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ITERATIVE AND METRIC ALGEBRAIC THEORIES

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Abstract

Three known classes of iterative theories related to the theory of computation are shown to be metrizable in a natural way. In the induced topology the theory operations of composition, source-tupling and iteration are continuous. Furthermore for any ideal morphism $f\colon n \to p$ in these theories the function $\xi \mapsto f\circ\xi$ is a contraction. It follows that for every ideal $f\colon n \to p+n$, and any $a\colon n \to p$, $f^{\dagger} = \lim_{k\to\infty} f^k\circ(I_p, a)$. We introduce a definition of the class of metric theories with these properties and show that not every iterative theory belongs to this class.

0. Introduction

In this paper the classes of iterative theories known until now (either implicitly or explicitly from [5], [6] and [7]) are briefly reviewed. These theories may be grouped into three classes: tree theories [7], theories of sequacious functions [5] and matrix and matricial theories [5], [6]. Each of these classes is related to the theory of computation and in fact these examples clearly suggest the definition of "iterative theory".

The defining property of iterative theories (see below) is that for any ideal morphism $f\colon n \to p+n$, there is a unique solution $f^{\dagger}\colon n \to p$ to the equation in the variable $\xi\colon n \to p$

(0.1) $$\xi = f\circ(I_p, \xi),$$

the "iteration equation" for f.

For example if $f\colon X \to X^{+}\times[2]\cup X^{\infty}$ is the "sequacious function" (see § 4) which records the "track" of the computations of a "machine" M

with two exists, then $f^\dagger$ records the track of the computations of the machine $M^\dagger$ obtained from M by identifying exit 2 with the begin state (see [5] for more details). Thus the solution of the iteration equation (0.1) expresses the meaning of "do-while" in a compact algebraic form.

In this paper it is shown that each of these known iterative theories is metrizable in a natural way so that the theory operations are continuous and more importantly, the ideal morphisms induce contraction mappings so that the unique solution of the iteration equation (0.1) for ideal f is a metric limit of (roughly) the powers of f.

Since in examples related to computation, the limits of the powers of f correspond to one's intuition of the meaning of $f^\dagger$, the notion of metric limit in algebraic theories may be a useful one.

We propose a definition of a metric algebraic theory in § 6 and obtain a few elementary properties of these theories. In § 7 an example is given to show that not every iterative theory is a ("ideal power complete") metric theory.

We assume the reader has some familiarity with algebraic and iterative theories, although we will provide all the necessary definitions to keep the paper self-contained.

1. Definitions and preliminary results

An *algebraic theory* T is a category whose objects are the nonnegative integers having, for each $n > 0$, n distinguished morphisms

$$\mathbf{i}\colon 1 \to n$$

$i \in [n]$ (where $[n] = \{1, 2, \ldots, n\}$; $[0] = \emptyset$). Furthermore, for each family of morphisms $f_i\colon 1 \to p$, $i \in [n]$, $n \geqslant 0$ there is a unique morphism $f\colon n \to p$ such that for each $i \in [n]$, f_i is the composition

$$f_i\colon 1 \xrightarrow{\mathbf{i}} n \xrightarrow{f} p. \tag{1.1}$$

The morphism f in (1.1) is called the *source-tupling* of the morphisms f_i, and is denoted $(f_1, f_2, \ldots, f_n)$. In case $n = 0$, this condition amounts to requiring the existence of a unique morphism $0_p\colon 0 \to p$. All morphisms $n \to p$ formed by source tupling the distinguished morphisms are called *base* morphisms. In the case that the distinguished morphisms $\mathbf{1}\colon 1 \to 2$ and $\mathbf{2}\colon 1 \to 2$ are distinct, the base morphisms $n \to p$ may be identified with the collection of functions $[n] \to [p]$. For example the function $[2] \to [3]$, defined by $1 \mapsto 3$, $2 \mapsto 1$, is identified with $(\mathbf{3}, \mathbf{1})$. (See [8] or [5] for more details.) The base morphism $n \to n$ corresponding to the identity function is denoted I_n.

We will denote the *composition* of $f\colon n \to p$ with $g\colon p \to q$ by either $f \circ g$, or $n \xrightarrow{f} p \xrightarrow{g} q$ (note that one arrowhead is missing).

It is convenient to extend source tupling to pairs of morphisms with arbitrary sources. If $f_i\colon n_i \to p$, $i = 1, 2$, then the *source pairing* (f_1, f_2) of f_1 and f_2 is the unique morphism $n_1 + n_2 \to p$ satisfying

$$\mathbf{i} \circ (f_1, f_2) = \begin{cases} \mathbf{i} \circ f_1 & \text{if} \quad i \in [n_1], \\ \mathbf{j} \circ f_2 & \text{if} \quad i = n_1 + j,\ j \in [n_2]. \end{cases}$$

The base morphisms corresponding to the injections $\varkappa\colon [p] \to [p+n]$ and $\lambda\colon [n] \to [p+n]$ (where $i\varkappa = i$, $i\lambda = p+i$) are denoted $I_p \oplus 0_n$ and $0_p \oplus I_n$ respectively.

The *powers* g^r of a morphism $g\colon n \to p+n$ are defined inductively.

$$g^0 = 0_p \oplus I_n; \qquad g^{r+1} = g^r \circ (I_p \oplus 0_n, g).$$

Two useful facts connected with g^r are

$$(I_p \oplus 0_n, g)^r = (I_p \oplus 0_n, g^r), \qquad \text{all } r \geqslant 0. \tag{1.2}$$

(1.3) If $\xi = g \circ (I_p, \xi)$, then $\xi = g^r \circ (I_p, \xi)$ all $r \geqslant 0$, where $\xi\colon n \to p$.

