UNIVERSAL ALGEBRA AND APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 9
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

LOCAL POLYNOMIAL FUNCTIONS: RESULTS AND PROBLEMS

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In all parts of analysis, functions are very important. In algebra, however, functions have for a long time been rather a tool than an independent subject. But in the last years there has grown up a theory of functions on algebraic structures which is steadily expanding. Within this theory, the problem of interpolation of functions by polynomial functions or by term functions is a recent question which has been investigated by several authors. By means of the concept of a local polynomial function or of a local term function one can describe to what extent a function can be interpolated by polynomial functions or by term functions. This paper gives a survey of results and open questions of local polynomial functions and local term functions.

Let A be a universal algebra and k a natural number. Let $F_k(A)$ be the algebra which is obtained if on the set of all k-place functions $\varphi\colon A^k \to A$ those operations are induced, which arise from pointwise definition of the operations of A. Thus $F_k(A)$ is an algebra of the same type as A, and if A belongs to a certain variety, then $F_k(A)$ also belongs to this variety. Moreover, in $F_k(A)$ one introduces, in a natural way, an additional $(k+1)$-place operation, namely composition of functions.

If one wants to know a function of $F_k(A)$, then in general one has to know its graph, that means its values at all "arguments" $(a_1, a_2, \ldots, a_k) = a \in A^k$. There are functions, however, which can be described in a much simpler way, namely the term functions and the polynomial functions. The *k-place term functions on A* are the elements of the subalgebra $T_k(A)$ of $F_k(A)$, which is generated by the projections $\xi_1, \xi_2, \ldots, \xi_k$ and the *k-place polynomial functions on A* are the elements of the subalgebra $P_k(A)$ of $F_k(A)$ which is generated by the projections and the constant functions. Hence the term functions are those functions, which can be represented by words in the projections, and the polynomial functions are those functions, which can be represented by words in the projections and the constant functions. Thus a graph is not required to determine a term function or a polynomial function. Clearly, $T_k(A) \subseteq P_k(A)$, and $T_k(A)$ and $P_k(A)$ are also subalgebras with respect to composition of functions.

There exist algebras A such that $T_k(A) = F_k(A)$ — these algebras are called *k-primal* — and algebras A such that $P_k(A) = F_k(A)$ — these algebras are called *k-polynomially complete*. In important classes of algebras such as groups, rings, etc., the primal and polynomially complete algebras are well known (see [6]). If A is not k-primal or k-polynomially complete, $T_k(A) \subset F_k(A)$ or $P_k(A) \subset F_k(A)$, respectively. In order to investigate the gap between $T_k(A)$ and $F_k(A)$ or the gap between $P_k(A)$ and $F_k(A)$ we introduce the concepts of a local term function and a local polynomial function:

Let s be a natural number. A function $\varphi \in F_k(A)$ is called an *s-local term function* (*s-local polynomial function*) if, for any s elements $a_i \in A^k$, $i = 1, 2, \ldots, s$ (not necessarily distinct), there exists a term function g of $T_k(A)$ (a polynomial function g of $P_k(A)$) such that

$$\varphi(a_i) = g(a_i), \quad i = 1, 2, \ldots, s.$$

A function φ, which is an s-local term function (s-local polynomial function) for every s, is called a *local term function* (*local polynomial function*).

Let us denote the set of all s-local term functions by $L_sT_k(A)$, the set of all s-local polynomial functions by $L_sP_k(A)$, the set of all local term functions by $LT_k(A)$, the set of all local polynomial functions by $LP_k(A)$. Clearly, $L_sT_k(A) \subseteq L_sP_k(A)$, $LT_k(A) \subseteq LP_k(A)$, and all these sets are subalgebras of $F_k(A)$, which are also closed with respect to composition of functions. Clearly, $L_1P_k(A) = F_k(A)$.

