.e. the set of maximal ideals of $B(C)$ [12].

LEMMA 1.7. $\bigcap_{I \in P(C)} IC = 0$.

Proof. Let $J = \bigcap_{J \in P(C)} IC$ and let $a \in J$. Suppose $a \neq 0$. Moreover, let $K = \{e \in B(C)\colon ea = 0\}$. It is easily seen that K is an ideal in $B(C)$ and $1 \notin K$. Let $I \in P(C)$ be such that $K \subset I$. Therefore $a \in IC$. By Lemma 1.6 there exists $e \in I$ such that $a = ea$, i.e. $1 - e \in K \subset I$. Hence $1 = e \vee 1 - e$ which is impossible.

THEOREM 1.8. *Let* $p \in U(C[X])$. *Then*

(1) $p \in G_1(C[X])$ *iff* $(\alpha_{IC}\otimes 1)(p) \in X$ *for every* $I \in P(C)$.

(2) $p \in G_2(C[X])$ *iff the support of* $(\alpha_{IC}\otimes 1)(p)$ *is a singleton for every* $I \in P(C)$.

(3) $p \in G_3(C[X])$ *iff* $(\alpha_{IC}\otimes 1)(p)$ *has, for every* $I \in P(C)$, *exactly one coefficient which is not nilpotent.*

Proof. (1) Let $p = \sum_{x\in X} p_x x \in G_1(C[X])$ and let $I \in P(C)$. Since non-zero elements p_x form a decomposition of the unity element into orthogonal idempotents, there exists exactly one element $y \in X$ such that $py \notin I$. Hence $1-py \in I$ and therefore

$$(\alpha_{IC}\otimes 1)(p) = (\alpha_{IC}\otimes 1)(p_y y) = \alpha_{IC}(y) = y \in X.$$

Now assume that $(\alpha_{IC}\otimes 1)(p) \in X$ for any $I \in P(C)$. Then Lemma 1.7 yields immediately that $\sum_{x\in X} p = 1$ and $p_x p_y = 0$ for $x \neq y$, i.e. $p \in G_1(C[X])$ by Theorem 1.3.

(2) If $p \in G_2(C[X])$, then (1) implies that a support of an element $(\alpha_{IC}\otimes 1)(p)$ is a singleton for any $I \in P(C)$. If we assume that, for any $I \in P(C)$, a support of $(\alpha_{IC}\otimes 1)(p)$ is a singleton, then for any $x, y \in X$ we obtain that $p_x p_y \in I$, and thus, by Lemma 1.7, $p_x p_y = 0$. Obviously, $\sum_{x\in X} p_x \in U(C)$, hence applying Theorem 1.3 we get $p \in G_2(C[X])$.

(3) If $p \in G_3(C[X])$, then by (2), $(\alpha_{IC}\otimes 1)(p)$ has exactly one non-nilpotent coefficient, for every $I \in P(C)$.

Now assume that, for any $I \in P(C)$, $(\alpha_{IC}\otimes 1)(p)$ has exactly one non-nilpotent coefficient. Of course, $\sum_{x\in X} p_x \in U(C)$. Let $x, y \in \operatorname{Supp}(p)$, $x \neq y$, $a = p_x p_y$ and let

$$I = \{e \in B(C) : \bigvee_{k\geqslant 1} a^k e = 0\}.$$

It is easily seen that I is an ideal in $B(C)$. Let us suppose that $I \neq B(C)$ and let M be the maximal ideal in $B(C)$ containing I. By hypothesis, the image of a in C/MC is nilpotent, i.e. there exists an integer n such that $a^n \in MC$. Therefore, by Lemma 1.6, there exists an $e \in M$ such that $a^n = ea^n$, i.e. $1-e \in I \subset M$. Hence $1 \in M$, which is impossible. Thus we have shown that $1 \in I$, i.e. $a = p_x p_y$ is a nilpotent element. By Theorem 1.3 we get that $p \in G_3(C[X])$.

2. u.p.-groups

We shall prove that if X is an u.p.-group, then $U(C[X]) = G_k(C[X])$, $k = 1, 2, 3$, and the result depends only on properties of the ring C.

Let us recall that if R is a ring, then a group $U(R[X])$ is said to be *trivial* if $U(R[X]) = U(R)X$ ([15])

LEMMA 2.1. $U(C[X]) = G_2(C[X])$ *iff C is the subdirect sum of rings C_i, $i \in I$, such that groups $U(C_i[X])$ are trivial for all $i \in I$.*

Proof. Let C be a subdirect sum of rings C_i, $i \in I$, such that all the groups $U(C_i[X])$ are trivial and let a_i, $i \in I$, be homomorphisms of the ring C onto the rings C_i. By hypothesis, if $u \in U(C[X])$, then $x \neq y$ implies that for every $i \in I$ either $a_i(u_x) = 0$ or $a_i(u_y) = 0$, i.e., for every $i \in I$, $a_i(u_x u_y) = 0$. Since $\bigcap_{i\in I} \ker a_i = 0$, $u_x u_y = 0$. Therefore, by Theorem 1.3, $u \in G_2(C[X])$.

Let us now suppose that $U(C[X]) = G_2(C[X])$. Let $M \in P(C)$ and let $J = MC$. We will show that the group $U(C/J)[X]$ is trivial. Let $p \in U(C/J)[X]$. There are elements $q_1, q_2 \in C[X]$ such that $(\alpha_J\otimes 1)(q_1) = p$, $(\alpha_J\otimes 1)(q_2) = p^{-1}$. Therefore $(\alpha_J\otimes 1)(q_1q_2-1) = 0$, i.e. $q_1q_2-1 \in J[X]$. Now, by Lemma 1.6, there exists an $e \in M$ such that $e(q_1q_2-1) = q_1q_2-1$, i.e. $(1-e)q_1q_2 = 1-e$. Let $q = (1-e)q_1+e$. Then $q \in U(R[X])$. Since $e \in M$, $(\alpha_J\otimes 1)(q) = (\alpha_J\otimes 1)(q_1) = p$. By hypothesis, $q \in G_2(C[X])$. By Theorem 1.8, the support of $(\alpha_J\otimes 1)(q) = p$ is a singleton, i.e. $U(C/J)[X]$ is trivial. From Lemma 1.7 it now follows that C is a subdirect product of rings C_i such that the groups $U(R_i[X])$ are trivial.

THEOREM 2.2. *Let* $X \neq \{1\}$ *be a u.p.-group. Then* $U(C[X]) = G_2(C[X])$ *iff* $N = 0$.

Proof. It is known that any commutative ring without non-trivial nilpotent elements is a subdirect sum of rings without zero divisors. It is also known ([15], [8]) that if R is a ring without zero divisors and x is a t.u.p.-group, then the group $U(R[X])$ is trivial.

Now, applying the equality of classes of t.u.p.-groups and u.p.-groups, proved by Strojnowski in [14], and Lemma 2.1, we obtain that if $N = 0$, then $U(C[X]) = G_2(C[X])$. The converse implication is clear.

Theorem 2.2 and Theorem 1.2 immediately imply the following results:

THEOREM 2.3. *If* $X \neq \{1\}$ *is a u.p.-group, then the group* $U(C[X])$. *is trivial iff the ring C has neither non-trivial idempotent nor nilpotent elements*)

THEOREM 2.4. *If* X *is a u.p.-group, then* $U(C[X]) = G_3(C[X])$.

