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THE FUNCTOR K_2 FOR THE RING OF INTEGERS OF A NUMBER FIELD

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Let O_F be the ring of integers of a number field F. In the present paper we give some results on the group K_2O_F, where K_2 is the functor of Milnor.

D. Quillen proved [7] that the sequence

$$0 \to K_2O_F \to K_2F \xrightarrow{\tau} \coprod_{v\,\mathrm{fin}} \bar{F}_v^* \to 0 \tag{1}$$

is exact, where the sum is extended over all finite places v of the number field F, and the homomorphism τ is defined by tame symbols:

$$\{a, b\} \mapsto (a, b)_v = (-1)^{v(a)v(b)} a^{v(b)} b^{-v(a)} (\mathrm{mod}\ v).$$

The group K_2O_F is finite (see H. Garland [4]). C. Moore proved [6] that the following sequence is exact

$$0 \to \mathfrak{R}_2F \to K_2F \xrightarrow{\eta} \coprod_{v\,\mathrm{non\,c.}} \mu_v \xrightarrow{d} \mu \to 0, \tag{2}$$

where the sum is extended over all non-complex places v of F, μ (resp. μ_v) is the group of roots of unity in F (resp. in the completion F_v); the homomorphism η is defined by Hilbert symbols $\{a, b\} \mapsto [a, b]_v$ (for the definition see e.g. [5], § 15), $\mathfrak{R}_2F = \mathrm{Ker}\,\eta$, and the homomorphism d is given by the formula

$$d\big((a_v)_{v\,\mathrm{non\,c.}}\big) = \prod_{v\,\mathrm{non\,c.}} a_v^{m_v/m},$$

where $m = |\mu|$, $m_v = |\mu_v|$.

It is known [1] that for any finite place v

$$[a, b]_v^{m_v/(Nv-1)} = (a, b)_v \tag{3}$$

holds, where $Nv = |\bar{F}_v|$ is the number of elements of the residue class field $\bar{F}_v$. Consequently $\operatorname{Ker}\eta \subset \operatorname{Ker}\tau$, i.e. the Hilbert kernel $\mathfrak{K}_2F$ is a subgroup of the tame kernel K_2O_F.

We shall give a description of the quotient group $K_2O_F/\mathfrak{K}_2F$ for any number field F.

First we determine places v satisfying $m_v = Nv-1$. For a finite place v let p be the prime number satisfying $v(p) > 0$; then we write $v|p$.

THEOREM 1. *For any field E satisfying $F \subset E \subset F_v$ let μ_E be the group of roots of unity in E, and let $m_E = |\mu_E|$. Then for every finite place v of F we have*

$$m_E \nmid Nv-1 \Leftrightarrow \zeta_p \in E,$$

where ζ_p is a primitive p-th root of unity and $v|p$.

Proof. $\Rightarrow$. Let ζ be a generator of μ_E, and $\eta = \zeta^{Nv-1}$. Then $\eta \equiv 1 \pmod{v}$ and $\eta \neq 1$ by assumption. Hence

$$0 = 1+\eta+\eta^2+\ldots+\eta^{m_E-1} \equiv m_E \pmod{v}.$$

Consequently $p|m_E$, i.e. $\zeta_p \in E$.

$\Leftarrow$. From $p|m_E$ and $p|Nv$ it follows that $m_E \nmid Nv-1$.

COROLLARY 1. (i) $m \nmid Nv-1 \Leftrightarrow \zeta_p \in F$,

(ii) $m_v \neq Nv-1 \Leftrightarrow m_v \nmid Nv-1 \Leftrightarrow \zeta_p \in F_v$.

Proof. Put in Theorem 1 $E = F$ and $E = F_v$, respectively. Moreover, from $Nv-1|m_v$ it follows the first part of (ii).