A scalar morphism $g\colon 1 \to n$ in an algebraic theory T is *ideal* if $g \circ f$ is not distinguished for any $f\colon n \to p$ in T. T itself is *ideal* if every scalar morphism is either distinguished or ideal. A vector morphism $g\colon n \to p$ in T is ideal if each component $\mathbf{i} \circ g$ is ideal, $i \in [n]$.

An *iterative theory* is an ideal theory T with the property that for each ideal morphism $g\colon n \to p+n$ in T, there is a unique morphism $g^\dagger\colon n \to p$ such that $g^\dagger = g \circ (I_p, g^\dagger)$.

2. Metrics on words and sets of words

The reader will recall that a *metric* d on a set X is a function from $X \times X$ into the nonnegative real numbers such that for all x, y, z in X

$$d(x, y) = d(y, x); \tag{2.1}$$

$$d(x, y) = 0 \quad \text{iff} \quad x = y; \tag{2.2}$$

$$d(x, z) \leqslant d(x, y) + d(y, z) \quad \text{(triangle inequality)}. \tag{2.3}$$

We will define two metrics: on $X^* \cup X^\infty$, the set of finite and infinite sequences of elements of X, and on $X^{*\wedge}$, the powerset of X^*.

To begin with, we consider an element of X^* to be a certain kind of function $N \to X_\perp$, where N is the set of nonnegative integers and $X_\perp$ is X augmented by a new element "$\perp$". If $u \in X^*$ and $ku = \perp$, then $nu = \perp$, all $n > k$; also, $ku = \perp$ for some k. The *length* of $u \in X^*$ is the least k such that $ku = \perp$. X^∞ is just the set of functions $N \to X$. Thus we may consider X^* and X^∞ to be subsets of $(X_\perp)^\infty$.

(2.4) Definition. For $f \neq g \in X^* \cup X^\infty$, $d(f, g) = 1/2^k$, where k is the least integer such that $kf \neq kg$. When $f = g$, $d(f, g) = 0$. It is easy to show that

(2.5) $X^* \cup X^\infty$ equipped with the function d of (2.4) is a complete metric space; i.e. every Cauchy sequence converges.

Now suppose $V \subseteq X^*$. We define the set (V_r) to be the set of words in V of length $\leqslant r$. Note that $V = \bigcup_{r \geqslant 0} (V)_r$.

(2.6) Definition. For $U, V \subseteq X^*$, let

$$d(U, V) = \begin{cases} 0 & \text{if } U = V; \\ 1/2^k, & \text{otherwise} \end{cases}$$

where k is least such that $(U)_k \neq (V)_k$.

(2.7) Proposition. *$X^{*\wedge}$ equipped with the function d of (2.6) is a complete metric space.*

The straightforward proof is omitted.

Later we will need to use the following well-known fact.

(2.8) If (X, d) is a complete metric space (where d is bounded) so is X^Y, the set of all functions $Y \to X$ with the metric $\bar{d}(f, g) = \sup\{d(yf, yg): y \in Y\}$.

If X and Y are metric spaces, a function $F: X \to Y$ is a *contraction mapping* if there is a real number c, $0 \leqslant c < 1$, such that for all $x, x' \in X$,

$$d(xF, x'F) \leqslant c \cdot d(x, x').$$

A sufficient condition for a contraction mapping $X \to X$ to have a unique fixed point is given by the well-known

(2.9) Banach Fixed Point Theorem. *If X is a complete metric space and $F: X \to X$ is a contraction, then there is a unique $\bar{x} \in X$ with $\bar{x}F = \bar{x}$; in fact, for any $y \in X$, $\bar{x} = \lim_{k \to \infty} yF^k$.*

It is useful to observe that any contraction map $X \to X$ has at most one fixed point; completeness of the space guarantees the existence of at least one.

3. Tree theories

Let Γ be a ranked set, i.e. Γ is the disjoint union $\bigcup_n \Gamma_n$ of the sets Γ_n, $n \geqslant 0$. By a *Γ-tree* $f: 1 \to p$ we mean a rooted tree f such that every vertex of f has a finite number of successors. The successors of a vertex are ordered,

so that one may speak of the first, or second, ... successor. Every vertex having n successors, $n > 0$, is labeled by an element of Γ_n. Every leaf is labeled by either an element of Γ_0 or an element of $[p] = \{1, 2, \ldots, p\}$. The number of vertices of a Γ-tree may be infinite.

A Γ-tree $n \to p$ is an n-tuple of Γ-trees $1 \to p$.

If $f: n \to p$ and $g = (g_1, \ldots, g_p): p \to q$ are Γ-trees, the *composition* $f \circ g: n \to q$ of f and g is the Γ-tree obtained by attaching to every leaf of f labeled $i \in [p]$ a copy of the tree g_i.

The distinguished tree $\mathbf{i}: 1 \to n$, $i \in [n]$ has one vertex, which is simultaneously a root and a leaf; the label on the leaf is i.

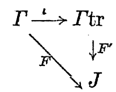
The collection $\Gamma\mathrm{Tr}$ of all Γ-trees is an iterative algebraic theory. (See [7] for a detailed study of Γ-trees.)

