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*Presented to the Semester*
*Universal Algebra and Applications*
*(February 15 – June 9, 1978)*


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UNIVERSAL ALGEBRA AND APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 9
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982


# ON BOUNDING CONGRUENCES IN SOME ALGEBRAS HAVING THE LATTICE STRUCTURE


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An algebra of the form $\mathfrak{A} = (X; +, \cdot, f_1, f_2, \ldots)$, where $(X; +, \cdot)$ is a lattice and all operations $f_1, f_2, \ldots$ are unary, we shall call an *algebra having the lattice structure with unary operations*. Obviously, there need not exist the greatest element or the smallest element in $X$. However, it can happen that there exists a congruence $R$ in $\mathfrak{A}$ such that any congruence class $[a]_R$ is bounded by the greatest element $1([a]_R)$ and the smallest element $0([a]_R)$ where $1([a]_R), 0([a]_R) \in [a]_R$ and $[a]_R$ is a subalgebra in $\mathfrak{A}$. Such a congruence $R$ we shall call a *bounding congruence* in $\mathfrak{A}$. In this case we can define two functions $0(x)\colon X \to X$ and $1(x)\colon X \to X$ as follows:

$$0(x) = 0([x]_R), \quad 1(x) = 1([x]_R).$$

We have:

(i) $$\bigwedge_{a,b \in X} aRb \Leftrightarrow 0(a) = 0(b).$$

In this paper we study connections between bounding congruences and couples of functions $0(x)$ and $1(x)$ discussed above and we give some descriptions of algebras having the lattice structure with unary operations in which bounding congruences exist. Finally, we give some examples of algebras beeing generalizations of Boolean algebras and examples in the graph theory. In [2] the notion of double system of lattices was introduced which we recall here. Let $I = (I; \wedge, \vee)$ be a lattice and $\{\mathfrak{A}_i\}_{i \in I}$ a family of pairwise disjoint lattices such that $\mathfrak{A}_i = (A_i; +_i, \cdot_i)$ for each $i \in I$, and $I \cap \bigcup_{i \in I} A_i = \emptyset$. We assume that for any $i, j \in I$, $i \leqslant j$ the times-homomorphism $g_i^j\colon \mathfrak{A}_j \to \mathfrak{A}_i$ and plus-homomorphism $h_i^j\colon \mathfrak{A}_i \to \mathfrak{A}_j$ are given and the following conditions hold:

(1) $g_i^i, h_i^i$ are identity mappings for $i \in I$,

(2) $g_i^j g_j^k = g_i^k, \quad h_j^k h_i^j = h_i^k \quad$ for any $i \leqslant j \leqslant k$, $i, j, k \in I$.

The system $\langle I, \{\mathfrak{A}_i\}_{i\in I}, \{g_i^j, h_i^j\}_{i\leqslant j, i,j\in I}\rangle$ will be called a *double system of lattices* $\mathfrak{A}_i$, $i \in I$, and will be denoted by $\mathscr{A}$. For every such a double system $\mathscr{A}$ we define an algebra $\mathrm{DS}(\mathscr{A})$ in the following way: $\mathrm{DS}(\mathscr{A}) = (\bigcup_{i\in I} A_i; +, \cdot)$ where plus and times are defined as follows:

(3) $x \cdot y = g_k^i(x) \cdot_k g_k^j(y), \qquad x + y = h_i^l(x) +_l h_j^l(y),$

where $x \in A_i$, $y \in A_j$, $k = i \wedge j$, $l = i \vee j$, $i, j \in I$.

The algebra $\mathrm{DS}(\mathscr{A})$ will be called the *sum of double system* $\mathscr{A}$.

## 1

Let $\mathfrak{A} = (X; +, \cdot, f_1, f_2, \ldots)$ be an algebra having the lattice structure with unary operations. Let $0(x)\colon X \to X$ and $1(x)\colon X \to X$ be functions.

THEOREM 1. *The following three conditions* (a), (b), (c) *are equivalent*:

(a) *The relation defined by* (i) *is a bounding congruence in* $\mathfrak{A}$ *such that for any congruence class* $[a]_R$ *the element* $0(a)$ *is the smallest and the element* $1(a)$ *is the greatest element in* $[a]_R$.