Related to the local term functions and the local polynomial functions are the conservative and the compatible functions. A function $\varphi \in F_k(A)$ is called *conservative* if it maps the kth power of any subalgebra of A into A, and φ is called *compatible* if, for any congruence θ on A, $a_i \equiv b_i \bmod \theta$, $i = 1, 2, \ldots, k$, implies $\varphi(a_1, a_2, \ldots, a_k) \equiv \varphi(b_1, b_2, \ldots, b_k) \bmod \theta$. The set of all conservative functions is a subalgebra $K_k(A)$ of $F_k(A)$, also with respect to composition of functions, and the set of all compatible functions is a subalgebra $C_k(A)$ of $F_k(A)$, also with respect to composition of functions As one can easily see, $K_k(A) = L_1T_k(A)$ and $C_k(A) \supseteq L_2P_k(A)$.

The subalgebras of $F_k(A)$ which we have just defined can be arranged in the following chains:

$$F_k(A) \supseteq K_k(A) \supseteq L_2T_k(A) \supseteq \ldots \supseteq LT_k(A) \supseteq T_k(A),$$

$$F_k(A) \supseteq C_k(A) \supseteq L_2P_k(A) \supseteq \ldots \supseteq LP_k(A) \supseteq P_k(A).$$

In most of the algebras A, for which these chains have been investigated, a fair number of members of the chains coincide. Thus, in general, there arises the problem to find conditions for equality or inequality of given members of the chains.

Certain classes of algebras are defined by the property that a given pair of members of one of these chains coincides. Thus, for example, an algebra A is called

k-(locally) primal	if $F_k(A) = T_k(A)$ $\big(= LT_k(A)\big)$,
k-(locally) polynomially complete	if $F_k(A) = P_k(A)$ $\big(= LP_k(A)\big)$,
k-(locally) semiprimal	if $K_k(A) = T_k(A)$ $\big(= LT_k(A)\big)$,
k-(locally) affine complete	if $C_k(A) = P_k(A)$ $\big(= LP_k(A)\big)$.

Especially, locally polynomially complete algebras have been studied by various authors (see [5]).

Most of the results on equality of the members of the chains are on the chain of local polynomial functions. One can arrange these results into two classes:

The one type of results are statements on the initial members of the chains. Clearly, $F_k(A) = K_k(A)$ if and only if A has no proper subalgebras, and $F_k(A) = C_k(A)$ if and only if A is simple. By results of Werner [8] and Hule–Nöbauer [2], for any algebra A of a congruence permutable variety, $C_k(A) = L_2P_k(A)$. There exist, however, lattices A such that $C_k(A) \supset L_2P_k(A)$ (see Dorninger–Nöbauer [1]). Istinger, Kaiser and Pixley [4] have proved: If, in a simple algebra A of a congruence permutable variety, $L_2P_k(A) \supset LP_k(A)$, then $L_2P_k(A) \supset L_3P_k(A)$ or $L_3P_k(A) \supset L_4P_k(A)$. That both cases can occur, can be observed for simple abelian groups: If $k > 1$, then $L_2 \supset L_3 = L$ for any abelian group of prime order $p > 2$, and $L_2 = L_3 \supset L_4 = L$ for the abelian group of order 2 (see Hule–Nöbauer [2]). The result of Istinger–Kaiser–Pixley can be restated as follows: If, for a simple algebra A of a congruence permutable class, $L_2P_k(A) = L_4P_k(A)$, then $L_2P_k(A) = LP_k(A)$. Now the problem arises: Given a class of algebras, does there exist a natural number t such that $L_2P_k(A) = L_tP_k(A)$ implies $L_2P_k(A) = LP_k(A)$ for all algebras of the class? Some results in this direction can be found in [5].

The other type of results are statements on the last members and on the length of the chains. Some results of this type are solutions of the problem: Given a class C of algebras, single out all algebras A of C such that $LP_k(A) = P_k(A)$ or all algebras such that $LT_k(A) = T_k(A)$ (of course, all finite algebras satisfy these equations). This problem, for example, has been solved for the class of abelian groups: Either equation does not hold if and only if A is periodic and not bounded (see Hule–Nöbauer [2]). Another problem of this type is as follows: Let C be a class of algebras.