By a "tree theory", we mean a subtheory of $\Gamma\mathrm{Tr}$ for some Γ.

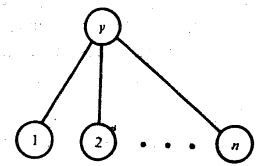
Two tree theories deserve special mention. The finite Γ-trees, $\Gamma\mathfrak{J}$, form an ideal subtheory of $\Gamma\mathrm{Tr}$. $\Gamma\mathfrak{J}$ is not iterative, but as shown in [8] $\Gamma\mathfrak{J}$ is the algebraic theory freely generated by Γ.

The subtheory $\Gamma\mathrm{tr}$ of $\Gamma\mathrm{Tr}$ consisting of those trees of "finite index", i.e. having (up to isomorphism) a finite number of descendency trees, is the iterative theory freely generated by Γ. In detail,

(3.1) [7] For any iterative theory J and any function $F: \Gamma \to J$ taking $\gamma \in \Gamma_n$ to an ideal morphism $\gamma F: 1 \to n$ in J, there is a unique theory morphism $F': \Gamma\mathrm{tr} \to J$ such that the diagram



commutes, where $\gamma\iota$ is the tree



, $\gamma \in \Gamma_n$.

We will introduce a metric on the set of Γ-trees $n \to p$.

(3.2) Definition. Let $g: 1 \to p$ be a Γ-tree. For any natural number $d \geqslant 0$, the *profile* of g at depth d, in symbols $P_d(g)$, is the sequence of elements in $\Gamma \cup [p]$

$$l(w_1),\ l(w_2),\ \ldots,\ l(w_k)$$

where $w_1, w_2, \ldots, w_k$, $k \geqslant 0$, is the sequence (from left to right) of the vertices of depth d and where $l(w_i) \in \Gamma \cup [p]$ is the label of w_i, $i \in [p]$.

Two trees have the same profile at every depth iff they are isomorphic. We will identify isomorphic trees so that

(3.3) PROPOSITION. *$g = g'$ iff $P_d(g) = P_d(g')$, all $d \geqslant 0$. More generally, g and g' are identical up to vertices of length $\leqslant k$ iff $P_d(g) = P_d(g')$ for all $d \leqslant k$.*

(3.4) DEFINITION. If $g \neq g'$ are Γ-trees $1 \to p$,

$$d(g, g') = 1/2^n$$

where n is the least integer such that $P_k(g) \neq P_k(d')$. If $g = g'$, we let $d(g, g') = 0$. For vector trees $g, g'\colon n \to p$ we define

$$d(g, g') = \max\{d(i \circ g,\ i \circ g'),\ i \in [n]\}.$$

(3.5) PROPOSITION [2]. *For each $n, p \geqslant 0$ the function d is a metric on the set $\Gamma\mathrm{Tr}_{n,p}$ of Γ-trees $n \to p$ in $\Gamma\mathrm{Tr}$. $\Gamma\mathrm{Tr}_{n,p}$ is complete in this metric.*

In [2] the following properties of this metric were proved.

(3.6) PROPOSITION. *Let $g_i\colon n \to p$, $h_i\colon p \to q$ be Γ-trees. Then*

(3.6.1) $$d(g_1 \circ h_1,\ g_2 \circ h_1) \leqslant d(g_1, g_2),$$

(3.6.2) $$d(g_1 \circ h_1,\ g_1 \circ h_2) \leqslant d(h_1, h_2),$$

(3.6.3) *if* $g, h\colon n \to p+n$, $\quad d(g^r, h^r) \leqslant d(g, h)$, $\quad$ *for any* $r \geqslant 0$.

From (3.6.1) and (3.6.2) it follows that composition is continuous: i.e. if $g_k \to g$ and $h_k \to h$, then $g_k \circ h_k \to g \circ h$, whenever these morphisms are composable (i.e. the sources and targets match). Indeed,

$$d(g_k \circ h_k,\ g \circ h) \leqslant d(g_k \circ h_k,\ g_k \circ h) + d(g_k \circ h,\ g \circ h)$$

by the triangle inequality. But by (3.6.1) and (3.6.2) the right-hand side is less than $d(h_k, h) + d(g_k, g)$, which goes to zero.

From Definition (3.4) it follows that source tupling is also continuous.

Ideal Γ-trees have an important property.

(3.7) PROPOSITION. *If $f\colon n \to p$ is an ideal Γ-tree, then for any q, the function taking $\xi\colon p \to q$ to $f \circ \xi\colon n \to q$ is a contraction; in fact*

$$d(f \circ \xi,\ f \circ \xi') \leqslant \tfrac{1}{2} d(\xi, \xi').$$

(3.8) COROLLARY. *Let $f\colon n \to p+n$ be an ideal tree in $\Gamma\mathrm{Tr}$. For any $a\colon n \to p$ the metric limit*

(3.8.1) $$\lim_{k \to \infty} f^k \circ (I_p, a)$$

exists and is the unique solution to the iteration equation

(3.8.2) $$\xi = f \circ (I_p, \xi)$$

for f; i.e.

$$f^\dagger = \lim_{k \to \infty} f^k \circ (I_p, a).$$

Proof. By (3.7) the function $\xi \mapsto f \circ (I_p, \xi)$ is a contraction mapping on the complete metric space $\Gamma\mathrm{Tr}_{n,p}$. By the Banach fixed point theorem, this map has a unique fixed point, which is given by (3.8.1) for any $a\colon n \to p$.