(b) *The following equalities hold*

(4) $x + 0(x) = x \cdot 1(x) = x,$

(5) $0(0(x)) = 0(1(x)) = 0(x),$

(6) $1(0(x)) = 1(1(x)) = 1(x),$

(7) $0(x+y) = 0(x) + 0(y),$

(8) $1(x \cdot y) = 1(x) \cdot 1(y),$

(9) $0(f_i(x)) = 0(x), \qquad i = 1, 2, \ldots,$

(10) $1(f_i(x)) = 1(x), \qquad i = 1, 2, \ldots$

(c) *The lattice* $(X; +, \cdot)$ *is the sum of a double system of its sublattices* $(X_i; +, \cdot)$, $i \in I$, *with the greatest element* $1_i$ *and the smallest element* $0_i$ *where* $1_i = 1(a)$ *and* $0_i = 0(a)$ *for some* $a \in X_i$ — *such that any* $X_i$ *is a subalgebra of* $\mathfrak{A}$.

*Proof.* (a) $\Rightarrow$ (b). Observe first that

(11) $\bigwedge_{a,b\in X} aRb \Leftrightarrow 1(a) = 1(b).$



Equalities (4), (5), (6) are obvious; (9) and (10) are also obvious because any congruence class of $R$ is a subalgebra. Since $R$ is a congruence in $\mathfrak{A}$ and $xR0(x)$, $yR0(y)$ (for $x, y \in X$), we have $0(x)+0(y)\,R\,x+y\;R\,0(x+y)$. Thus

(12) $0(x+y) \leqslant 0(x) + 0(y).$

Further, write $0(x)\cdot 0(x+y) = x^*$, $0(y)\cdot 0(x+y) = y^*$. So

$$x^* \in [0(x)]_R, \qquad y^* \in [0(y)]_R$$

and we have

$$0(x) = 0(x)\cdot x^* = 0(x)\cdot 0(x+y),$$

$$0(y) = 0(y)\cdot y^* = 0(y)\cdot 0(x+y).$$

The last equalities give $0(x) \leqslant 0(x+y)$, $0(y) \leqslant 0(x+y)$; hence $0(x) + 0(y) \leqslant 0(x+y)$ which together with (12) gives (7). Analogously, using (11), we can prove (8).

(b) $\Rightarrow$ (c). Define $x \circ_1 y = x \cdot 1(y)$, $x \circ_2 y = x + 0(y)$. Further put $x \cdot y = F_1(x, y)$, $x + y = F_2(x, y)$, $T_1 = \{1\}$, $T_2 = \{2\}$. Now, using (4)–(10), it is enough to check that $\circ_1$ and $\circ_2$ is a companied couple (see [4]) for the lattice $(X; +, \cdot)$, and to use a theorem from [4] to obtain the first statement of (c). In any sublattice $X_i$ defined by decomposition from [4] we have

(13) $x, y \in X_i \Leftrightarrow x + 0(y) = x,\; y + 0(x) = y \Leftrightarrow x \cdot 1(y) = x,\; y \cdot 1(x) = y.$

By (4), (5) we have $0(x)$, $x$, $y$, $0(y) \in X_i$ and $0(x) = 0(x) + 0(0(y)) = 0(x) + 0(y)$, $0(y) = 0(y) + 0(x)$. Hence $0(x) = 0(y)$ and, by (4), $0(x)$ is the smallest element in $X_i$. Analogously we can prove, by using (4), (6) and (13), that $1(x)$ is the greatest element in $X_i$. By (9) and (4) we have $x + 0(f_i(x)) = x + 0(x) = x$, $f_i(x) + 0(x) = f_i(x) + 0(f_i(x)) = f_i(x)$. Thus $x \in X_i \Leftrightarrow f_i(x) \in X_i$.

(c) $\Rightarrow$ (a). The fact that $R$ is a congruence in $\mathfrak{A}$ follows from the definition of the sum of double system of lattices. Now the proof that (c) $\Rightarrow$ (a) is obvious.

### *Applications*

1. Let $\mathfrak{A} = (X; +, \cdot, 0(x), 1(x))$ be an algebra such that $(X; +, \cdot)$ is a lattice and equalities (4)–(8) hold. Then by our theorem $(X; +, \cdot)$ is the sum of double system of bounded sublattices where in any sublattice $X_i$ the elements $0(a)$ and $1(a)$ $(a \in X_i)$ are the smallest and the greatest element, respectively.