If $p \in G_2(C[X])$ is an element of a finite order, then one may verify directly that $p \in U(C)$. Providing some additional assumptions on C and X, we get a more general result:

LEMMA 2.5. *Let us suppose that the additive group of the nilradical N of the ring C is torsion-free and let $p \in G_3(C[X])$. Moreover, suppose that* *there exists an $n \geqslant 1$ such that $p^n \in G_2(C[X])$. Then $p \in G_2(C[X])$ if any of the following conditions holds:*

(1) *X is linearly ordered group;*

(2) *the element p and $\delta(p)$ commute.*

Proof. Let $p = uv + t$ be the canonical form of an element p. It is enough to show that $t = 0$. Suppose for a moment that $v \in X$. Moreover, let I be an ideal of C generated by the coefficients of t. Clearly, I is a nilpotent ideal. Let $I^k = 0$, $I^{k-1} \neq 0$. If $k = 1$, then $t = 0$, obviously. If $k > 1$, consider $J = \{c \in N\colon mc \in I^{k-1}$ for some $m \geqslant 1\}$. Additive group of a ring $\overline{N} = N/J$ is torsion-free. Now, an element $(\alpha_J \otimes 1)(p)$ has a canonical form $(\alpha_J\otimes 1)(u)(\alpha_J\otimes 1)(v) = (\alpha_J\otimes 1)(t)$ and ideal of C/J generated by the coefficients of $(\alpha_J\otimes 1)(t)$ is of nilpotency index $\leqslant k-1$.

By induction hypothesis we get that $(\alpha_J\otimes 1)(t) = 0$, i.e. $t \in J[X]$. Since the additive group of the ring N is torsion-free, from the definition it follows that $J^2 = 0$; hence $I^2 = 0$. Then

$$p^n = u^n v^n + \sum_{i=0}^{n-1} u^i v^i t u^{n-i-1} v^{n-i-1}.$$

As $v \notin \operatorname{Supp}(t)$ and $p^n \in G_2(C[X])$, then

$$0 = \sum u^i v^i t u^{n-i-1} v^{n-i-1} = u^{n-1} \sum v^i t v^{n-i-1},$$

i.e.

$$\sum v^i t v^{n-i-1} = 0.$$

Now let $x \in \operatorname{Supp}(t)$. Then there exists a $v' \in \operatorname{Supp}(t)$ such that

$$x^{n-1} v = x'^i \vee v'^{n-i-1} \quad \text{for a certain } i \neq n-1.$$

Now it is easy to show that every element of a support of t commutes with a certain power of the element v of a natural exponent. By imposed assumptions every element of $\operatorname{Supp}(t)$ commutes with v and thus equality $\sum v^i t v^{n-i-1}$ implies $0 = \sum v^i t v^{n-i-1} = n v^{n-1} t$. As the additive group of N is torsion-free and v is invertible, we have $t = 0$, which ends the proof in this case.

Now if $v \in G_1(C[X])$, then there exists an $E \in D(C)$ such that $v \in X_E$. Let $e \in E$. Then $ep = euv + et = eue \cdot x_e + et$ is, as it is easy to verify, a canonical form of an element $ep \in G_3(eC[X])$. The first part of the proof applied to the ring $eC[X]$ yields that $et = 0$. Since e is an arbitrary element of E, we have $t = 0$, which ends the proof.

THEOREM 2.6. *Let an additive group of the ring N be torsion-free and let X be a u.p.-group. Then if $p \in U(C[X])$ is an element of a finite order, then $p \in U(C)$.*

Proof. By Theorem 2.4 we obtain that $p \in G_3(C[X])$. Applying Theorem 1.5, we get that $\delta(p)$ is an element of a finite order in $G_1(C[X])$, and hence $\delta(p) = 1$. Let $p = uv + t$ be the presentation of p in a canonical form. Since $v = \delta(p)$ and t commute, Lemma 2.5 yield that $t = 0$ and $p = u \in U(C)$.

Remark. Let R be a ring in which every idempotent is central and the set of all nilpotent elements is a locally nilpotent ideal of R. It is known that there exist non-commutative rings possessing such properties [3]. If X is a semigroup, then, similarly as in § 1, we may define a sequence of subgroups $G_k(R[X])$ and, replacing u.p.-property with t.u.p.-property, we may prove counterparts of all the results we have obtained hitherto. The results of this kind have been announced in [2].

3. Categories of group rings

Now we shall study homomorphisms of group rings with not necessarily commutative coefficients. We shall use simple notions of a category theory in order to make formulations more convenient.

If R is a ring, then $R[\cdot]$ will denote a category whose the objects are group rings with coefficients in R, and whose class of morphisms is a class of all homomorphisms of such rings constant on R.

Henceforth we shall assume that A is a ring whose centre is C. First we shall show that it is possible to reduce the study of categorical properties of $A[\cdot]$ to the study of analogical properties of $C[\cdot]$.

THEOREM 3.1. *There exists a natural isomorphism $\mathcal{R}$ between the categories $A[\cdot]$ and $C[\cdot]$.*

Proof. Put $\mathcal{R}(A[X]) = C[X]$. If $\varphi \in \operatorname{Hom}(A[X], A[Y])$, then define $\mathcal{R}(\varphi) = \varphi|C[X]$. To show that $\mathcal{R}$ is a functor it suffices to prove that $\varphi(X) \subseteq U(C[Y])$.

Let us observe first that $U(A[Y]) \cap C[Y] = U(C[Y])$, as $C[Y]$ is a centralizer of A in $A[Y]$.

Now, let $x \in X$. Then it is clear that $\varphi(x) \in U(A[Y])$, and, for any $a \in A$, the elements $a = \varphi(a)$ and $\varphi(x)$ commute. Therefore $\varphi(x) \in C[Y]$, i.e. $\varphi(x) \in U(A[Y]) \cap C[Y] = U(C[X])$.

From the definition of $A[\cdot]$ it follows that $\mathcal{R}$ is an injection of morphisms. Now, let $\psi \in \operatorname{Hom}(C[X], C[Y])$, and let

$$\varphi\Big(\sum a_i x_i\Big) = \sum a_i \psi(x_i), \qquad a_i \in A,\ x_i \in X.$$

It is easy to verify that $\varphi \in \operatorname{Hom}(A[X], A[Y])$ and $\mathcal{R}(\varphi) = \psi$. Hence $\mathcal{R}$ is an isomorphism.

Now, let $\varphi \in \mathrm{Hom}(A[X], A[Y])$. If φ is an injection, then so is $\mathscr{R}(\varphi)$. Moreover, if $\mathscr{R}(\varphi)$ is a surjection, so is φ. Answers to the following questions remain unknown to the author: are the following statements true?

(1) if $\mathscr{R}(\varphi)$ is an injection, so is φ;

(2) if φ is a surjection, so is $\mathscr{R}(\varphi)$;

(3) if φ is a monomorphism, then φ is an injection;

(4) if φ is an epimorphism, then φ is a surjection.

If the answer to (3) is positive, so is the answer to (1). Positive answers to the first two questions would simplify further considerations, which would help, among others things, to find criteria for a given homomorphism of group rings to be an injection or a surjection.

