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REGULAR ELEMENTS IN COMPLETE UNIQUELY COMPLEMENTED LATTICES

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Since Dilworth [2] discovered that any lattice may be isomorphically embedded in a suitable uniquely complemented lattice (i.e. in a lattice with 0 and I in which every element has one and only one complement), not much about this class of close-to-Boolean lattices has become known. For example, it is an open problem if such lattice may be complete without being distributive. Birkhoff and Ward [1] have proved that it does not have to be atomic and in [4] it was shown that the property of being compactly generated also implies distributivity. Another fact concerning complete uniquely complemented lattices was stated in [5], namely that every such lattice is isomorphic to a direct product of a complete atomic Boolean lattice and a complete atomless uniquely complemented lattice.

In the present note we deal with so-called regular elements in a uniquely complemented lattice. It is shown that regular elements form a complete Boolean sublattice (Theorem 1) and the above-mentioned result on direct decomposition is generalized in a natural way (Theorem 2).

Let L be a uniquely complemented lattice (sometimes a "UC-lattice" to be short). An element $a \in L$ is called *regular* if

$$(\forall x, y \in L)\ \big(a \wedge x = a \wedge y = 0 \Rightarrow a \wedge (x \vee y) = 0\big).$$

Obviously, 0 and I are regular elements. And here is a less trivial example.

PROPOSITION. *Every atom of a uniquely complemented lattice is a regular element.*

Proof. Let a be an atom of a UC-lattice L. Assume that a is not regular. Then there exists an $x \in L$ such that $a \wedge x = 0$ and x is not contained in a'. Take an element $y = (x \wedge a')' \wedge x$. It is easy to see that $y \neq 0$ because otherwise $(x\wedge a')'$ would have two different complements: $x\wedge a'$ and x.

Since a' is evidently coatom, we find y to be a complement for a' and different from a (because $a\wedge y=0$).

So we have a contradiction, which proves that a is regular.

We shall often use the fact which is stated in the following

Remark. $(\forall x,y\in L)\ (x<y\ \Rightarrow x'\wedge y\neq 0)$.

(Indeed, if $x'\wedge y=0$ then x' has two different complements: x and y.)

Now let $R(L)$ denote the set of all regular elements of a UC-lattice L.

LEMMA 1. $(\forall a\in R(L),\ x\in L)\ (a\wedge x=0\Rightarrow x\leqslant a')$.

Proof. If x is not contained in a', let us take $y=(a'\wedge x)'\wedge x$.

Then $a\wedge a'=0=a\wedge y$ but $a\wedge(a'\vee y)\neq 0$ because otherwise $a'\vee y$ would be a complement for a different from a'.

So we get a contradiction with the regularity of a.

LEMMA 2. $(\forall a\in R(L),\ x\in L)\ \big((a\wedge x)\vee(a'\wedge x)=x\big)$.

Proof. Let us assume that $(a\wedge x)\vee(a'\wedge x)=y<x$. Then $x\wedge y'\neq 0$ (see Remark) and

$$a\wedge(x\wedge y')=(a\wedge x)\wedge y'\leqslant y\wedge y'=0.$$

Thus $a\wedge a'=0=a\wedge(x\wedge y')$ and so we have a contradiction with the regularity of a because $a\wedge\big(a'\vee(x\wedge y')\big)\neq 0$ (equality would imply the existence of two complements for a, namely a' and $a'\vee(x\wedge y')$, which are different since $a'\wedge(x\wedge y')=0$).

LEMMA 3. $(\forall a\in R(L),\ x\in L)\ (x\leqslant a\Leftrightarrow x'\geqslant a')$.

Proof. Suppose that $x\leqslant a$ but x' does not contain a', that is, $a'\wedge x'<a'$. Then using Lemma 2 we get

$$\begin{aligned}a\vee(a'\wedge x')&\geqslant\big(x\vee(a\wedge x')\big)\vee(a'\wedge x')\\&=x\vee\big((a\wedge x')\vee(a'\wedge x')\big)=x\vee x'=I.\end{aligned}$$

Now it follows that $a'\wedge x'$ is a complement for a, which is impossible.