2. Let $\mathfrak{A} = (X; +, \cdot, ')$ be an algebra such that $(X; +, \cdot)$ is a lattice and the following equalities hold:

$$(x')' = x, \tag{14}$$

$$(x+x')' = xx', \tag{15}$$

$$(x+y)(x+y)' = xx'+yy', \tag{16}$$

$$(xy)+(xy)' = (x+x')(y+y'). \tag{17}$$

Then we can put $0(x) = xx'$, $1(x) = x+x'$ and use our theorem. In this way we get that $(X; +, \cdot)$ is the sum of a double system of sublattices $X_i$ where the operation $x'$ is an algebraic operation in $X_i$ assigning to every $x$ some complement.

The lattice in Figure 1, where $0' = 1$, $1' = 0$, $a_i' = b_i$, $b_i' = a_i$ for $i = 1, 2$, is an example of the last algebra.

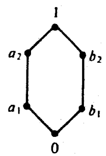


Fig. 1

3. An algebra $\mathfrak{A} = (X; +, \cdot, ')$ we shall call a *locally Boolean algebra* iff $(X; +, \cdot)$ is a distributive lattice and there exists in $\mathfrak{A}$ a congruence such that any class $[a]$ of this congruence is a Boolean algebra with respect of $+, \cdot, '$. It is easy to see that this congruence must be identical with the congruence $R$ from (i) where $0(x) = xx'$.

Let us add to our theorem the assumption that $(X; +, \cdot)$ is a distributive lattice. Then from our theorem we deduce that locally Boolean algebras form an equational class described by equalities of distributive lattices and equalities (14)–(17).

4. Let $X$ be a non-empty set and denote by $B$ the set of all pairs $\langle A, B\rangle \in 2^X \times 2^X$ such that $B \subseteq A$. Consider an algebra $\mathfrak{A} = (B; +, \cdot, ')$ where $+, \cdot, '$ are defined as follows:

$$\langle A, B\rangle + \langle C, D\rangle = \langle A\cup C, B\cup D\rangle,$$

$$\langle A, B\rangle \cdot \langle C, D\rangle = \langle A\cap C, B\cap D\rangle,$$

$$(\langle A, B\rangle)' = \langle A, A\cap (X\setminus B)\rangle.$$

Then $\mathfrak{A}$ is a locally Boolean algebra where the class of the congruence $R$ is of the form $\{\langle A, Y\rangle\}$, $Y \subseteq A$. Such algebras were considered as models in logic by K. Hałkowska (see [3]).



5. By a *graph* we mean (see [1]) a pair $G = (V; R)$ where $V$ is a non-empty set and $R$ is a binary relation in $V$. The set $V$ is called the *set of vertices* and $R$ the *set of arcs*. $V$ need not be finite but if we consider some properties of $G$ it is enough sometimes to check them locally, i.e. to consider finite subgraphs of $G$ or even partial subgraphs. By a *subgraph* we mean a pair $\big(V_0; (V_0\times V_0)\cap R\big)$ where $\emptyset \neq V_0 \subseteq V$. By a *partial subgraph* we mean a pair $(V_0; S)$ where $S \subseteq (V_0\times V_0)\cap R$. Denote by $P_G$ the family of the finite partial subgraphs of a given graph $G$ and write $P = P_G \cup \{(\emptyset, \emptyset)\}$. In the family $P$ we can consider three operations $\cup, \cap, '$ defined as follows:

$$(V_1; S_1)\cup(V_2; S_2) = (V_1\cup V_2; S_1\cup S_2),$$

$$(V_1; S_1)\cap(V_2; S_2) = (V_1\cap V_2; S_1\cap S_2),$$

$$(V_1; S_1)' = \big(V_1, (V_1\times V_1)\cap(V_1; S^*)\big) \quad \text{where} \quad S^* = (V_1\times V_1)\cap(R\setminus S).$$

Then the algebra $(P, \cup, \cap, ')$ is an example of locally Boolean algebra where any family of partial subgraphs $(V_0; S)$ ($V_0$ is fixed) is a congruence class of the congruence $R$.

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*Presented to the Semester*
*Universal Algebra and Applications*
*(February 15–June 9, 1978)*