COROLLARY 2. *If $m_v \neq Nv-1$, then $p-1$ divides the ramification index $e_v(F/Q)$. Consequently $p = 2$ or v ramifies.*

Proof. From Corollary 1 we have $\zeta_p \in F_v$. Since $e_v(F_v/F) = 1$ and $e_v(Q(\zeta_p)/Q) = p-1$, in view of the diagram

$$\begin{array}{ccc} Q(\zeta_p) & \longrightarrow & F_v \\ \uparrow & & \uparrow \\ Q & \longrightarrow & F \end{array}$$

we obtain $p-1|e_v(F/Q)$.

COROLLARY 3. *For almost all places v the Hilbert symbol $[a, b]_v$ is equal to the tame symbol $(a, b)_v$.*

THEOREM 2. *The group $K_2O_F/\mathfrak{K}_2F$ is isomorphic to the abelian group defined by the generators g_v, where v runs through all real places of F, and such finite places that $\zeta_p \in F_v$ for $v|p$, and relations*

$$g_v^2 = 1 \quad \text{for } v \text{ real},$$

$$g_v^{m_v/(Nv-1)} = 1 \quad \text{for } v \text{ finite},$$

$$\prod_{v \text{ real}} g_v \cdot \prod_{\substack{v \text{ finite} \\ \zeta_p \in F \text{ for } v|p}} g_v^{m_v/m} = 1.$$

Remark. In the last relation in general not all generators occur, because it can happen that $\zeta_p \in F_v$ and $\zeta_p \notin F$ for $v|p$.

Proof. We define a homomorphism

$$\lambda\colon \prod_{v \text{ non c.}} \mu_v \to \prod_{v \text{ fin.}} \bar{F}_v^*,$$

$$\lambda(a_v) = \begin{cases} 1 & \text{if } v \text{ is real}, \\ a_v^{m_v/(Nv-1)} & \text{if } v \text{ is finite}, \end{cases}$$

where $a_v \in \mu_v$.

From (1) and (2) we obtain the commutative diagram with exact rows and columns:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & & & \\ & & \downarrow & & \downarrow & & & & \\ 0 & \to & \mathfrak{K}_2F & \longrightarrow & K_2F & \to & \operatorname{Im}\eta & \to & 0 \\ & & \downarrow & & \downarrow \mathrm{id} & & \downarrow \lambda & & \\ 0 & \to & K_2O_F & \to & K_2F & \to & \prod_{v \text{ fin.}} \bar{F}_v^* & \to & 0 \\ & & & & \downarrow & & \downarrow & & \\ & & & & 0 & & 0 & & \end{array}$$

By the snake lemma we conclude that the groups $K_2O_F/\mathfrak{K}_2F$ and $\operatorname{Im}\eta \cap \operatorname{Ker}\lambda$ are isomorphic.

Let ζ_v be a generator of the group μ_v for non-complex v. We may assume that $\zeta_v^{m_v/m} = \zeta_m$, where ζ_m is a fixed generator of the group μ. From the definition of λ it follows that the group $\mu_v \cap \operatorname{Ker}\lambda$ is generated by

$$\zeta_v, \quad \text{if } v \text{ is real},$$

$$\zeta_v^{Nv-1}, \quad \text{if } v \text{ is finite}.$$

Consequently

$$\operatorname{Ker}\lambda = \prod_{v \text{ real}} \mu_v \times \prod_{v \text{ fin.}} \mu_v^{Nv-1}.$$

From Corollary 2 it follows that the group $\operatorname{Ker}\lambda$ is finite. The sequence

(4) $$0 \to \operatorname{Ker} d_1 \to \operatorname{Ker}\lambda \xrightarrow{d_1} \mu$$

is exact, where d_1 is the homomorphism $d|\operatorname{Ker}\lambda$. Hence from (2) we deduce that

$$\operatorname{Ker} d_1 = \operatorname{Ker} d \cap \operatorname{Ker}\lambda = \operatorname{Im}\eta \cap \operatorname{Ker}\lambda.$$

By the Z-injectivity of Q/Z from (4) we obtain

(5) $$0 \leftarrow \operatorname{Hom}_Z(\operatorname{Ker} d_1, Q/Z) \leftarrow \operatorname{Hom}_Z(\operatorname{Ker}\lambda, Q/Z) \xleftarrow{d_1^*} \operatorname{Hom}_Z(\mu, Q/Z),$$

where $d_1^*(\varphi) = \varphi\circ d_1$ for $\varphi \in \operatorname{Hom}_Z(\mu, Q/Z)$.