Let I be an ideal of a ring A, let X, Y be groups, and let $\varphi \in \mathrm{Hom}(A[X], A[Y])$. Now, put $\mathscr{F}_I(A[X]) = (A/I)[X]$ and put $\mathscr{F}_I(\varphi)$ to be the only morphism from $(A/I)[X]$ into $(A/I)[Y]$ such that the diagram

$$\begin{array}{ccc} A[X] & \longrightarrow & A[Y] \\ {\scriptstyle a_I\otimes 1}\downarrow & & \downarrow{\scriptstyle a_I\otimes 1} \\ (A/I)[X] & \xrightarrow{\mathscr{F}_I(\varphi)} & (A/I)[Y] \end{array}$$

commutes.

Immediately, from the definition it follows that $\mathscr{F}_I$ is a functor from the category $A[\cdot]$ into the category $(A/I)[\cdot]$. It is also clear that functor $\mathscr{F}_I$ preserves surjections and isomorphisms.

Lemma 3.2. *Let I be an ideal of A. If I is left or right T-nilpotent, then $\mathscr{F}_I$ reflects surjections.*

Proof. Let $\varphi \in \mathrm{Hom}(A[X], A[Y])$ and let $\mathscr{F}_I(\varphi)$ be a surjection. Assume that I is left T-nilpotent. Since $\mathscr{F}_I(\varphi)$ is a surjection, we have

$$A[Y] = \varphi(A[X]) + I[Y] = \varphi(A[X]) + I\cdot A[Y].$$

If M is a quotient left A-module $A[X]/\varphi(A[X])$, then by the above equality $M = IM$. The left T-nilpotency of M implies now that $M = (0)$ ([11]). Therefore $A[Y] = \varphi(A[X])$. If I is right T-nilpotent, the proof is analogous.

Remark. The following result may be proved in the same way as Lemma 1.6 and Lemma 1.7.

Lemma 3.3. *Let I be an ideal of the Boolean algebra $B(C)$. Then*

(1) *if S is a finite subset of IA, then there exists an $e \in I$ such that $es = s$ for any $s \in S$;*

(2) $I(A) \cap B(C) = I$ *and* $IA \cap C = IC$.

Moreover,

(3) $\bigcap_{I\in B(C)} IA = (0)$.

Theorem 3.4. *Let $\varphi \in \mathrm{Hom}(A[X], A[Y])$. Then*

(1) *φ is an injection iff $\mathscr{F}_{IA}(\varphi)$ is an injection for any $I \in P(C)$;*

(2) *φ is a surjection iff $\mathscr{F}_{IA}(\varphi)$ is a surjection for any $I \in P(C)$;*

(3) *φ is an isomorphism iff $\mathscr{F}_{IA}(\varphi)$ is an isomorphism for any $I \in P(C)$.*

Proof. (1) Let φ be an injection and let $I \in P(C)$. If $p + IA[X] \in \ker \mathscr{F}_{IA}(\varphi)$, then $\varphi(p) \in IA[Y]$. Hence, by Lemma 3.3, there exists an idempotent $e \in I$ such that $\varphi(p) = e\varphi(p) = \varphi(ep)$. Therefore $\varphi(p - ep) = 0$. Since φ is an injection, $p = ep$ and this means that $p \in IA[X]$. Hence $\mathscr{F}_{IA}(\varphi)$ is an injection. Now, if $\mathscr{F}_{IA}(\varphi)$ is an injection for every $I \in P(C)$, then it is easily verified that

$$\ker\varphi = \bigcap_{I\in P(C)} IA[X] = \Big(\bigcap_{I\in P(C)} IA\Big)[X].$$

But, by Lemma 3.3, $\bigcap_{I\in P(C)} IA = 0$ and therefore φ is an injection.

(2) If φ is a surjection, then, of course, $\mathscr{F}_{IA}(\varphi)$ is a surjection for every $I \in P(C)$.

Let us now suppose that $\mathscr{F}_{IA}(\varphi)$ is a surjection for every $I \in P(C)$. Let $q \in A[Y]$ and let $J_q = \{e \in B(C)\colon eq \in \varphi(A[X])\}$. Of course, $0 \in J_q$. If $e \in J_q$ and $f \leqslant e$, then $f \in J_q$. Moreover, if $e_1, e_2 \in J_q$ are such that $e_1 q = \varphi(p_1)$, $e_2 q = \varphi(p_2)$, then

$$(e_1 \vee e_2)q = e_1 q + e_2 q - e_1 e_2 q = \varphi(p_1) + \varphi(p_2) - \varphi(e_1 p_2) = \varphi(p_1 + p_2 - e_1 p_2).$$

Thus, we have shown that J_q is an ideal of $B(C)$. Let us suppose $J_q \neq B(C)$. Let M be a maximal ideal of $B(C)$ containing J_q. By hypothesis, $\mathscr{F}_{MA}(\varphi)$ is a surjection, i.e. there exists a $p \in A[X]$ such that $\varphi(p) \in q + MA[Y]$. Therefore $q - \varphi(p) \in MA[Y]$. By Lemma 3.3 there exists an $e \in M$ such that $q - \varphi(p) = e\big(q - \varphi(p)\big)$, i.e.

$$(1-e)q = q - eq = \varphi(p) - e\varphi(p) = \varphi\big((1-e)p\big).$$

Therefore $1 - e \in J_q \subseteq M$, i.e. $1 = (1-e) \vee e \in M$ which is impossible. Therefore $J_q = B(C)$ and so $q \in \varphi(A[X])$. This shows that φ is a surjection.

(3) follows directly from (1) and (2).

Proposition 3.5. *Let $\varphi, \psi \in \mathrm{Hom}(A[X], A[Y])$. Then if $\mathscr{F}_{IA}(\varphi) = \mathscr{F}_{IA}(\psi)$ for every $I \in P(C)$, then $\varphi = \psi$.*

Proof. It is easily seen that for every $p \in A[X]$ the element $\varphi(p) - \psi(p)$ belongs to $\bigcap_{I\in P(C)} IA[Y] = (0)$.

If $\varphi \in \mathrm{Hom}(A[X], A[Y])$, then φ will be called a G_k-*homomorphism*

$(k = 0, 1, 2, 3)$ whenever $\varphi(X) \in G_k(C[Y])$. The set of G_k-homomorphisms will be denoted by $\mathrm{Hom}_k(A[X], A[Y])$.

THEOREM 3.6. *Let Y be an u.p.-group and let X be any group. Then*

(1) $\mathrm{Hom}(A[X], A[Y]) = \mathrm{Hom}_3(A[X], A[Y])$,

(2) *if* $N = 0$, *then* $\mathrm{Hom}(A[X], A[Y]) = \mathrm{Hom}_2(A[X], A[Y])$.

The proof follows directly from Theorem 2.2 and Theorem 2.4.

THEOREM 3.7. *For every $k = 0, 1, 2, 3$, group rings with coefficients in A together with G_k-homomorphisms as morphisms form a category. Moreover, if $\varphi \in \mathrm{Hom}_k(A[X], A[Y])$ and φ is an isomorphism in $A[\cdot]$, then $\varphi^{-1} \in \mathrm{Hom}_k(A[X], A[Y])$.*

Proof. For $k = 0$, the theorem is obvious. Now, let φ and ψ be G_k-homomorphisms such that $\varphi\psi$ is defined. If $I \in P(C)$, then $\mathscr{F}_{IA}(\varphi\psi) = \mathscr{F}_{IA}(\varphi)\mathscr{F}_{IA}(\psi)$. Now, Theorem 1.8 yields the rest of the proof.

