ON SYSTEM OF SUBOBJECT FUNCTORS IN THE CATEGORY OF ORDERED SETS

MILAN SEKANINA

Department of Algebra and Geometry, Faculty of Science of JEP University,
66291 Brno, Czechoslovakia

Let \( \mathcal{S} \) be the category of all non-empty sets with mappings as morphisms, \( \mathcal{U} \) the category of all non-empty (partially) ordered sets with isotone maps as morphisms.

Let \( \text{Exp} \) be the endofunctor \( \mathcal{S} \to \mathcal{S} \) with

\[
\text{Exp}X = \{ Y : Y \subseteq X, Y \neq \emptyset \}
\]

and

\[
[\text{Exp}f](Y) = f(Y) = \{ f(y) : y \in Y \}
\]

for all sets \( X \neq \emptyset \), \( Y \subseteq X \) and all maps \( f \). Defining \( \eta_X(x) = \{ x \} \) for \( x \in X \) and \( \eta_{\mathcal{S}}(\mathcal{S}) = \bigcup Y \) for \( \mathcal{S} \in \text{Exp Exp}X \), we get a monad \( (\text{Exp}, \eta, m) \) (see [3], p. 138). We look now for such functors \( \mathcal{T} : \mathcal{U} \to \mathcal{U} \) for which the diagram

\[
\begin{array}{ccc}
\mathcal{U} & \xrightarrow{\eta} & \mathcal{U} \\
\downarrow U & & \downarrow U \\
\mathcal{S} & \xrightarrow{\text{Exp}} & \mathcal{S}
\end{array}
\]

(a)

\((U \text{ is the forgetful functor})\) is commutative (so \( \mathcal{T}(A, \varphi) = (\text{Exp}A, \mathcal{T}(\varphi)) \), where \( \mathcal{T}(\varphi) \) is a partial order on \( \text{Exp}A \) for each \( (A, \varphi) \in \mathcal{U} \).

(b)

\[
Y_1 \subseteq Y_2 \subseteq A \Rightarrow Y_1 \mathcal{T}(\varphi) Y_2
\]

as \( T \) is a functor, we get for any isotone mapping \( f : (A, \varphi) \to (B, \psi) \)

\[
Y_1 \mathcal{T}(\varphi) Y_2 \Rightarrow f(Y_1) \mathcal{T}(f(\varphi)) f(Y_2).
\]

We shall call such \( T \) a functor lifting \( \text{Exp} \) and extending inclusion. A description of these liftings was considered in [4]. The system of all considered functors \( \mathcal{T} \) will be denoted by \( \mathcal{T}^e \).

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for all \((A, \varrho)\).

Now \(T_M\) will be the subsystem of \(T\) formed by these functors \(T \in T\) for which \((T, \eta, \rho)\) is a monad (denoted simply by \(T\)). Having \((A, \varrho) \in \mathcal{U}\), \(\eta\), and \(\rho\) are now considered as morphisms in \(\mathcal{U}\); i.e., they must be isomorphisms. This is in fact the only condition on \(T \in T\) to yield such a monad as needed commutativity of involved diagrams follows obviously from the fact that \(T\) lifts \(\operatorname{Exp}_A, T_M\) has the smallest element \(T_1\) and the greatest element \(T_2\) (see [5], Theorem 2) defined as follows:

Let \((A, \varrho) \in \mathcal{U}\). \(T_{\varrho}(x)\) is the transitive hull of set-inclusion on \(\operatorname{Exp}_A\) and the relation \(\{(a, b) : a \in \varrho b, a, b \in A\}; X T_{\varrho}(x) \subseteq Y \Longleftrightarrow X \leq Y \subseteq Y\) = \(\{x\}; X = \sup X \in (A, \varrho)\) and there exists \(y \in Y\), for which \(x \leq y\) (here \(X, Y \in \operatorname{Exp}_A\)). The present paper should contribute to the study of \(T\), especially with respect to \(T_M\). Next lemma picks up one special situation for \(T \in T_M\), which will repeatedly occur in the sequel.