Let us consider elements $g_v \in \operatorname{Hom}_Z(\operatorname{Ker}\lambda, Q/Z)$ defined by the conditions

$$g_v(\zeta_w) = \begin{cases} \frac{1}{2} & \text{for } w = v, \\ 0 & \text{for } w \neq v, \end{cases} \quad \text{if } w \text{ is real,}$$

$$g_v(\zeta_w^{Nw-1}) = \begin{cases} \dfrac{Nv-1}{m_v} & \text{for } w = v, \\ 0 & \text{for } w \neq v, \end{cases} \quad \text{if } w \text{ is finite.}$$

Almost all elements g_v are trivial, and $\operatorname{Hom}_Z(\operatorname{Ker}\lambda, Q/Z)$ is the direct sum of groups generated by elements g_v. The element g_v is of order 2, if v is real, and of order $\dfrac{m_v}{Nv-1}$, if v is finite.

The group $\operatorname{Hom}_Z(\mu, Q/Z)$ is generated by the element g satisfying $g(\zeta_m) = 1/m$. Consequently the group $\operatorname{Im} d_1^*$ is generated by the element $d_1^*(g) = g\circ d_1$. By a direct computation one can verify that

(6) $$d_1^*(g) = \prod_{v \text{ non c.}} g_v^{m_v/m}.$$

It suffices to compare the values of both sides of (6) on the generators of the group $\operatorname{Ker}\lambda$.

From (5) it follows that the group $\operatorname{Hom}_Z(\operatorname{Ker} d_1, Q/Z)$, isomorphic to $\operatorname{Ker} d_1$, is also isomorphic to $\operatorname{Hom}_Z(\operatorname{Ker}\lambda, Q/Z)/\operatorname{Im} d_1^*$. Hence it is isomorphic to the abelian group defined by the generators g_v, where v runs through non-complex places of F, and relations:

(7) $$g_v^2 = 1 \quad \text{for } v \text{ real}, \qquad g_v^{m_v/(Nv-1)} = 1 \quad \text{for } v \text{ finite},$$

(8) $$\prod_{v \text{ non c.}} g_v^{m_v/m} = 1.$$

If v is real, then $m_v = m = 2$; if v is finite and $m|Nv-1$, then $\dfrac{m_v}{Nv-1} \Big| \dfrac{m_v}{m}$. Thus by (7) we have $g_v^{m_v/m} = 1$ and (8) takes the form

$$\prod_{v \text{ real}} g_v \cdot \prod_{\substack{v \text{ finite} \\ m \nmid Nv-1}} g_v^{m_v/m} = 1.$$

In view of Corollary 1 the result follows.

Applying Theorem 2 we determine the group $K_2O_F/\Re_2F$ in the following cases: $F = Q$, $F = Q(\sqrt{d})$ and $F = Q(\zeta_p)$, where p is an odd prime number. The results are given in Table 1 (pp. 193–194). In the column denoted by "v" there are given all real places, and all finite places satisfying $\zeta_p \in F_v$ for $v|p$. If there is only one such place, we denote it by ∞ or p, if there are several such places we denote them by $\infty_1, \infty_2, \ldots$ or $p_1, p_2, \ldots$

Let $F = Q(\sqrt{d})$, where d is a squarefree integer. Since $e_v(F/Q) \leqslant 2$, it follows from Corollary 2 that $\zeta_p \in F_v$ implies $p \leqslant 3$. Evidently $\zeta_2 = -1 \in F_v$ for every v. It is easy to verify that $\zeta_3 \in F_v$ for $v|3$ if and only if $d \equiv -3 \pmod 9$. Thus in the case $d \not\equiv -3 \pmod 9$ the group $K_2O_F/\Re_2F$ has generators g_v, where v is real or $v|2$. In the case $d \equiv -3 \pmod 9$ there is one more generator g_v corresponding to the place v satisfying $v|3$.