Further on, $A_k[\cdot]$ will denote categories referred to in Theorem 3.7. Immediately from the definition it follows that $\mathscr{R}$ can be considered as an isomorphism of categories $A_k[\cdot]$ and $C_k[\cdot]$, for any $k = 0, 1, 2, 3$.

Moreover, by Theorem 1.2, it follows that, for every ideal I if A, the functor $\mathscr{F}_I$ can be considered as a functor of the category $A_k[\cdot]$ into the category $(A/I)_k[\cdot]$.

4. G_0-homomorphisms and G_1-homomorphisms

Let $\beta\colon X \to Y$ be a homomorphism of groups. Let $1\otimes\beta$ denote the homomorphism of a ring $A[X]$ into $A[Y]$ defined by the rule:

$$1\otimes\beta\Big(\sum a_i\cdot x_i\Big) = \sum a_i\beta(x_i).$$

Obviously, every morphism from $\mathrm{Hom}_0(A[X], A[Y])$ is of the form $1\otimes\beta$ for some homomorphism $\beta\colon X \to Y$ of groups. Moreover, β is determined uniquely. The following result is inserted exclusively for the completeness.

THEOREM 4.1. *Let $\varphi = 1\otimes\beta \in \mathrm{Hom}_0(A[X], A[Y])$. Then*

(1) *φ is an injection iff β is an injection;*

(2) *φ is a surjection iff β is a surjection;*

(3) *φ is an isomorphism iff β is an isomorphism.*

Now we shall turn to the investigation of G_1-homomorphism. The following is an immediate consequence of Theorem 1.8.

PROPOSITION 4.2. *If $\varphi \in \mathrm{Hom}_1(A[X], A[Y])$, then for every $I \in P(A)$ there is a unique homomorphism $\beta_I = \beta_I(\varphi)\colon X \to Y$ such that $\mathscr{F}_{IA}(\varphi) = 1\otimes\beta_I$.*

Proposition 3.5 implies that, if φ and ψ are G_1-homomorphisms, then $\varphi = \psi$ iff, for any $I \in P(C)$, $\beta_I(\varphi) = \beta_I(\psi)$. Therefore, we shall often write $\varphi = \{(I, \beta_I(\varphi))\colon I \in P(C)\}$.

However, not every family of homomorphisms from the group X into the group Y determines a homomorphism from the ring $A[X]$ into the ring $A[Y]$.

For every $e \in B(C)$ let us put $P_e(C) = \{I \in P(C)\colon e \notin I\}$. We then have

THEOREM 4.3. *Let $\{(I, \gamma_I)\colon I \in P(C)\}$ be a set of pairs such that $\gamma_I\colon X \to Y$. Then this set determines a G_1-homomorphism $\varphi\colon A[X] \to A[Y]$ iff for any $x \in X$ there exists an $E \in D(C)$ such that $\gamma_I(x) = \gamma_J(x)$ for any $e \in E$ and $I, J \in P_e(C)$.*

The mapping φ is then given by:

$$\varphi(x) = \sum_{e\in E} e\gamma_{I_e}(x),$$

where $E \in D(C)$ is a decomposition of unity chosen for x and $I_e \in P_e(C)$.

Proof. Suppose that φ is a homomorphism such that $\beta_I(\varphi) = \gamma_I$ for $I \in P(C)$. Let $x \in X$. Then $\varphi(x) \in G_1(C[Y])$, i.e. there is an $E \in D(C)$ and there are elements y_e, $e \in E$, such that $\varphi(x) = \sum_{e\in I} ey_e$.

Let $e_0 \in E$ and let $I \in P_{e_0}(C)$. Then

$$\begin{aligned}\big(\mathscr{F}_{IC}(\varphi)\big)(x) &= \sum_{e\in E} ey_e + IC[Y]\\ &= e_0y_{e_0} + IC[Y] = e_0y_{e_0} + (1-e_0)y_{e_0} + IC[Y] = y_{e_0} + IC[Y],\end{aligned}$$

i.e.

$$\gamma_I(x) = \big(\beta_I(\varphi)\big)(x) = y_{e_0}.$$

Now, if $J \in P(C)$, then, as before, we have $\gamma_J(x) = y_{e_0} = \gamma_I(x)$. Therefore the family $\{\gamma_I\}_{I\in P(C)}$ of homomorphisms satisfies the conditions of the theorem.

Suppose now that the family $\{\gamma_I\}_{I\in P(C)}$ satisfies the condition of the theorem. If $x \in X$, then, by hypothesis, there exists an $E \in D(C)$ and there are ideals I_e, $e \in E$, such that the element $\varphi_E(x) = \sum_{e\in E} e\cdot\gamma_{I_e}(x)$ does not depend upon the choice of the ideals $I_e \in P_e(C)$. If $F \in D(C)$, $E \leqslant F$, then it is easily verified that the element $\varphi_F(x)$ is well defined and $\varphi_F(x) = \varphi_E(x)$. Since the set $D(C)$ is direct, we can put $\varphi(x) = \varphi_E(x)$. Since the definition of the element $\varphi(x)$ does not depend upon the choice of a sufficiently large decomposition of the unity, a simple verification shows that φ is a homomorphism from the group X into $G_1(C[Y])$, i.e. φ is a homomorphism from $A[X]$ into $A[Y]$. It is easy to see that $\beta_I(\varphi) = \gamma_I$ for every $I \in P(C)$, which ends the proof.

In a topological language, Theorem 4.3 can be stated as follows: *If we introduce a discrete topology in X, and a standard topology in $P(C)$ [12], then continuous functions from $P(C)$ into X form a group with respect to the action on values.* Now a set of pairs $\{(I, \gamma_I): I \in P(C), \gamma_I: A[X] \to A[Y]\}$ determines a G_1-homomorphisms iff the homomorphism of X into $Y^{P(C)}$, induced by this family, is a homomorphism of X into the group of continuous functions on $P(C)$.

THEOREM 4.4. *Let φ be a G_1-homomorphism from the ring $A[X]$ into the ring $A[Y]$. Then*

(1) *φ is an injection iff $\{\beta_I(\varphi): I \in P(C)\}$ is a set of injections,*

(2) *φ is a surjection iff $\{\beta_I(\varphi): I \in P(C)\}$ is a set of surjections,*

(3) *φ is an isomorphism iff $\{\beta_I(\varphi): I \in P(C)\}$ is a set of isomorphisms.*

The proof follows directly from Theorems 3.4, 4.2 and 4.1.