**Lemma 1.** Let \((T, \eta, \mu) \in T_M\), and \(A\) be a set, \((\operatorname{Exp}_A, \sigma) \in \mathcal{U}\), \(\varphi \in \operatorname{Exp}_A\), \(\Sigma \subseteq \varphi\sigma \subseteq \varphi\). Then \(\varphi \sigma \subseteq \varphi \subseteq \varphi\). There exist \(Y \in \mathcal{U}\) such that \(X \subseteq Y \subseteq Y\) \(\operatorname{Exp}_A\), \(\mu_{\varphi}(\varphi) \subseteq \mu_{\varphi}(\varphi)\). Then \(\varphi \subseteq \varphi \subseteq \varphi\). The assertion of the lemma follows immediately from the fact that \(T \subseteq T_1\) and from the definition of \(T_{\varrho}\).

In Propositions 1-4 some constructions are described which applied to subsystems of \(T (T_M \text{ resp.})\) or to an element of \(T (T_M \text{ resp.})\) yield again an element of \(T (T_M \text{ resp.})\).

**Proposition 1.** Let \(T \subseteq T (T \subseteq T_M \text{ resp.})\). Let \(T_{\varrho}(x) = \bigcap_{\varphi \in \varphi}(x)\). Then \(T_{\varrho}(x) = \bigcap_{\varphi \in \varphi}(x)\). Put \(F(A, \varrho) = \bigcap_{\varphi \in \varphi}(x)\). Then \(F(A, \varrho) = \bigcap_{\varphi \in \varphi}(x)\). This is clear.

**Proposition 2.** Let \(m \in \operatorname{Exp}_A, T \in T (T \in T_M \text{ resp.})\), \((A, \varrho) \in \mathcal{U}\). Put for \(Y \in \operatorname{Exp}_A\)

\[
X = Q(A, \varrho) Y = X \subseteq Y \text{ or } X T_{\varrho}(Y), Y, \quad \operatorname{card}X \subseteq m,
\]

\[
X \equiv Q(A, \varrho) Y = X \subseteq Y \text{ or } X T_{\varrho}(Y), Y, \quad \operatorname{card}X \subseteq m.
\]

Let \(F_i(A, \varrho) = \operatorname{Exp}_A, (Q_i(A, \varrho)), F_i(f) = F_i, i = 1, 2\). Then \(F_i \subseteq T (F_i \subseteq T_M \text{ resp.})\).

**Proof.** We shall prove Proposition 2 for \(T \in T_M\) and \(F_i\).

(a) \(F_i(A, \varrho)\) is an order on \(\operatorname{Exp}_A\). Reflexivity and antisymmetry are clear. Transitivity (put \(Q_i\) instead of \(Q_i(A, \varrho)\)):

- \(X \equiv Q_i Y \subseteq Z\); if \(X \subseteq Y \subseteq Z\), then \(X \subseteq Z\); so \(X \equivalent Q_i Y\).

So \(X = Y \subseteq Z\). If \(X \neq Y \subseteq Z\), then \(X \neq Y \subseteq Z\); if \(X \equivalent Y \subseteq Z\), then \(\operatorname{card}X \subseteq m\); and \(\operatorname{card}Y \subseteq Z\). If \(X \neq Y \subseteq Z\), then \(\operatorname{card}Y \subseteq m\); so \(\operatorname{card}X \subseteq m\) and \(\operatorname{card}Y \subseteq Z\).

(b) It is evident that \(F_i\) is an endofunctor in \(\mathcal{U}\).

\(F_i\) is an isotone mapping from \((A, \varrho)\) into \((A, \varrho)\). This is clear.

(c) \(\rho_{\varrho}\) is an isotone mapping from \((A, \varrho)\) into \((A, \varrho)\). This is clear.