Does there exist an integer t such that $L_tP_k(A) = LP_k(A)$ for all algebras A of C? This question has been answered in the affirmative for the class of abelian groups since Hule–Nöbauer [2] have proved that for all algebras A of this class $L_3P_1(A) = LP_1(A)$ and $L_4P_k(A) = LP_k(A)$ for any $k \geqslant 1$. A similar answer has been found for the class C of lattices since $L_2P_k(A) = LP_k(A)$ for any lattice A and any $k \geqslant 1$ (see Dorninger–Nöbauer [1]). The answer is negative, however, if C is the class of commutative rings with identity since in a recent paper Lausch–Nöbauer [7] have proved: Let $A = Z/(p^e)$ be the factor ring of the integers modulo a prime power p^e $(e > 1)$, and let t be the greatest positive integer such that $t+\varepsilon(t) \leqslant e$, where $\varepsilon(t)$ is the exponent of the greatest power of p dividing $t!$ (especially $t = e$ of $p > e$); then

$$L_1P_1(A) \supset L_2P_1(A) \supset \ldots \supset L_tP_1(A) \supset L_{t+1}P_1(A) = P_1(A).$$

Furthermore the paper by Lausch–Nöbauer shows that for the ring Z of integers all members of the chain of local polynomial functions are distinct, i.e.

$$L_1P_1(Z) \supset L_2P_1(Z) \supset L_3P_1(Z) \supset \ldots \supset LP_1(Z) \supset P_1(Z).$$

Another problem concerning the chain of local polynomial functions refers to the cardinalities of the members of the chain: Let, as usual, $|S|$ denote the cardinal number of the set S, and let A be an infinite algebra. Then $|F_k(A)| > |A|$, but $|P_k(A)| = |A|$. Therefore, in the chain of cardinalities of the members of the chain of local polynomial functions, there must occur at least one inequality. Where in the chain does this inequality occur? As examples show, not all algebras A behave in the same way. If, for example, A is an infinite commutative field, then, as it is well known, $P_k(A) \subset LP_k(A) = F_k(A)$, but if A is the additive group of the rational numbers, then $P_k(A) = LP_k(A) = L_3P_k(A) \subset L_2P_k(A) \subset F_k(A)$ (see Hule–Nöbauer [3]; it is not known where inequality occurs in the chain of cardinalities).

As one could see from the just quoted results on the chains of local polynomial functions and local term functions, the properties of these chains heavily depend on the class of algebras to which the considered algebra belongs. There are, however, some general results which are valid in any class of algebras, and we now shall mention some of these results and problem related to them.

The first result (which can be proved quite easily) shows what connections exist between the chains of local polynomial functions in l variables and in k variables: If, in the chain of local polynomial functions in k variables, two members are equal, then the corresponding members are also equal in the chain of local polynomial functions in l variables for any $l \leqslant k$, and a similar result holds for the chain of local term functions. Hence from an inequality in the chain in l variables one can conclude that inequality holds also in the chain in $k \geqslant l$ variables. In certain cases it is also possible to derive from an equality in the chain in l variables the corresponding equality in the chain in $k \geqslant l$ variables. For example, by a well-known theorem of Sierpiński, if $F_2(A) = L_tP_2(A)$, then $F_k(A) = L_tP_k(A)$ for any k, and also $F_2(A) = L_tT_2(A)$ implies $F_k(A) = L_tT_k(A)$ for any k. Furthermore, if A is an abelian group, then $L_tP_2(A) = LP_2(A)$ implies $L_tP_k(A) = LP_k(A)$ for any k, and if A is a distributive lattice, then $LP_1(A) = P_1(A)$ implies $LP_k(A) = P_k(A)$ for any k. In all so far known examples of chains of local polynomial functions all equalities, which are valid in the chain of 2-place functions, are also valid in the chain of k-place functions for any k, but there is no proof that this holds in general.

In some classes of algebras one knows relations between the chain of local term functions and the chain of local polynomial functions. Clearly, $T_k(A) = P_k(A)$ (which holds if the nullary operations of A generate A) implies that both chains concide. If A is any abelian group, then $L_sT_k(A) = T_k(A)$ implies $L_{s+1}P_k(A) = P_k(A)$, and $L_sP_k(A) = LP_k(A)$ implies $L_sT_k(A) = LT_k(A)$.