THEOREM 4.5. *Let I be an ideal of the ring A and let φ be a G_1-homomorphism from the ring $A[X]$ into the ring $A[Y]$. Then*

(1) *if φ is an injection, so is $\mathscr{F}_I(\varphi)$.*

Moreover, if $I \cap B(C) = 0$, then

(2) *if $\mathscr{F}_I(\varphi)$ is an injection, so is φ;*

(3) *if $\mathscr{F}_I(\varphi)$ is a surjection, so is φ;*

(4) *if $\mathscr{F}_I(\varphi)$ is an isomorphism, so is φ.*

Proof. If $\alpha: A \to A'$ is a homomorphism of rings, then α determines homomorphism from $B(C)$ into $B(C')$, where $C' = Z(A')$ and so also a continuous transformation of spaces $\alpha^*: P(C') \to P(C)$. By Theorem 4.4, it is sufficient to prove the following

LEMMA 4.6. *Let I be an ideal of the ring A, let $\alpha_I = \alpha$ and let $\varphi: A[X] \to A[Y]$ be a G_1-homomorphism. Let C' denote the centre of the ring $A' = A/I$. Then*

(1) *if $J \in P(C')$, then $\beta_J(\mathscr{F}_I(\varphi)) = \beta_{\alpha^*(J)}(\varphi)$,*

(2) *if $I \cap B(C) = 0$, then α^* is a surjection.*

Proof. (1) Let $J \in P(C')$. Then $\alpha^{-1}(J) \supseteq \alpha^*(J) \in P(C)$. Hence $A'/A'J \simeq A/\alpha^{-1}(J)A$ is an image of the ring $A/\alpha^*(J)A$. Since φ is a G_1-homomorphism, $\mathscr{F}_{\alpha^*(J)A}(\varphi) = 1 \otimes (\beta_{\alpha^*(J)}(\varphi))$ and similarly, $\mathscr{F}_{JA}(\mathscr{F}_I(\varphi)) = 1 \otimes \beta_J(\mathscr{F}_I(\varphi))$. Since $\alpha^*(J) \subseteq \alpha^{-1}(J)$, $\beta_J(\mathscr{F}_I(\varphi)) = \beta_{\alpha^*(J)}(\varphi)$.

If X and Y are groups, then the set $\mathrm{Hom}_1(A[X], A[Y])$ is described by Theorem 4.3. But this description is rather uncomfortable in application. With some additional hypotheses the description of the set $\mathrm{Hom}_1(A[X], A[Y])$ becomes more readable.

Let $\varphi: A[X] \to A[Y]$ be a G_1-homomorphism and let $E \in D(C)$.

We admit $\varphi \in \mathrm{Hom}_E(A[X], A[Y])$ if there are homomorphisms $\alpha_E: X \to Y$, $e \in E$, such that $\varphi(x) = \sum_{e\in E} e\alpha_e(x)$ for every $x \in X$, i.e. $\varphi(x) \in Y^E$. Therefore the set $\mathrm{Hom}_E(A[X], A[Y])$ can be identified in a natural way with the set $(\mathrm{Hom}(X, Y))^E$. Moreover, if $E \leqslant F$, then $\mathrm{Hom}_E(A[X], A[Y]) \subseteq \mathrm{Hom}_F(A[X], A[Y])$ and, respectively, $(\mathrm{Hom}(X, Y))^E \subseteq (\mathrm{Hom}(X, Y))^F$.

We shall say that a G_1-homomorphism φ from the ring $A[X]$ into the ring $A[Y]$ is *bounded* if there exists $E \in D(C)$ such that $\varphi \in \mathrm{Hom}_E(A[X], A[Y])$. Therefore, a G_1-homomorphism φ is bounded iff $\varphi \in \bigcup_{E\in D(C)} \mathrm{Hom}_E(A[X], A[Y])$.

We shall now show that if we introduce the discrete topology to $A[X]$ and $A[Y]$, then the set of bounded homomorphisms from $A[X]$ to $A[Y]$ is dense in the set $\mathrm{Hom}_1(A[X], A[Y])$ with the compact-open topology.

THEOREM 4.7. *If $\varphi: A[X] \to A[Y]$ is a G_1-homomorphism and if X_0 is a finitely generated subgroup in X, then there exists a bounded homomorphism $\psi: A[X] \to A[Y]$ such that $\varphi|_{A[X_0]} = \psi|_{A[X_0]}$.*

Proof. Let $x_1, \ldots, x_n$ be generators of the group X_0. By Theorem 4.3 there exists an $E \in D(C)$ and, for every $a \in E$, there exists a homomorphism $\alpha_e: X \to Y$ such that

$$\varphi(x_i) = \sum_{e\in E} e\alpha_e(x_i), \quad i = 1, \ldots, n.$$

Let $\psi(x) = \sum_{e\in E} e\alpha_e(x)$ for $x = X$. It is easily seen that $\varphi|_{A[X_0]} = \psi|_{A[X_0]}$.

THEOREM 4.8. *Let X and Y be groups. Then any of the following conditions implies that*

$$\mathrm{Hom}_1(A[X], A[Y]) = \bigcup_{E\in D(C)} \mathrm{Hom}_E(A[X], A[Y]).$$

(1) *$D(C)$ contains a maximal element;*

(2) *X is a finitely generated group;*

(3) *X is an abelian group of a finite rank and Y can be linearly ordered.*

Proof. (1) and (2) follow directly from the definition.

Let $\varphi \in \mathrm{Hom}_1(A[X], A[Y])$ and let $X_0 =$ be a finitely generated subgroup of X such that X/X_0 is a torsion group. Then, by Theorem 4.7, there exists an $E \in D(C)$ such that $\varphi|_{A[X_0]} \in \mathrm{Hom}_E(A[X_0], A[Y])$, i.e. for some α_e, $e \in E$, we have $\varphi(y) = \sum_{e\in E} e\cdot\alpha_e(y)$ for $y \in Y$. Now, if $x \in X$, then from Theorem 4.2 it follows that there exists an $F \in D(C)$ and a homomorphisms β for $f \in F$ such that $\varphi(x) = \sum_{f\in F} f\cdot\beta_J$. We can assume that

$E \leqslant F$. If $x^n \in X_0$, then

$$\varphi(x^n) = \Big(\sum_{f\in F} f\cdot\beta_f\Big)^n = \sum_{f\in F} f\cdot\beta_f(x)^n.$$

On the other hand, $\varphi(x^n) = \sum_{e\in E} e\cdot e_a(x^n)$. Now, if $f = ef$, then we get $\big(a_e(x)\big)^n = a_e(x^n) = \beta_f(x^n) = \big(\beta_f(x)\big)^n$ and therefore $a_e(x) = \beta_f(x)$. Thus we have shown that $\varphi(x) = \sum_{e\in E} e\cdot a_e(x)$, i.e. $\varphi \in \mathrm{Hom}_E(A[X], A[Y])$.

If we replace all G_1-homomorphism by injections, surjections or isomorphisms, then we get a description analogous to the above using only injections, surjections or isomorphisms of groups. This follows from Theorem 4.4.

We will show that there exists a G_1-homomorphism which is not bounded.

EXAMPLE 4.9. Let $X_1, X_2, \ldots$ be a sequence of groups such that $\overline{\overline{X}}_i > 2$ and let $X \subset \mathop{\mathsf{X}}\limits_{i=1}^{\infty} X_i$ be the set of all elements almost everywhere equal to 1. Then, for every i, $X_i \subset X$ in a natural way. Since $\overline{\overline{X}}_i > 2$, there exists a non-trivial automorphism σ_i of the group X_i. Now, let A be a ring containing an infinite sequence of distinct central idempotent elements $e_1, e_2, \ldots$ Let us put $\varphi(x) = e_i x + (1-e_i)\sigma_i(x)$ for every $x \in X_i$. Since the groups X_i generate X and pairwise centralize each other, φ is a homomorphism from the group X into the group $G_1(A[X])$ and therefore φ is a G_1-endomorphism of the ring $A[X]$. It is easily seen that φ is a G_1-automorphism. If the orders of all the automorphisms σ_i have a common upper bound, then φ is an automorphism of finite order. If, in particular, we assume that the groups X_i are abelian, then we can choose automorphisms of order 2 and φ will be a G_1-automorphism of order 2 which is not bounded.