Of course, from (3.8) it follows that for any iterative tree theory T (i.e. any iterative subtheory of $\Gamma\mathrm{Tr}$, for some Γ) the iterate $f^\dagger$ of an ideal tree $n \to p+n$ is given by (3.8.1). Indeed, the limit (3.8.1) exists in $\Gamma\mathrm{Tr}$ and is the unique solution of (3.8.2); but since T is iterative, this limit belongs to T.

From (3.8) and (3.6.3) it follows that the operation of iteration $(f \mapsto f^\dagger)$ is also continuous. This is proved in a general setting in § 6.

4. Theories of sequacious functions

The algebraic theory of sequacious functions was introduced and studied in [5]. We repeat the definition here. Let X be a non-empty set. A *sequacious function* $f\colon n \to p$ is a function (1) $f\colon X^+ \times [n] \cup X^\infty \to X^+ \times [p] \cup X^\infty$ with the properties that

(i) if $u \in X^\infty$, $uf = u$ (i.e. X^∞ is kept pointwise fixed);

(ii) if $x \in X$, $i \in [n]$ and $xif = x_1, x_2, \ldots$ in $X^+ \times [p] \cup X^\infty$, then $x_1 = x$;

(iii) if $u \in X^*$, $x \in X$, $i \in [n]$,

$$(uxi)f = u(xif)$$

(i.e. f is determined by its values on $X \times [n]$).

The *composition* $f \circ g$ of sequacious functions is function composition. The *source-tupling* of the sequacious functions $f_i\colon X^+ \cup X^\infty \to X^+ \times [p] \cup \cup X^\infty$, $i \in [n]$ is the function $f\colon X^+ \times [n] \cup X^\infty \to X^+ \times [p] \cup X^\infty$ defined by:

$$uif = uf_i, \qquad u \in X^+.$$

The distinguished morphism $i\colon 1 \to n$ is the sequacious function

$$X^+ \cup X^\infty \to X^+ \times [n] \cup X^\infty,$$

determined by $x \mapsto xi$, $x \in X$.

(As usual, we have identified X^+ with $X^+ \times [1]$.)

(1) X^+ is X^*-{null word}.

A sequacious function $f\colon n \to p$ is *positive* if for each $xi \in X \times [n]$, if $xif = x_1x_2 \dots x_t j \in X^+ \times [p]$, then $t \geqslant 2$. The least subtheory of all sequacious functions containing the positive sequacious functions is denoted $\mathrm{Seq}(X)$. In [5] it was shown that $\mathrm{Seq}(X)$ is an iterative algebraic theory.

We will introduce a metric on $\mathrm{Seq}(X)$ in a natural way. First note that the value of a sequacious function $f\colon n \to p$ on $xi \in X \times [n]$ is an element of $Y^* \cup Y^\infty$, where $Y = X \cup [n]$. $Y^* \cup Y^\infty$ *has* a metric (say d_0) on it, defined in (2.1). Thus, we may define, for $f, g\colon n \to p$ in $\mathrm{Seq}(X)$, the function d by

(4.1) DEFINITION.
$$d(f, g) = \sup\{d_0(uf, ug)\colon u \in X^+ \times [n] \cup X^\infty\}.$$

Thus the set of sequacious functions $n \to p$ forms a metric space. Note that as a result of Definitions (2.3) and (4.1), if $f, g\colon n \to p$ are sequacious functions and $d(f, g) < 1/2^k$, then for all $x \in X$, $i \in [n]$ the values of xif and xig are sequences (finite or infinite) which agree at least up to the kth position. (Recall that we are regarding a finite sequence as a special kind of infinite one, see § 2.)

It follows that the function which is the limit of a Cauchy sequence of sequacious functions (which exists by (2.5) and (2.8)) is also a sequacious function.

(4.2) PROPOSITION. *Let $g_i\colon n \to p$, $h_i\colon p \to q$, $i = 1, 2$, be morphisms in* $\mathrm{Seq}(X)$. *Then*

(4.2.1) $\quad d(g_1 \circ h_1,\ g_2 \circ h_1) \leqslant d(g_1, g_2)$,

(4.2.2) $\quad d(g_1 \circ h_1,\ g_1 \circ h_2) \leqslant d(h_1, h_2)$,

(4.2.3) $\quad$ *for* $g, h\colon n \to p+n$, $\quad d(g^r, h^r) \leqslant d(g, h)$, *all* $r \geqslant 0$.

Proof. We prove only the first statement. But the first statement follows from the fact that if $xig_1 = x_1x_2, \dots, x_k j$ then $xi(g_1 \circ h_1) = x_1 x_2, \dots \dots, x_{k-1}(x_k j)h_1$, and thus $d_0(xig_1, xig_1h) \leqslant 1/2^k$.

Thus, as in the case of tree theories, it follows that composition is continuous, as is source tupling.

The iterate of an ideal (= positive) sequacious function is again a metric limit, for the same reason as for the tree theories, as we now show.

A "Theory of sequacious functions" is a subtheory of $\mathrm{Seq}(X)$, for some X. Thus in any iterative theory of sequacious functions, the iterate $f^\dagger$ of an ideal morphism f satisfies
$$f^\dagger = \lim_{k\to\infty} f^k \circ (I_p, a)$$
for any $a\colon n \to p$, by (4.4) below.