On the other hand, let us assume that $x'\geqslant a'$ but x is not contained in a, that is, $a\wedge x<x$. Then for $y=(a\wedge x)'\wedge x$ it follows that $a'\wedge y\leqslant x'\wedge y\leqslant x'\wedge x=0$, and so we must have $a\wedge(a'\vee y)\neq 0$, which prevents a from having two different complements (a' and $a'\vee y$). But $a\wedge a'=0=a\wedge y$, and so we have got a contradiction with the regularity of a.

LEMMA 4. $(\forall a\in R(L);\ x,y\in L)\ \big(a\wedge(x\vee y)=(a\wedge x)\vee(a\wedge y)\big)$.

Proof. Let us assume that $(a\wedge x)\vee(a\wedge y)=z<a\wedge(x\vee y)$. Then, using Lemmas 3 and 2, we get what follows:

$$\begin{aligned}\big(a\wedge(x\vee y)\big)'\vee z&=\Big(\big(a\wedge(x\vee y)\big)'\vee a'\Big)\vee z\\&=\big(a\wedge(x\vee y)\big)'\vee(a'\vee z)\\&\geqslant\big(a\wedge(x\vee y)\big)'\vee\Big[\big((a'\wedge x)\vee(a'\wedge y)\big)\vee z\Big]\\&=\big(a\wedge(x\vee y)\big)'\vee\big[(a'\wedge x)\vee(a'\wedge y)\vee(a\wedge x)\vee(a\wedge y)\big]\\&=\big(a\wedge(x\vee y)\big)'\vee(x\vee y)\geqslant\big(a\wedge(x\vee y)\big)'\vee\big(a\wedge(x\vee y)\big)=I.\end{aligned}$$

Thus z is a complement for $\big(a\wedge(x\vee y)\big)'$ different from $a\wedge(x\vee y)$, which is impossible.

THEOREM 1. *The set of all regular elements of a complete uniquely complemented lattice forms its complete Boolean sublattice.*

Proof.

1) $\big(\forall a\in R(L)\big)\ \big(a'\in R(L)\big)$.

Indeed, let $a\in R(L)$ and $a'\wedge x=0$. Assume that $a\wedge x<x$. Then $y=(a\wedge x)'\wedge x\neq 0$ and $a\wedge a'=0=(a\wedge x)'\wedge(a\wedge x)=\big((a\wedge x)'\wedge x\big)\wedge a=y\wedge a$. But $a\wedge(a'\vee y)\neq 0$, which contradicts the regularity of a. So $a'\wedge x=0$ implies $x\leqslant a$. This means that a' is a regular element.

2) $\big(\forall(a_i)_{i\in I}\subset R(L)\big)\ \big(\bigvee_{i\in I}a_i\in R(L)\big)$.

Indeed, let $(a_i)_{i\in I}$ be any set of regular elements of L and let $a=\bigvee_{i\in I}a_i$. Then, using Lemma 3, we get

$$\bigwedge_{i\in I}a_i'\leqslant a_i'\Rightarrow\big(\bigwedge_{i\in I}a_i'\big)'\geqslant a_i\Rightarrow\big(\bigwedge_{i\in I}a_i'\big)'\geqslant\bigvee_{i\in I}a_i=a$$

and

$$a_i\leqslant\bigvee_{i\in I}a_i\Rightarrow a_i'\geqslant\big(\bigvee_{i\in I}a_i\big)'\Rightarrow\bigwedge_{i\in I}a_i'\geqslant\big(\bigvee_{i\in I}a_i\big)'=a'.$$

So $a'\leqslant\bigwedge_{i\in I}a_i'$ and $a\leqslant\big(\bigwedge_{i\in I}a_i'\big)'$.

Assuming now that $a'<\bigwedge_{i\in I}a_i'$, we shall get two different complements for a, namely a' and $\bigwedge_{i\in I}a_i'$.

Thus $\big(\bigvee_{i\in I}a_i\big)'=\bigwedge_{i\in I}a_i'$.

Then for an arbitrary $x\in L$

$$a\wedge x=0\Rightarrow(\forall i\in I)(a_i\wedge x=0)\Rightarrow(\forall i\in I)(x\leqslant a_i')\Rightarrow x\leqslant\bigwedge_{i\in I}a_i'=a',$$

from which it follows that $a=\bigvee_{i\in I}a_i$ is regular.

We have proved that regular elements form in L a complete sublattice which is closed under complementation. Moreover, De Morgan laws being established for this uniquely complemented (sub)lattice, we find also that distributivity holds in it.