5. G_2- and G_3-homomorphisms

In this section G_2- and G_3-homomorphisms will be investigated. Theorem 1.5 enables us to relate a certain G_1-homomorphism to every G_3-homomorphism. Let $\varphi\colon A[X] \to A[Y]$ be a G_3-homomorphism and let $\big(\vartheta(\varphi)\big)(x) = \delta\big(\varphi(x)\big) \in G_1(C[Y])$. Then $\vartheta(\varphi)\colon A[X] \to A[Y]$ is a G_1-homomorphism. Since $\delta = \delta^2$, $\vartheta\big(\vartheta(\varphi)\big) = \vartheta(\varphi)$. Moreover, if φ is a G_1-homomorphism, then $\vartheta(\varphi) = \varphi$. Similarly, if I is an ideal of A and $\psi \in \mathrm{Hom}_3\big((A/I)[X], (A/I)[Y]\big)$, then let $\big(\vartheta_I(\psi)\big)(x) = \delta_{(I\cap C)A}\big(\psi(x)\big)$. Obviously, $\vartheta = \vartheta_{(0)}$.

THEOREM 5.1. *The mapping ϑ_I is a functor from the category $(A/I)[\cdot]_3$ onto the category $(A/I)[\cdot]_1$. If $I = JA$, where $J \in P(C)$, then $\mathscr{F}_I\vartheta$ is a functor onto the category $(A/I)[\cdot]_0$. Moreover, the diagram*

$$\begin{array}{ccc} A[\cdot]_3 & \xrightarrow{\ \vartheta\ } & A[\cdot]_1 \\ \downarrow{\scriptstyle \mathscr{F}_I} & & \downarrow{\scriptstyle \mathscr{F}_I} \\ (A/I)[\cdot]_3 & \xrightarrow{\ \vartheta_I\ } & (A/I)[\cdot]_1 \end{array}$$

commutes.

Proof. Let $\varphi \in \mathrm{Hom}_3(A[X], A[Y])$, $\psi \in \mathrm{Hom}_3(A[Y], A[Y])$. If $J \in P(C)$, then by Theorem 1.8 we get that for any $x \in X$ the equality $\delta_{JA}\big((\psi\varphi)\big)(x) = \big(\delta_{JA}(\psi)\,\delta_{JA}(\varphi)\big)(x)$ holds. Therefore, for any $x \in X$ we have $\delta\big(\psi\varphi((x))\big) = \delta(\psi)\,\delta(\varphi)(x)$. Thus $\vartheta(\psi\varphi) = \vartheta(\psi)\vartheta(\varphi)$ and thereby ϑ is a functor. Similarly it can be proved that ϑ_I is a functor. Clearly, it is a functor from the category $(A/I)[\cdot]_3$ into the category $(A/I)[\cdot]_1$. Commutativity of diagram follows directly from Theorem 1.5. The remaining part of the proof follows by Theorem 1.8.

Now, let γ be a homomorphism from the group X into $U(C)$. This homomorphism induces the automorphism γ^* of the ring $A[X]$ given by $\gamma^*(x) = \gamma(x)x$ for $x \in X$ ([9]). Of course, γ^* is a G_2-automorphism. If $\varphi\colon A[X] \to A[Y]$ is a G_1-homomorphism then, as it follows from Theorem 3.7, $\varphi\gamma^*\colon A[X] \to A[Y]$ is a G_2-homomorphism and $(\varphi\gamma^*)(x) = \gamma(x)\cdot\varphi(x)$ for every $x \in X$. Now, if $\gamma'\colon X \to U(C)$ is a group homomorphism and if $\varphi'\colon A[X] \to A[Y]$ is a ring homomorphism such that $\varphi\gamma^* = \varphi'(\gamma')^*$, then $\gamma = \gamma'$ and $\varphi = \varphi'$. This follows from Lemma 1.4. Of course, in this situation $\vartheta(\varphi\gamma^*) = \varphi$. If $\psi\colon A[X] \to A[Y]$ is a G_2-homomorphism, then, putting $\gamma(x) = \sum_{y\in Y}\big(\psi(x)\big)_y$, we obtain $\psi = \vartheta(\psi)\gamma^*$.

The above considerations imply the following

THEOREM 5.2. *Let $\varphi\colon A[X] \to A[Y]$ be a G_2-homomorphism. Then there exists a unique group homomorphism $\gamma\colon X \to U(C)$ such that $\varphi = \vartheta(\varphi)\gamma^*$ and the set* $\mathrm{Hom}_2(A[X], A[Y])$ *can be identified in a natural way with* $\mathrm{Hom}\big(X, U(C)\big)\times\mathrm{Hom}_1(A[X], A[Y])$.

Since γ^* is an automorphism of the ring $A[X]$, the following holds:

COROLLARY 5.3. *Let φ be a G_2-homomorphism; then*

(1) *φ is an injection iff $\vartheta(\varphi)$ is an injection;*

(2) *φ is a surjection iff $\vartheta(\varphi)$ is a surjection;*

(3) *φ is an isomorphism iff $\vartheta(\varphi)$ is an isomorphism.*

It is easily verified that not all G_3-homomorphisms have decomposition into automorphisms and G_1-homomorphisms.

Let $\varphi\colon A[X] \to A[Y]$ be a G_3-homomorphism. By Lemma 1.4 we then have the canonical presentation $\varphi(x) = u_x\cdot\big(\vartheta(x)\big)(x) + \theta(\varphi)(x)$ for every $x \in X$, where $\theta(\varphi)(x) \in N[Y]$. Let $T(\varphi)$ be an ideal in A generated by all the coefficients of elements $\theta(\varphi(x))$ for $x \in X$. Of course, $T(\varphi) \subseteq NA$. It can be verified that $T(\varphi)$ is the smallest ideal among those for which $\mathscr{F}_I(\varphi)$ is a G_2-homomorphism.

THEOREM 5.4. *Let $\varphi\colon A[X] \to A[Y]$ be a G_3-homomorphism. Then*

(1) *φ is an injection iff $\vartheta(\varphi)$ is an injection;*

(2) *if φ is a surjection, so is $\vartheta(\varphi)$;*

(3) *if $\vartheta(\varphi)$ is a surjection and $T(\varphi)$ is a left or right T-nilpotent ideal, then φ is a surjection;*

(4) *if $\vartheta(\varphi)$ is an isomorphism and $T(\varphi)$ is a left or right T-nilpotent ideal, then φ is an isomorphism.*

Proof. For a moment, let $T(\varphi) = 0$. Then φ is a G_2-homomorphism and the theorem follows from Corollary 5.3. Now, let $T(\varphi) = I$. Then, of course, $T(\mathscr{F}_I(\varphi)) = 0$. Moreover, by Theorem 5.1, $\vartheta(\mathscr{F}_I(\varphi)) = \mathscr{F}_I(\vartheta(\varphi))$.