(4.3) PROPOSITION. *Let $f\colon n \to p$ be an ideal morphism in* $\mathrm{Seq}(X)$. *Then for each $q \geqslant 0$, the function taking $\xi\colon p \to q$ to $f \circ \xi\colon n \to q$ is a contraction.*

(4.4) COROLLARY. *If $f\colon n \to p+n$ is an ideal morphism in* $\mathrm{Seq}(X)$, $f^\dagger = \lim_{k\to\infty} f^k \circ (I_p, a)$, *for any $a\colon n \to p$ in* $\mathrm{Seq}(X)$.

The proof of (4.4) is identical to the proof of (3.8) (using (4.3) instead of (3.7)).

5. Matrix and matricial theories

The collection $X^{*\wedge}$ of subsets of X^* forms a semiring in two different ways. We let XR_C denote the semiring whose elements are subsets of X^*, $+$ is union, and where multiplication is given by

(5.1) $\quad$ (in XR_C) $\qquad U \cdot V = \{uv \mid u \in U, v \in V\}$.

The additive identity 0 is the empty set and the multiplicative identity 1 is the unit set consisting of the empty word.

We let XR_F denote the semiring with the same elements and $+$ as XR_C, but with multiplication defined by

(5.2) $\quad$ (in XR_F) $\qquad U \cdot V = \{uxv \mid ux \in U,\ xv \in V\}$

where $u, v \in X^*$, $x \in X$. This multiplication was called *fusion* in [6]. The subscripts C and F suggest the words "concatenation" and "fusion". The multiplicative identity in XR_F is X, the set of all words of length one.

We will mention two kinds of algebraic theories based on the semiring XR_C. There are similar theories based on XR_F.

Let $M(X)$ be the algebraic theory having the set of all $n \times p$ matrices with entries in XR_C as morphisms $n \to p$. Matrix multiplication (using the $+$ and $\cdot$ in XR_C) is the theory operation of composition; the source tupling of the row matrices $f_i\colon 1 \to p$, $i \in [n]$ is the $n \times p$ matrix whose ith row is f_i. The distinguished morphism $\boldsymbol{i}\colon 1 \to n$ is the row matrix
$$[0, 0, \dots, 1, 0, \dots, 0]$$
whose only nonzero entry is a 1 in the ith position.

We extend the metric on $X^{*\wedge}$ defined in (2.6) to the morphisms $n \to p$ in $M(X)$ by:

for $(U_{ij}), (V_{ij})\colon n \to p$ in $M(X)$
$$d\big((U_{ij}), (V_{ij})\big) = \max\{d(U_{ij}, V_{ij})\colon i \in [n],\ j \in [p]\}.$$

By (2.8), the set of morphisms $n \to p$ in $M(X)$ is a complete metric space. Moreover, as in the previous theory we can prove

(5.3) PROPOSITION. *For* $g_i\colon n \to p$, $h_i\colon p \to q$, $i = 1, 2$, *in* $M(X)$,

(5.3.1) $$d(g_1 \circ h_1,\ g_2 \circ h_1) \leqslant d(g_1, g_2);$$

(5.3.2) $$d(g_1 \circ h_1,\ g_1 \circ h_2) \leqslant d(h_1, h_2);$$

(5.3.3) *for* $g, h\colon n \to p+n$, $d(g^r, h^r) \leqslant d(g, h)$, *any* $r \geqslant 0$.

(5.4) COROLLARY. *Composition and source tupling in* $M(X)$ *are continuous.*

The subtheory $\mathrm{Mat}(X)$ generated by the row matrices $[U_1, \ldots, U_n]$ such that each set $U_i \subseteq X^*$ consists of words of positive length is an iterative theory. If $f\colon 1 \to n$ is an ideal morphism in $\mathrm{Mat}(X)$, write $f = [U_1, \ldots, U_n]$. If $(a_i)\colon n \to 1$ is any column matrix over XR_C, $f \circ a = U_1 \cdot a_1 + U_2 \cdot a_2 + \ldots + U_n \cdot a_n$. If $d(a, a') = 1/2^k$, then $(a_i)_r = (a_i')_r$ for $r < k$ (see Definition (2.6)). Thus $(U_i \cdot a_i)_r = (U_i \cdot a_i')_r$ for $r \leqslant k$, since each set U_i is empty or consists of words of positive length. It follows that $d(f \circ a, f \circ a') < \frac{1}{2} d(a, a')$, since $(V+W)_r = (V)_r + (W)_r$, for any $V, W \subseteq X^*$. From this special case, one can easily prove

(5.5) PROPOSITION. *If* $f\colon n \to p$ *is an ideal morphism in* $\mathrm{Mat}(X)$ *and* $q \geqslant 0$, *the function taking* $\xi\colon p \to q$ *to* $f \circ \xi\colon n \to q$ *is a contraction.*

(5.6) COROLLARY. *Let* $f\colon n \to p+n$ *be an ideal morphism in* $\mathrm{Mat}(X)$. *For any* $a\colon n \to p$ *in* $\mathrm{Mat}(X)$,

$$f^\dagger = \lim_{k \to \infty} f^k \circ (I_p, a).$$

A closely related theory is $M'(X)$. (2) A morphism $n \to p$ in $M'(X)$ is a pair $(A; a)$ consisting of morphisms $A\colon n \to p$ and $a\colon n \to 1$ in $M(X)$. The composition of $(A; a)\colon n \to p$ with $(B; b)\colon p \to q$ is

(5.6.1) $$(A; a) \circ (B; b) = (AB; Ab + a).$$

The "+" in (5.6.1) yields the column matrix (c_i) where $c_i = (Ab)_i + a_i$. Source-tupling is as in $M(X)$ (extended to matrix-vector pairs) and the distinguished morphism $\boldsymbol{i}\colon 1 \to n$ is the pair

$$([0 \ldots 1 \ldots 0];\ [0]).$$

The metric on $M(X)$ extends to $M'(X)$ in the obvious way. Again, the set of morphisms $n \to p$ in $M'(X)$ form a complete metric space and we have

(5.7) The statements (5.3) and (5.4) hold for $M'(X)$.