(d) Isotonicity of \(\mu_{\varphi}\) for \(F_i\). Let \(X \in \operatorname{Exp}_A\), \(\varphi \in \operatorname{Exp}_A\), \(\mu_{\varphi}(\varphi) \subseteq \varphi\) and \(\mu_{\varphi}(\varphi) \subseteq \varphi\). Then \(\mu_{\varphi}(\varphi) \subseteq \varphi\). The only case, which needs a consideration, is \(\mu_{\varphi}(\varphi) \subseteq \varphi\). As \(\mu_{\varphi}(\varphi) \subseteq \varphi\), it is \(X = Y \subseteq \varphi\). When there exists \(Y \in \varphi\) such that \(X \subseteq Y \subseteq Y\) \(\operatorname{Exp}_A\), \(\mu_{\varphi}(\varphi) \subseteq \varphi\), \(X_1 \subseteq X_2\) (see Lemma 1). Then \(\operatorname{card}X_1 \subseteq m\). As \(X_1 \subseteq X_2\), \(\mu_{\varphi}(\varphi) \subseteq \varphi\). Then \(\mu_{\varphi}(\varphi) \subseteq \varphi\). Therefore, for all these \(X\)'s we have \(\operatorname{card}X \subseteq \operatorname{card}X_1\). As \(\operatorname{card}X \subseteq \operatorname{card}X_1\), we have \(\operatorname{card}X \subseteq m\). Hence \(\operatorname{card}X \subseteq \operatorname{card}X_1\). If \(X = \varphi\), then \(\mu_{\varphi}(\varphi) \subseteq \varphi\).

**Definition 1.** Let \(m, T, F_1, F_2\) be as in Proposition 2. We put

\[
r_{F_2}(T) = F_1, \quad r_{F_2}(T) = F_2.
\]

**Definition 2.** Let \((A, \varrho) \in \mathcal{U}\). Put \(t(A, \varrho)\) (briefly \(t(A)\)) = \(\operatorname{sup}(\operatorname{card}X); X \in A, X\) is an antichain in \((A, \varrho)\) (one-point set is taken as an antichain).

**Proposition 3.** Let \(m \in \operatorname{Exp}_A, T \subseteq T (T \subseteq T_M \text{ resp.})\), \((A, \varrho) \in \mathcal{U}\). Put for \(Y \in \operatorname{Exp}_A\)

\[
X = Q(A, \varrho) Y = X \subseteq Y \text{ or } X T_{\varrho}(Y), Y, \quad t(X) \subseteq m,
\]

\[
X = Q(A, \varrho) Y = X \subseteq Y \text{ or } X T_{\varrho}(Y), Y, \quad t(X) \subseteq m.
\]
Let $F_i(A, g) = (\text{Exp}A, q(A, g), F_i(f)) = \text{Exp}f$ for $i = 3, 4$. Then $F_i \in \mathcal{T}$ ($F_i \in \mathcal{T}_M$, resp.).

Proof. Take again the case $\mathcal{T} \in \mathcal{T}_M$ and prove $F_i \in \mathcal{T}_M$. The proof runs along the same lines as the proof of Proposition 2. We can concentrate ourselves to the proof of the isotonicity of $\mu_3$ for $F_i$.

Let $\mathcal{X}, \mathcal{Y} \in \text{Exp} \mathcal{A}$. Suppose $\mu_3(\mathcal{X}) \nonc \mu_3(\mathcal{Y})$. Then $\mathcal{X} \circ \mathcal{Y} \nonc \mathcal{Y}$. For every $\mathcal{Z} \in \mathcal{Y}$ there exists $\mathcal{W} \in \mathcal{X}$ such that $\mathcal{W} \in \mathcal{Z}$ and $\mathcal{W} \nonc \mathcal{Y}$. Therefore $\mathcal{W} \in \mathcal{X} \circ \mathcal{Y}$. We take $\mathcal{Z} \in \mathcal{Y}$ as $\mathcal{Z} \circ \mathcal{Y}$. If $\mathcal{X} \circ \mathcal{Y} = \mathcal{Z}$ then $\mathcal{X} \circ \mathcal{Y} = \mathcal{Z}$ and $\mathcal{X} \nonc \mathcal{Y}$.

DEFINITION 5. Let $m, F_1, F_2$ be as in Proposition 3. Then we put $r_i^\alpha(T) = F_3$, $r_i^\alpha(T) = F_4$.