(2) If φ is a surjection, then $\mathscr{F}_I(\varphi)$ is a surjection, therefore $F_I(\vartheta(\varphi)) = \vartheta(F_I(\varphi))$ is a surjection. Since $I \cap P(C) = 0$, it follows from Theorem 4.5 that $\vartheta(\varphi)$ is a surjection.

(3) Let $\vartheta(\varphi)$ be a surjection and let I be a left or right T-nilpotent ideal. Then, $\mathscr{F}_I(\vartheta(\varphi)) = \vartheta(\mathscr{F}_I(\varphi))$ is a surjection. Since $\mathscr{F}_I(\varphi)$ is a G_2-homomorphism, $\mathscr{F}_I(\varphi)$ is a surjection as follows from Corollary 5.3. The ideal I is left or right T-nilpotent and therefore φ is a surjection (see Lemma 3.2).

(1) Suppose for a moment that $\vartheta(\varphi) = 1\otimes\beta$ for a certain β. If β is not an injection, then there is $1 \neq x \in X$ such that $\beta(x) = 1$, i.e. $\varphi(x) = u_x + \sum a_k y_k$ for some $a_k \in N$ and $y_k \in Y$. Let S be the subring without unity generated by the elements a_k. Since S is a nilpotent ring, there are $0 \neq a \in A$ such that $aS = 0$. Then $\varphi(ax) = a\varphi(x) = a(u_x + \sum a_k y_k) = au_x = \varphi(au_x)$, i.e. $\varphi(ax - au_x) = 0$ and φ is not an injection. Now, if β is an injection, then it can be verified [8] that the elements $\varphi(x_1), \ldots, \varphi(x_n)$ are independent over A whenever $x_1, \ldots, x_n$ are distinct elements of the group X, i.e. φ is an injection which ends proof of (1) in this case. Now, if φ is arbitrary, then from Lemma 3.2 it follows that for any $J \in P(C)$, $\vartheta(\mathscr{F}_{JA}(\varphi)) = 1\otimes\beta_J$, where $\beta_J \in \mathrm{Hom}(X, Y)$. Now, (1) follows from the first part of the proof and from Theorems 3.4 and 4.1.

The proof of (4) follows from (3) and (1).

PROPOSITION 5.5. *If $\varphi \in \mathrm{Hom}_3(A[X], A[Y])$, then $T(\varphi)$ is a T-nilpotent ideal in each of the following cases:*

(1) *N is a T-nilpotent ring;*

(2) *the group X is finitely generated;*

(3) *Y can be linearly ordered, X is abelian of a finite rank and the additive group of N is torsion-free.*

Proof. From the characterization of T-nilpotent ideals it easily follows that the T-nilpotency of N implies the T-nilpotency of the ring NA ([10]). Therefore it is enough to prove our Proposition in the case of $A = C$.

(1) is obvious since $T(\varphi) \subset N$.

(2) Let $x_1, \ldots, x_n$ be the generators of X. Let I be an ideal in C generated by coefficients of the elements $\theta(\varphi(x_i))$ for $i = 1, \ldots, n$. From the definition of the canonical presentation it easily follows that $\theta(\varphi(x)) \in I[Y]$ and therefore $T(\varphi) = I$ is even a nilpotent ideal.

(3) Let $W \subset X$ be a finitely generated subgroup of X such that X/W is torsion-free. If $\psi = \varphi|_{C[W]}$, then by (2) it follows that $T(\psi)$ is a nilpotent ideal. Let $I = \{c \in N\colon k \subset T(\psi)$ for some $k \geqslant 1\}$. Since an additive group of N is torsion-free, I is a nilpotent ideal and the additive group of N/I is torsion-free. Considering homomorphism $\mathscr{F}_I(\varphi)$, we get that φ and $\mathscr{F}_I(\varphi)$ are G_3-homomorphisms, as Y is an ordered group. Then for every $x \in W$ we obtain that $\mathscr{F}_I(\varphi)(x) \in G_2(C/I)[Y]$. Since X/W is a torsion group, it follows, by Lemma 2.5, that $\mathscr{F}_I(\varphi)(x) \in G_2(C/I)[Y]$ for any $x \in X$. Hence $T(\varphi) \subseteq I$, and therefore $T(\varphi)$ is a nilpotent ideal which ends the proof.

We give below an example concerning our results connected with T-nilpotency.

EXAMPLE 5.6. Let A be a ring such that N is not a T-nilpotent ring and let $a_1, a_2, \ldots$ be elements of N such that $a_1 \cdot \ldots \cdot a_k \neq 0$ for every $k \geqslant 1$. Moreover, let X be either a free group or a free abelian group with the set of free generators $x_1, x_2, \ldots$ Then the mapping $\varphi\colon X \to G_3(A[X])$ given by $\varphi(x_i) = x_i + a_i x_{i+1}$ is a homomorphism of groups. Therefore, φ determines a G_3-endomorphism of the ring $A[X]$. By the choice of the elements $a_i \in N$, $\vartheta(\varphi) = 1$. Therefore φ is an injection. But it is not a surjection, as $x_1 \notin \varphi(A[X])$.

If for the same ring A and for the same group X we put $\psi(x_1) = x_1$, $\psi(x_i) = x_i + a_i x_{i-1}$ for $i > 1$, then ψ determines an automorphism of the ring $A[X]$ for which $T(\psi)$ is not a T-nilpotent ideal.

Now as a result of Theorem 5.4 we obtain the following

THEOREM 5.7. *For any u.p.-group Y and for an arbitrary group X the following statements hold:*

(1) *if there exists an injection $A[X] \to A[Y]$, then there is an injection $X \to Y$;*

(2) *if there exists a surjection $A[X] \to A[Y]$, then there is a surjection $X \to Y$;*

(3) *if there is an isomorphism* $A[X] \to A[Y]$, *then there is an isomorphism* $X \to Y$.

Proof. (1) Let $\varphi\colon A[X] \to A[Y]$ be an injection; so φ is a G_3-homomorphism. Then, as it follows from Theorem 5.4, $\vartheta(\varphi)\colon A[X] \to A[Y]$ is also an injection. Therefore, by Theorem 4.4, for any $I \in P(C)$, $\beta_I(\vartheta(\varphi))\colon X \to Y$ is an injection. The proof of (2) and (3) is analogous.

The category $A[\cdot]$ is concrete. Therefore every injection in $A[\cdot]$ is a monomorphism.

THEOREM 5.8. *Let* Y *be a u.p.-group and let* X *be any group. Let* $\varphi\colon A[X] \to A[Y]$ *be a monomorphism in the category* $A[\cdot]$. *Then* φ *is an injection.*

Proof. Since φ is a G_3-homomorphism, the proof easily follows from the following

LEMMA 5.9. *Let* $\varphi\colon A[X] \to A[Y]$ *be a* G_3-*homomorphism. If* φ *is not an injection, then there exists* $1 \neq u \in U(C[X])$ *such that* $\varphi(u) = 1$.