(2) $M'(X)$ and $\mathrm{Mat}'(X)$ are examples of matricial theories studied in [6].

The subtheory $\mathrm{Mat}'(X)$ of $M'(X)$ generated by the pairs $(A; a)\colon 1 \to n$ where A is in $\mathrm{Mat}(X)$ is an iterative algebraic theory. Furthermore:

(5.8) The Propositions (5.5) and (5.6) hold for $\mathrm{Mat}'(X)$.

We end this section with two remarks.

Remark 1. Using the semiring XR_F instead of XR_C one obtains "fusion-theories" analogous to $M(X)$ and $M'(X)$. The subtheories generated by the ideal elements, (in the case of $M(X)$, the row matrices containing sets of words of length at least 2 are the ideal morphisms) are iterative, and the same metric has the properties (5.3)–(5.6).

Remark 2. Every tree theory is isomorphic to a subtheory of $\mathrm{Mat}'(X)$, for some X (see [7]). Every theory of sequacious functions is isomorphic to a subtheory of "fusion-$\mathrm{Mat}'(X)$". One has to modify slightly the argument of [6] to prove this.

6. Metric theories

We introduce the very general notion of a metric algebraic theory and study some further properties that might be imposed on such theories.

(6.1) DEFINITION. An algebraic theory T is a *metric theory* if, for each $n, p \geqslant 0$ there is a metric d on the set $T_{n,p}$ of morphisms $n \to p$ such that

(6.1.1) composition is continuous; i.e. if

$$\lim_{k \to \infty} f_k = f \text{ in } T_{n,p} \quad \text{and} \quad \lim_{k \to \infty} g_k = g \text{ in } T_{p,q}$$

then

$$\lim_{k \to \infty} f_k \circ g_k = f \circ g \text{ in } T_{n,q};$$

furthermore

(6.1.2) the metric on $T_{n,p}$ is determined by that on $T_{1,p}$:

$$d(f, g) = \max\{d(\boldsymbol{i} \circ f, \boldsymbol{i} \circ g)\colon i \in [n]\}, \quad \text{for} \quad f, g \in T_{n,p}.$$

Condition (6.1.2) implies source-tupling is also continuous.

Of course any algebraic theory may be made a metric theory in a trivial way by defining $d(f, g) = 1$ if $f \neq g$.

Consider some properties a metric theory T may possess.

(P1) $d(g_1 \circ h_1, g_1 \circ h_2) \leqslant d(h_1, h_2)$, for all $g_1\colon n \to p$, $h_i\colon p \to q$, $i \in [2]$.

(P2) $d(g_1 \circ h_1,\ g_2 \circ h_1) \leqslant d(g_1, g_2)$, for all $g_i\colon n \to p$, $h_1\colon p \to q$, $i \in [2]$.

(P3) $d(g^r, h^r) \leqslant d(g, h)$, for all $g, h\colon n \to p+n$, and all $r \geqslant 0$.

(P4) T is an ideal theory and each ideal morphism $f\colon n \to p$ induces a contraction mapping $\xi \mapsto f\circ\xi$, for $\xi\colon p \to q$.

(P5) T is an ideal theory and for each ideal $f\colon n \to p+n$, and each $\alpha, \beta\colon n \to p$, both limits $\lim\limits_{k\to\infty} f^k\circ(I_p, \alpha)$, $\lim\limits_{k\to\infty} f^k\circ(I_p, \beta)$ exist and are equal.

(P6) Each set $T_{n,p}$ is a complete metric space.

All of the theories discussed in §§ 3–5 had properties (P1)–(P5). ΓTr, Seq(X), Mat(X) and Mat$'(X)$ also had property (P6).

We will deduce some elementary consequences of these properties. First we prove

(6.2) Proposition. *Let $f\colon n \to p+n$ be a morphism in the metric theory T. Whenever the limit $\lim\limits_{k\to\infty} f^k\circ(I_p, \alpha)$ exists (for some $\alpha\colon n\to p$) it is a solution of the iteration equation for f and any solution of this equation may be expressed as such a limit.*

Proof. If $g = f\circ(I_p, g)$, then by (1.2), $g = f^k\circ(I_p, g)$. So when $\alpha = g$, the constant sequence $f^k\circ(I_p, \alpha)$ has limit g.

Conversely, suppose $g = \lim\limits_{k\to\infty} f^k\circ(I_p, \alpha)$, for some $\alpha\colon n \to p$. Then

$$\begin{aligned} f\circ(I_p, g) &= \lim_{k\to\infty} f\circ\big(I_p, f^k\circ(I_p, \alpha)\big), \quad \text{by continuity;}\\ &= \lim_{k\to\infty} f\circ(I_p\oplus 0_n, f^k)\circ(I_p, \alpha)\\ &= \lim_{k\to\infty} f^{k+1}\circ(I_p, \alpha) = g, \end{aligned}$$

so g is a solution of the iteration equation for f.