It remains to prove

1. $r_i^\alpha(T) = r_i^\alpha(T)$ for $i = 2, 3, 5, 6, 7, 8$.
2. $r_i^\alpha(T) = r_i^\alpha(T)$ for $i = 2, 3, 5, 6, 7, 8$.
3. $r_i^\alpha(T) = r_i^\alpha(T)$ for $i = 2, 3, 5, 6, 7, 8$.
4. $r_i^\alpha(T) = r_i^\alpha(T)$ for $i = 2, 3, 5, 6, 7, 8$.

The proof of assertions (a), (b) consists in constructing a set $(A, g)$ such that $r_4^\alpha(T) \in \mathcal{A}$. This construction is a generalization of an example due to A. Kurepa (22, 9). We shall proceed as follows. Take the smallest number $m$, with the property $m > m$. We have $m \leq m$. Let $B$ be a set with card $B = m$, and let us order the set $B$ by a well-ordering of the corresponding initial type. This type will be denoted by $\beta$. We can put $B = \beta$ and consider $A = -\beta$, where $-\beta$ is the system of all maps of $\beta$ in the set $\{0, 1\}$. (0 \in \omega) ordered lexicographically (this ordering is denoted as $\leq \omega$). Let $\gamma < \beta$, $\gamma$ an ordinal. Then $\text{card} \gamma \leq m$. As $\gamma < \beta$, we can suppose that $\beta$ is a subset of $\omega$ (e.g. the maps from $\omega$ are extended to those of $\beta$ by assigning 0 to the elements of $\beta - \gamma$). The set $D = \bigcup \beta$ is dense in $A$, i.e. for $a, b \in A, a \leq b$ there exists $c \in D$ such that $a \leq c \leq b$. As card $\omega \leq m$ for $\gamma < \beta$, we get card $D \leq m$. 

Let $X_i \in X_i$ (so $a_1 \leq a_2$, or $X_i = X_i$) and again $a_1 \leq a_2$. Put $\mathcal{Z} \circ \mathcal{Y} = \mathcal{Z}$. Then $\mathcal{X} \circ \mathcal{Y} = \mathcal{X} \circ \mathcal{Y}$.

DEFINITION 6. Let $m, F_1, F_2$ be as in Proposition 3. Then we put $r_i^\alpha(T) = F_3$, $r_i^\alpha(T) = F_4$.

It remains to prove

1. $r_i^\alpha(T) = r_i^\alpha(T)$ for $i = 2, 3, 5, 6, 7, 8$.
2. $r_i^\alpha(T) = r_i^\alpha(T)$ for $i = 2, 3, 5, 6, 7, 8$.
3. $r_i^\alpha(T) = r_i^\alpha(T)$ for $i = 2, 3, 5, 6, 7, 8$.
4. $r_i^\alpha(T) = r_i^\alpha(T)$ for $i = 2, 3, 5, 6, 7, 8$.
Now, order $A$ by a well-order $<_a$. Put $\sigma = \leq_a \land \leq_a$. Let $Z \subset (A, \sigma)$ be a chain. This chain must be well-ordered, say $Z = \{z_1, z_2, z_3, \ldots\}$, and this chain is also a chain in $(A, <_a)$. As $D$ is a dense set in $(A, <_a)$ and card $D \leq m$, the cardinality of $Z$ is $\leq m$. As card $A = \sigma > m$, it follows that $\sigma(A, \sigma) > m$.

Let $Z$ be an antichain in $(A, \sigma)$, $Z = \{z_1, z_2, z_3, \ldots\}$, $z_i < z_j$ for all $i < j$. For the same reason as before, card $Z \leq m$. So $(A, \sigma) \leq m$.

Proof of (a). Let $(A, \sigma)$ be the set just constructed and take $a, b$ non $\in A$. Put $T = (A, \sigma) \uplus \{\emptyset\} \uplus \{\emptyset\}$ where the ordinal sum. Put $G_1 = A \cup \{\emptyset\}$, $G_2 = A \cup \{\emptyset\}$. It is $G_1 = \{\emptyset\}$. The proof of (b) is similar.

Proof of (c). Let $M$ be an antichain of the cardinality $m$, $a, b$ non $\in M$. Put $A = M \uplus \{\emptyset\} \uplus \{\emptyset\}$. Put $G_1 = M \cup \{\emptyset\}$, $G_2 = M \cup \{\emptyset\}$. It is $G_1 = \{\emptyset\}$. The proof of (b) is similar.