COROLLARY 1. *A complete uniquely complemented lattice is distributive if and only if each of its non-zero elements contains some non-zero regular element.*

Proof. It is evident that every element of a Boolean lattice is regular, and so we need to prove only one implication.

Let x be an arbitrary non-zero element of such a lattice and let a denote the join of all regular elements contained in x. If $a < x$, then $y = a' \wedge x$ is not zero and does not contain any regular element. This contradiction shows that every element of the lattice in question is regular, and so this lattice is Boolean.

COROLLARY 2 (Birkhoff and Ward; [1]). *Every complete atomic uniquely complemented lattice is distributive.*

Proof. Follows from Proposition and Corollary 1.

(*Note.* Now we prove directly the statement of Corollary 1 without the assumption of completeness for the lattices in question (and then Corollary 2 may be modified into the Ogasawara–Sasaki Theorem [3]).

So let L be any UC-lattice in which every non-zero element contains some non-zero regular element and let x, y, z be arbitrary elements of L. Assume that $t = (x \wedge z) \vee (y \wedge z) \neq (x \vee y) \wedge z = u$ (that is, $t < u$). Now

$$(t' \wedge u) \wedge x \leqslant (t' \wedge z) \wedge x = t' \wedge (x \wedge z) \leqslant t' \wedge t = 0$$

and similarly $(t' \wedge u) \wedge y = 0$. So, if a is a regular element contained in $t' \wedge u$, then $a \wedge (x \vee y) = 0$. But $a \leqslant u < x \vee y$. Thus $a = 0$ and we conclude that $t' \wedge u$ ($\neq 0$ by Remark) contains no non-zero regular elements, which is impossible.)

LEMMA 5. $(\forall a \in R(L))\,([0, a]$ *is a UC-lattice*$)$.

Proof. Let $x \in [0, a]$. We shall show that $a \wedge x'$ is a unique complement for x in $[0, a]$. In fact, assuming that $x \vee (a \wedge x') < a$, we get using Lemmas 3 and 2

$$\begin{aligned}\big(x \vee (a \wedge x')\big) \vee a' &= x \vee \big((a \wedge x') \vee a'\big) = x \vee \big((a \wedge x') \vee (a' \wedge x')\big)\\ &= x \vee x' = I.\end{aligned}$$

But then a' has $x \vee (a \wedge x')$ as a complement, which is impossible. Thus $a \wedge x'$ is a complement for x in $[0, a]$.

Let y be another complement for x in $[0, a]$, that is, let $x \wedge y = 0$, $x \vee y = a$ and $y \neq a \wedge x'$. Then according to Lemma 4 $(a' \vee y) \wedge a = y$, and so

$$x \wedge (a' \vee y) = (x \wedge a) \wedge (a' \vee y) = x \wedge \big(a \wedge (a' \wedge y)\big) = x \wedge y = 0.$$

But $x \vee (a' \vee y) = a \vee a' = I$, which means that $a' \vee y$ is a complement for x in L different from x' because $(a' \vee y) \wedge a = y \neq x' \wedge a$.

The contradiction we have got proves the uniqueness of complements in $[0, a]$.

THEOREM 2. *Let a be any regular element of a complete uniquely complemented lattice L. Then L is isomorphic to the direct product of the complete uniquely complemented lattices* $[0, a]$ *and* $[0, a']$.

Proof. Consider a mapping $f\colon L \to [0, a] \times [0, a']$ such that $f(x) = (a \wedge x, a' \wedge x)$ for every $x \in L$. According to Lemma 2, f is one-to-one. Further, if $x \in [0, a]$ and $y \in [0, a']$, then by Lemma 4 $a \wedge (x \vee y) = x$ and $a' \wedge (x \vee y) = y$, so that $(x, y) = f(x \vee y)$, providing the surjectivity of f.

It is evident that f preserves meets and, by Lemma 4, f is also a join-preserving map.

COROLLARY (Salii [5]). *Every complete uniquely complemented lattice is isomorphic to a direct product of a complete atomic Boolean algebra and a complete atomless uniquely complemented lattice.*

Proof. Let L be a complete uniquely complemented lattice which is not distributive and contains atoms. Then the join of all atoms (which are regular by the Proposition) is a regular element and now we can use Theorem 2 to provide direct decomposability and the Corollary 1 for proving distributivity of the atomic component.

References

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