(6.3) Corollary. *If T is an ideal metric theory satisfying* (P5) *then T is an iterative theory.*

(6.4) Proposition. *If T is an iterative theory with property* (P4), *then T also has property* (P5).

Proof. Let F be the function $T_{n,p} \to T_{n,p}$ taking $\xi\colon n\to p$ to $\xi F = f\circ(I_p, \xi)$, where $f\colon n \to p+n$ is ideal in T. F is a contraction map, by (P4). Let $c<1$ be a real number such that $d(\xi F, \xi' F) \leqslant c\cdot d(\xi, \xi')$. It easily follows that $d(\xi F^k, \xi' F^k) \leqslant c^k\cdot d(\xi, \xi')$ for any $k \geqslant 1$ (where F^k is F composed with itself k times). But an easy induction shows $\xi F^k = f^k\circ(I_p, \xi)$, for each ξ. Thus letting $\alpha\colon n \to p$ be arbitrary, for any $k\geqslant 1$

$$d\big(f^\dagger, f^k\circ(I_p, \alpha)\big) = d\big(f^k\circ(I_p, f^\dagger), f^k\circ(I_p, \alpha)\big) \leqslant c^k\cdot d(f^\dagger, \alpha).$$

Hence $\lim\limits_{k\to\infty} f^k\circ(I_p, \alpha) = f^\dagger$, proving (P5).

In order to state the next proposition we need some terminology. Call a metric theory with property (P6) a *complete* theory. A *morphism* $F\colon T \to T'$ between metric theories is a theory morphism (i.e. F preserves composition and the distinguished morphisms) which is continuous (i.e. if $f = \lim f_k$ in T, then $\lim f_k F$ exists in T' and $\lim f_k F = fF$) and which preserves Cauchy sequences.

Properties (P1) and (P2) seem to be natural ones, in view of the following proposition. I do not see how to prove (6.5) without these assumptions.

(6.5) Proposition. *Let T be a metric theory satisfying* (P1) *and* (P2). *There is a complete metric theory TC satisfying* (P1) *and* (P2) *and a metric theory morphism $T \to TC$ with the following universal property. If $F\colon T\to T'$ is any metric theory morphism from T to a complete theory T', there is a unique morphism $F'\colon TC \to T'$ such that the diagram*

$$\begin{array}{ccc} T & \longrightarrow & TC \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle F'} \\ & & T' \end{array}$$

commutes. The morphism $T \to TC$ is injective and distance preserving, so that T is isometrically isomorphic to a subtheory of TC.

This fact is proved by showing that the standard completion of a metric space using equivalence classes of Cauchy sequences works for theories. The reason it works, briefly, is that theories may be considered as equationally defined many-sorted algebras (as in [1]). Such classes are closed under products, subalgebras and homomorphic images.

The collection of Cauchy sequences (f_k) in $T_{n,p}$ (each $n, p\geqslant 0$) forms a subtheory of T^∞, the countable product of T with itself. The usual equivalence relation $\sim$ on Cauchy sequences induces a theory congruence on this subtheory. The quotient theory is TC. The morphism $T\to TC$ takes $f\colon n\to p$ in T to the equivalence class of the constant sequence (f_k), $f_k = f$. Conditions (P1) and (P2) are used to show the Cauchy sequences are a subtheory of T^∞ and that $\sim$ is a theory congruence. We omit the details.

Note that it is possible that an iterative metric theory satisfies (P4) but not (P6)—indeed the free iterative theory Γtr is not complete.

Although I do not have an example, it seems possible *a priori* that there is an ideal theory T satisfying (P5) (and thus by (6.3), T is iterative) but not (P4).*

(6.6) Proposition. *If T is an iterative theory satisfying* (P2)–(P3)

* (Added in proof: A. Arnold found such an example.)

and (P5), *the operation* $f \mapsto f^\dagger$ *is continuous, i.e. if* f, f_k: $n \to p+n$ *are ideal,* $k \geq 1$, *and* $\lim f_k = f$, *then* $\lim f_k^\dagger = f^\dagger$.

Proof. Using the triangle inequality, for any k, r and a: $n \to p$:

$$d(f_k^\dagger, f^\dagger) \leq d(f_k^\dagger, f_k^r \circ (I_p, a)) + \\ + d(f_k^r \circ (I_p, a),\ f^r \circ (I_p, a)) + d(f^r \circ (I_p, a), f^\dagger).$$

By (P2) and (P3), $d(f_k^r \circ (I_p, a), f^r \circ (I_p, a)) \leq d(f_k, f)$. First choose k such that $d(f_k, f)$ is small. For that choice of k, we can find a large enough value for r such that both $d(f_k^\dagger, f_k^r \circ (I_p, a))$ and $d(f^r \circ (I_p, a), f^\dagger)$, are small, by (P5) and (6.2).

By (6.6) iteration is continuous in all of the theories of §§ 3–5.

In [3] it was shown that if T is an iterative tree theory (so that T has properties (P1)–(P5)).

(P7) for every (not necessarily ideal) f: $n \to p+n$ and every $\perp$: $1 \to 0$ in T, $\lim_{k\to\infty} f^k \circ (I_p, \bar{a})$ exists, where

$$\bar{a}:\ n \xrightarrow{\text{base}} 1 \xrightarrow{\perp} 0 \xrightarrow{0_p} p.$$

The argument given there in fact proves

(6.7) Proposition. *If* T *is an iterative metric theory satisfying* (P4) *and* (P2), *then* (P7) *holds as well.*

From [3] it follows that the operation $f \mapsto f^\perp = \lim_k f^k \circ (I_p, \bar{a})$, defined on all morphisms $n \to p+n$ in iterative metric theories having property (P4), will satisfy all equations that the iteration operation does.