Remark. Result of Dilworth [1] implies $r(T) = r(T)$.

One of the needed information on $T$ or $T_M$ is the answer to the question whether the classes or hyperclasses. Proposition 5 relates to this question.

**Proposition 5.** Let $T$ be the full subcategory of $T$ consisting of all ordered finite non-empty sets, $T'$, $T''$ be the system defined for $T$ in the same way as $T$ and $T_M$ for $T$. Then card $T'' = 2^n$, card $T_M'' = 2^n$.

**Proof.** card $T'' = 2^n$ can be proved along the same lines as Proposition 3 in [5]. Let us prove that card $T'' = 2^n$. Exp is now considered as an endofunctor in $T''$. Let

1. $m_1 < m_2 < \ldots < m_n$
2. $n_1 < n_2 < \ldots < n_m$
3. $s_1 < s_2 < \ldots < s_n$

be sequences of positive integers with

4. $m_1 < s_1 < n_1 < m_2 < s_2 < n_2 < \ldots < m_n < s_n < n_m$

Put $t_i = \{n_i, m_i\}$

Let $G_1$ be a set with $2 + s_i = t_i$ elements $b_1, a_1, b_1, \ldots, b_2, a_2, \ldots, a_2$. For this set $a_2$ is a mapping from the system $B_0$ of all subsets with $m_i$ elements from the set $\{a_1^1, \ldots, a_2^1\}$ onto $\{b_1^1, \ldots, b_2^1\}$. Let $b_1$ be the ordering of $G_1$ generated by pairs $(a^1_1, b^1_1)$, where $a_1^1 \in D_1$, $a_2^1(D_1) = b_1, (b^1_1, a^1_2)$ for all $j$, $a_2, b_1$ (see Fig. 1). Put $A_2 = (a_1^1, a_2^1, \ldots, a_2^2)$, $B_2 = G_1 - \{z_0\}$. By the same symbol (and by $G_1$ as well) also the corresponding ordered sets with the restrictions of $a_2$ to these sets will be denoted.

Let $P$ be a non-empty subset of the set $\{x_1, x_2, \ldots, x_n, \ldots\}$. Let $(M, \sigma)$ be any finite ordered set. We shall define the order $\varphi_{\sigma}(M, \sigma)$ on $\text{Exp} M$ in the following way.

For $X, Y \in M$, $X \neq \emptyset \neq Y$ we put $X \varphi_{\sigma}(M, \sigma) Y$ iff $X \subseteq Y$ or there exists $s_1, \ldots, s_\ell \in P$ and isotope maps $h_j : G_0 \rightarrow (M, \sigma)$, $j = 1, \ldots, \ell$ such that

4. $X \subseteq h_1(A_1), h_1(B_1) \subseteq h_2(A_2), h_2(B_2) \subseteq \ldots$, $h_{\ell-1}(B_{\ell-1}) \subseteq h_\ell(A_\ell), h_\ell(B_\ell) \subseteq Y$.

It is easy to prove that $\varphi_{\sigma}(M, \sigma)$ is an order on $\text{Exp} M$ and that $G_{\varphi_{\sigma}(M, \sigma)} = (\text{Exp} M, \varphi_{\sigma}(M, \sigma))$, $G_{\varphi}(f) = G_{\text{Exp} M}$ is an endofunctor in $T''$. In proving these facts it is sufficient to observe, in which case the cardinality of $h_1(A_1)$ and $h_{\ell-1}(B_{\ell-1})$ or of $h_1(A_1)$ and $h_\ell(B_\ell)$ respectively, are the same and to use the observation for proving antisymmetry for $\varphi_{\sigma}(M, \sigma)$.

Anyway, one can use [4], Theorem 1 as e.g. $\varphi_{\sigma}(M, \sigma) \subset \text{Exp} E_{\sigma}(M, \sigma)$.

Let us now prove one auxiliary statement.