Proof. By Theorem 5.4, $\vartheta(\varphi)$ is not an injection. Therefore there exists $J \in P(C)$ such that $\beta_J = \beta_J(\vartheta(\varphi))$ is not an embedding of a group, i.e. there exists $1 \neq x \in X$ such that $\beta_J(x) = 1$. By the definition of a G_3-homomorphism, there exists $c \in U(C)$, $E \in D(C)$, $t \in N[Y]$ and there are elements $y_e \in Y$, $e \in E$, such that $\varphi(x) = c\sum_{e\in E} e\cdot y_e + t$. There exists $f \in E$ such that $f \notin J$, i.e. $y_f = \beta_J(x) = 1$. Therefore $\varphi(f(x)) = cf + ft$.

If $ft \neq 0$, then the subring without unity generated by all the coefficients of ft is nilpotent, i.e. there exists $b \in C$ such that b is nilpotent, $bf = b$ and $bft = 0$. Therefore $\varphi(bx) = \varphi(bfx) = b(cf+ft) = bcf = bc \neq 0$. Put $u = 1 - bc + bx$. Of course, $u \neq 1$, $\varphi(u) = 1$ and $u = 1 - b(c-x)$ is a unit, as b is nilpotent.

6. Endomorphisms and automorphisms

In this section we shall assume that X is a u.p.-group. Apparently if $A[X]$ is either hopfian or cohopfian object of $A[\cdot]$, then X is either hopfian or cohopfian group, respectively.

THEOREM 6.1 (cf. [6]). *Let* X *be a hopfian group. Then the ring* $A[X]$ *is hopfian in* $A[\cdot]$.

Proof. Let $\varphi\colon A[X] \to A[X]$ be a surjection. Then φ is a G_3-homomorphism. Therefore $\vartheta(\varphi)\colon A[X] \to A[X]$ is also a surjection (Theorem 5.4). From Theorem 4.4 and from the fact that the group X is hopfian it follows that, for any $I \in P(C)$, $\beta_I(\varphi)$ is an automorphism of the group X, which means that $\vartheta(\varphi)$ is an injection. By Theorem 5.4, so is φ.

THEOREM 6.2. *Let* X *be a cohopfian group. Then the ring* $A[X]$ *is cohopfian in each of the following cases:*

(1) N *is a* T-*nilpotent ring;*

(2) X *is a finitely generated group;*

(3) X *is an abelian group of a finite rank and an additive group of* N *is torsion-free.*

Proof. As in the proof of the preceding theorem, it can be shown that if $\varphi\colon A[X] \to A[X]$ is an injection, then the mapping $\vartheta(\varphi)$ is defined and it is an automorphism of $A[X]$. Let $\psi = \varphi(\vartheta(\varphi))^{-1}$. By Theorem 5.1 it then follows that $\vartheta(\psi)$ is an identity on $A[X]$. From Theorem 5.7, it then follows in three cases that $T(\psi)$ is a T-nilpotent ideal. Theorem 5.4 now implies that ψ is a surjection and therefore an automorphism. Therefore, also φ is an automorphism.

Now we shall give an example related to the assumptions of Theorem 6.2 and the assumption on the T-nilpotency in our other results.

EXAMPLE 6.3. Let k be a field of prime characteristic p and let C be the factor algebra of $k[t_1, t_2, \ldots]$ by the ideal I generated by the elements $t_1^p, t_2^p - t_1, t_3^p - t_2, \ldots$ Let $a_n = t_n + I \in C$ for any $n \geqslant 1$. Let X be a group isomorphic with the additive group of rationals and let $1 \neq x \in X$. Then every element of X can be represented uniquely in the form x^r, where r is a rational number. Let $X_1 = \{x^{a/b}\colon (a, b) = 1,\ p \nmid b\}$. Let $X_2 = \{x^{a/b}\colon (a, b) = 1 \wedge (\exists n \geqslant 1)\ b = p^n\}$. Of course, X_1 and X_2 are subgroups of X such that $X_1 \cdot X_2 = X$ and $X_1 \cap X_2 = \langle x\rangle$. Let us take a sequence of generators of X_2: $x_n = x^{1/p^n}$. Now, let $\alpha_2\colon X_2 \to U(C[X])$ be a homomorphism such that $\alpha_2(x_0) = x_0$, $\alpha_2(x_n) = x_n + a_n x_{n+1}$. If $\alpha_1\colon X_1 \to X \subset U(C[X])$ is a natural embedding, then $\alpha_1|_{\langle X\rangle} = \alpha_2|_{\langle X\rangle}$, whence there exists a homomorphism $\alpha\colon X \to U(C[X])$ which extends α_1 and α_2. Therefore, α can be regarded as an homomorphism of the ring $C[X]$. Of course, α is an injective G_3-homomorphism such that $\vartheta(\alpha) = \mathrm{id}_X$. If we had $X_1 \in \alpha(C[X])$ we would have $a_1 \ldots a_n = 0$, for some n, which is impossible. Therefore, although X is a hopfian and cohopfian group, α is not a surjection.

Studying the structure of a semigroup of endomorphisms and a group of automorphisms, we can use Lemma 3.2 and restrict ourselves to the case of $A = C$. Since X is an u.p.-group, it follows by Theorem 5.1 that functor ϑ induces a homomorphism of a semigroup $\mathrm{End}(C[X])$ onto $\mathrm{End}_1(C[X])$. Moreover, a function which maps every $\gamma \in \mathrm{Hom}(X, U(C))$ onto an automorphism γ^* of a ring $C[X]$ is an injection of groups.

THEOREM 6.4 (cf. [4]). *If* $N = 0$, *then groups* $\mathrm{Aut}_1(C[X])$ *and* $\mathrm{Aut}[X]$ *are isomorphic.*

Proof. By Theorem 5.2 and Corollary 5.3 it follows that $\mathrm{Aut}(C[X])$ is the semidirect product of $\mathrm{Hom}(X, U(C))$ and $\mathrm{Aut}_1(C[X])$. Now if $B(C) = \{0, 1\}$, then from the definition follows that $G_1(C[X]) = X$ and therefore $\mathrm{Aut}_1(C[X]) \simeq \mathrm{Aut}\, X$.

In the case where $N \neq 0$ the group $\mathrm{Aut}(C[X])$ is the semidirect product of a subgroup $\mathrm{Aut}_1(C[X])$ and a normal subgroup $\ker\vartheta$.

Under some additional assumptions on the group X and the ring C we can describe the set $\ker\vartheta$ more precisely. Let X be a torsion-free abelian group, $Y = \{u+t\colon u \in U(C),\ t \in N[X]\}$. Under this notation we have the following

LEMMA 6.5. *Let $\psi \in \mathrm{Aut}(X)$ be such that $\vartheta(\psi) = 1$. Then there exists exactly one homomorphism $\varphi\colon X \to Y$ such that $\psi(x) = \varphi(x)x$ for every $x \in X$.*

Proof. It is enough to set $\varphi(x) = x^{-1}\psi(x)$ for every $x \in X$.

LEMMA 6.6. *Let $\varphi \in \mathrm{Hom}(X, Y)$ and let φ^* be an endomorphism of $C[X]$ such that $\varphi^*(x) = \varphi(x)x$ for any $x \in X$. Then φ^* satisfies the equality $\vartheta(\varphi^*) = 1$. Moreover, any of the following conditions implies that φ^* is an automorphism:*

(1) *N is T-nilpotent;*

(2) *X is a finitely generated group;*

(3) *additive group of N is torsion-free and X is of a finite rank.*

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