We plan to investigate other properties of metric theories in another paper. In particular we wish to use metric theories in the semantics of recursive program schemes.

7. An example

We show by means of an example that not every iterative theory is metrizable so that (P5) of § 6 holds.

For any set X, the algebraic theory $\mathrm{Pow}(X)$ has as its set of morphisms $n \to p$ all functions $X^p \to X^n$. Composition is function composition (f: $n \to p$, g: $p \to q$ compose to give $f \circ g$: $X^q \xrightarrow{g} X^p \xrightarrow{f} X^n$) and the distinguished morphism i: $1 \to n$ is the ith projection function $X^n \to X$.

Let $X = N \cup \{a, b\}$ where N is the set of natural numbers and a, b are distinct points not in N. Let f, g: $X \to X$ be the functions defined as follows:

$$nf = ng = n+1, \quad \text{all } n \in N;$$
$$(7.1)\qquad af = bf = a; \quad ag = bg = b.$$

Let T be the least subtheory of $\mathrm{Pow}(X)$ containing the morphisms f, g: $1 \to 1$ and all the constants n: $1 \to 0$, $n \in N$, a, b: $1 \to 0$. (Note a morphism $1 \to 0$ in $\mathrm{Pow}(X)$ is a function from the singleton set X^0 to X; we identify such a function with its value.)

(7.2) Proposition. *T is an iterative theory.*

This can be proved by showing that if h: $1 \to n$ is a nonconstant morphism in T, either h is a projection function, or h is a projection function $X^n \to X$ composed with a function $X \to X$ of the form $f^{k_1} g^{k_2} \ldots f^{k_n} g^{k_{n+1}}$, where $\sum_i k_i \geq 1$, $0 \leq k_i$. We omit the details.

(7.3) Proposition. *T cannot be made into a metric theory satisfying* (P5).

Proof. Clearly, the iterate of f: $1 \to 1$ is the morphism a: $1 \to 0$, and the iterate of g: $1 \to 1$ is b: $1 \to 0$. Suppose T is a metric theory satisfying (P5). Then for any morphism a: $1 \to 0$, the limits $\lim_{k\to\infty} f^k \circ a$ and $\lim_{k\to\infty} g^k \circ a$ exist and

$$(7.4)\qquad f^\dagger = \lim_{k\to\infty} f^k \circ a \quad \text{and} \quad g^\dagger = \lim_{k\to\infty} g^k \circ a.$$

Let a be the constant say 5: $1 \to 0$ (i.e. the function $X^0 \to X$ with value 5). Then

$$f^k \circ a:\ X^0 \xrightarrow{a} X \xrightarrow{f^k} X$$

is the function with value $5+k$ (by (7.1)). Similarly, $g^k \circ a = f^k \circ a$. Thus by (7.4)

$$a = f^\dagger = g^\dagger = b,$$

a contradiction.

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ON SYSTEM OF SUBOBJECT FUNCTORS IN THE CATEGORY OF ORDERED SETS

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Let $\mathscr{S}$ be the category of all non-empty sets with mappings as morphisms, $\mathscr{U}$ the category of all non-empty (partially) ordered sets with isotone maps as morphisms.

Let Exp be the endofunctor $\mathscr{S} \to \mathscr{S}$ with

$$\operatorname{Exp} X = \{Y\colon\ Y \subset X,\ Y \neq \emptyset\}$$

and

$$[\operatorname{Exp} f](Y) = f(Y) = \{f(y)\colon\ y \in Y\}$$

for all sets $X \neq \emptyset$, $Y \subset X$ and all maps f. Defining $\eta_X(x) = \{x\}$ for $x \in X$ and $\mu_X(\mathscr{X}) = \bigcup_{Y \in \mathscr{X}} Y$ for $\mathscr{X} \in \operatorname{Exp}\operatorname{Exp} X$, we get a monad $(\operatorname{Exp}, \eta, \mu)$ (see [3], p. 138). We look now for such functors $T\colon \mathscr{U} \to \mathscr{U}$ for which the diagram

(a)
$$\begin{array}{ccc} \mathscr{U} & \xrightarrow{T} & \mathscr{U} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ \mathscr{S} & \xrightarrow{\operatorname{Exp}} & \mathscr{S} \end{array}$$

(U is the forgetful functor) is commutative (so $T(A, \varrho) = (\operatorname{Exp} A,\ T(\varrho))$, where $T(\varrho)$ is a partial order on $\operatorname{Exp} A$ for each $(A, \varrho) \in \mathscr{U}$).

(b)
$$Y_1 \subset Y_2 \subset A \Rightarrow Y_1 T(\varrho) Y_2$$

as T is a functor, we get for any isotone mapping $f\colon (A, \varrho) \to (B, \sigma)$

$$Y_1\, T(\varrho)\, Y_2 \Rightarrow f(Y_1)\, T(\sigma)\, f(Y_2).$$

We shall call such T a *functor lifting* Exp and extending inclusion. A description of these liftings was considered in [4]. The system of all considered functors T will be denoted by $\boldsymbol{T}$. Having two elements T',