Sometimes it is possible to reduce the chain of local polynomial functions or the chain of local term functions on a direct product of algebras to the corresponding chains on the factors. The main result in this direction is as follows: Suppose that $A = A_1 \times A_2$. Then there exists a monomorphism φ: $C_k(A_1 \times A_2) \to C_k(A_1) \times C_k(A_2)$ (see [1]). If now $\varphi P_k(A_1 \times A_2) = P_k(A_1) \times P_k(A_2)$, then also $\varphi L_tP_k(A_1 \times A_2) = L_tP_k(A_1) \times L_tP_k(A_2)$ and $\varphi LP_k(A_1 \times A_2) = LP_k(A_1) \times LP_k(A_2)$. Hence, if φ maps $P_k(A_1 \times A_2)$ onto $P_k(A_1) \times P_k(A_2)$, then two members of the chain of local polynomial functions for $A = A_1 \times A_2$ coincide if and only if the corresponding members of both the chains for A_1 and A_2 coincide. A similar result holds for local term functions. The just mentioned hypothesis on φ is, for example, satisfied for all direct products of commutative rings with identity and for direct products of groups of coprime orders. In these cases therefore, the chain of local polynomial functions for a direct product can be reduced to the chains of local polynomial functions for the factors.

There exist, so far, only a few results on the number of t-local polynomial functions on a given finite algebra A and on the structure of the semigroup of t-local polynomial functions (in one variable) with respect to composition of functions. To investigate both problems seems to be worthwile.

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Presented to the Semester
Universal Algebra and Applications
(February 15 – June 9, 1978)

UNIVERSAL ALGEBRA AND APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 9
PWN–POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

ON BOUNDING CONGRUENCES IN SOME ALGEBRAS HAVING THE LATTICE STRUCTURE

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An algebra of the form $\mathfrak{A} = (X; +, \cdot, f_1, f_2, \ldots)$, where $(X; +, \cdot)$ is a lattice and all operations $f_1, f_2, \ldots$ are unary, we shall call an *algebra having the lattice structure with unary operations*. Obviously, there need not exist the greatest element or the smallest element in X. However, it can happen that there exists a congruence R in $\mathfrak{A}$ such that any congruence class $[a]_R$ is bounded by the greatest element $1([a]_R)$ and the smallest element $0([a]_R)$ where $1([a]_R), 0([a]_R) \in [a]_R$ and $[a]_R$ is a subalgebra in $\mathfrak{A}$. Such a congruence R we shall call a *bounding congruence* in $\mathfrak{A}$. In this case we can define two functions $0(x)\colon X \to X$ and $1(x)\colon X \to X$ as follows:

$$0(x) = 0([x]_R), \quad 1(x) = 1([x]_R).$$

We have:

(i) $$\bigwedge_{a,b \in X} aRb \Leftrightarrow 0(a) = 0(b).$$

In this paper we study connections between bounding congruences and couples of functions $0(x)$ and $1(x)$ discussed above and we give some descriptions of algebras having the lattice structure with unary operations in which bounding congruences exist. Finally, we give some examples of algebras beeing generalizations of Boolean algebras and examples in the graph theory. In [2] the notion of double system of lattices was introduced which we recall here. Let $I = (I; \wedge, \vee)$ be a lattice and $\{\mathfrak{A}_i\}_{i \in I}$ a family of pairwise disjoint lattices such that $\mathfrak{A}_i = (A_i; +_i, \cdot_i)$ for each $i \in I$, and $I \cap \bigcup_{i \in I} A_i = \emptyset$. We assume that for any $i, j \in I$, $i \leqslant j$ the times-homomorphism $g_i^j\colon \mathfrak{A}_j \to \mathfrak{A}_i$ and plus-homomorphism $h_i^j\colon \mathfrak{A}_i \to \mathfrak{A}_j$ are