**Lemma 3.** Let $x_1 \notin P$. Then $A_{x_1} \varphi_{\sigma}(G_0)$ $B_x$.

**Proof.** Suppose $A_{x_1} = h_1(A_1), \ldots, h_\ell(B_\ell) \subset B_x$ is a sequence of type (4). First of all we prove

5. $A_{x_1} = h_1(B_1)$.
6. $h_{\ell-1}(a_{x_1}) \in \{a_{x_1}\}$.

Suppose (6) does not hold. So $h_{\ell-1}(a_{x_1}) \notin \{a_{x_1}\}$. We have $h_{\ell-1}(a_{x_1}) = a_{x_1}$. and

7. $h_1(B_1) \varphi_{\sigma}(a_{x_1}), h_1(B_1) \varphi_{\sigma}(a_{x_1})$ for all $i$.---
We have \( k < l \) and \( m_0 > n_0 \). For any choice of \( a_i^n \in h_i^{-1}(a_i^n) \), \( i = 1, \ldots, n \) (such choice clearly exists) there exists \( b_i^n \) so that \( a_i^n b_i^n = a_i^n b_i^n \). Therefore \( a_i^n b_i^n \in h_i^{-1}(a_i^n) \), which gives \( h_i(b_i^n) = a_i^n b_i^n \). This is a contradiction to (7). Therefore (6) is valid and hence \( B_k \subseteq h_i(B_i) \). By induction we get (5). (5) together with \( B_\infty(B_i) = B_k \) implies \( A_k \subseteq B_k \) which is a contradiction to the definitions of \( A_k \) and \( B_k \).

From Lemma 3 we can deduce

\textbf{Lemma 4.} \( F_1 \neq F_2 \Rightarrow F_{P_1} \neq F_{P_2} \).

The proof is immediate, as by Lemma 3 \( s_k \in P_1 \neq P_2 \) \( (s_k \in P_1 \neq P_2) \) implies \( \varepsilon_{P_1}(C_3, a_3) \neq \varepsilon_{P_2}(C_3, a_3) \). card \( T^* = 2^c \) follows obviously from Lemma 4, the definition of \( P \) and from the evident upper bounds \( \text{card} \ T^* \leq 2^c \).

References


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HOMOMORPHISMS OF GROUP RINGS

JAN KEEMPA

Institute of Mathematics, Warsaw University, Warsaw, Poland

Introduction

Homomorphisms of group rings with the ring of integers as the coefficient ring and torsion-free group were first investigated by Higman [1]. These investigations were continued among others, by Snirnov [13]. Parmerter and Seigal considered automorphisms of the group ring \( A(C) \) for infinite cyclic group \( G \) and arbitrary ring of coefficients \( A \) ([7], [8]). Lautze [4] described automorphisms of group rings of free abelian groups of finite rank with commuting coefficients.

The aim of this paper is to present a new method of investigation of a group of units and homomorphisms of group rings. For this purpose we shall investigate in \( \S 1 \) properties of some subgroups of a group \( U(C(G)) \) of units of a group ring \( C(G) \), where \( C \) is a commutative ring. In \( \S 2 \) a structure of the group \( U(C(G)) \) is described in the case where \( G \) is a u.p. group. In \( \S 3 \) we introduce 4 classes of homomorphisms of group rings related to subgroups defined in \( \S 1 \). They are called \( G_0 \)-homomorphisms, \( (G_0 \neq G_1, (G_2, G_3) \) and it is shown that in the case of u.p.-groups every homomorphism is a \( G_0 \)-homomorphism. In \( \S 4 \) a structure of \( G_0 \)- and \( G_1 \)-homomorphisms is described. In \( \S 5 \) we investigate properties of \( G_0 \)- and \( G_1 \)-homomorphisms using in the essential way results concerning \( G_0 \)-homomorphisms. In \( \S 6 \) some criteria for a homomorphism to be an injection, a surjection or an automorphism are given. In \( \S 6 \) our results are applied to the description of the structure of group of automorphisms and homogeneity and cohomogeneity of group rings of u.p.-groups.

The paper is written in such a way that it is possible to extend all the results on u.p.-groups to the arbitrary torsion-free group after showing the triviality of the group of units of group algebras of such groups over fields.