In the case $F = Q(\zeta_p)$ the argument is similar.

The description of the group $K_2O_F/\Re_2F$ in the case $F = Q(\zeta_p)$ has been applied by St. Chaładus [2] to obtain an estimation from below of order of $K_2(ZC_p)$, where ZC_p is the group ring of the cyclic group C_p of order p. He proved that

$$|K_2(ZC_p)| \geqslant 2^r,$$

where r is the number of prime ideals in $Q(\zeta_p)$ dividing 2. For a non-cyclic elementary abelian p-group G an estimation of order of $K_2(ZG)$ has been given by R. K. Dennis, M. E. Keating and M. R. Stein [3].

For any finite abelian group A let $r_2(A) = \dim_{F_2}(F_2\otimes_Z A)$, and let A_2 be the subgroup of A consisting of elements of order $\leqslant 2$. Evidently $r_2(A) = r_2(A_2)$.

For a number field F let $\mathrm{Cl}(F)$ be its class group, and let $\mathrm{Cl}_2(F)$ be the subgroup of $\mathrm{Cl}(F)$ generated by classes containing prime ideals dividing 2. Let $j = r_2\big(\mathrm{Cl}(F)/\mathrm{Cl}_2(F)\big)$.

Denote by $S(2)$ the set of all archimedean places of F and of finite places v satisfying $v|2$. Let r_1, r_2, r be respectively the number of real places, of complex places, and of finite places dividing 2.

THEOREM 3. *For every number field F we have*

$$r_2(K_2O_F) = r_1 + r - 1 + j.$$

Proof. Let $\Gamma = \{a \in F^*\colon \{-1, a\} \in \operatorname{Ker}\tau\}$ and $\Delta = \{a \in F^*\colon \{-1, a\} = 1\}$. J. Tate proved [9] that every element of order 2 in K_2F has the form $\{-1, a\}$ for $a \in F^*$. Thus the groups Γ/Δ and $(K_2O_F)_2$ are isomorphic, and $\Gamma \supset \Delta \supset F^{*2}$. Consequently Δ/F^{*2} and Γ/F^{*2} are linear spaces over F_2.

J. Tate proved [9] that $r_2(\Delta/F^{*2}) = r_2+1$. Consequently it is sufficient to prove that $r_2(\Gamma/F^{*2}) = r_1+r_2+r+j$.

Let A_i $(i = 1, 2, \ldots, j)$ be a basis of $\big(\mathrm{Cl}(F)/\mathrm{Cl}_2(F)\big)_2$, and let $\mathfrak{A}_i \in \mathrm{Cl}(F)$ represent A_i and let an ideal $\mathfrak{a}_i$ of the ring O_F belong to the class $\mathfrak{A}_i$. Then

$$(9) \qquad \mathfrak{a}_i^2 = (a_i)\mathfrak{b}_i,$$

where $a_i \in F^*$ and $v(\mathfrak{b}_i) = 0$ for $v \notin S(2)$.

From the Dirichlet–Hasse–Chevalley theorem on units it follows that the group

$$U(2) = \{a \in F^*\colon v(a) = 0 \text{ for } v \notin S(2)\}$$

is the direct sum of the group μ and a free abelian group of the rank $t = r_1 + r_2 + r - 1$. Let $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t$ be a free basis of this group. We shall prove that the elements

$$(10) \qquad \zeta_m, \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_t, a_1, a_2, \ldots, a_j$$

form a basis of Γ/F^{*2}.

For any finite place v we have $(-1, \zeta_m)_v = 1$, $(-1, \varepsilon_k)_v = 1$ and $(-1, a_i)_v = 1$, because $v(a_i)$ is even for $v \notin S(2)$ in view of (9). Consequently elements (10) belong to Γ. We shall prove that they are linearly independent over $\boldsymbol{F}_2$.

Suppose that for some $\alpha, \alpha_1, \alpha_2, \ldots, \alpha_t, \beta_1, \beta_2, \ldots, \beta_j = 0$ or 1 and $b \in F^*$ we have

$$(11) \qquad \zeta_m^{\alpha}\varepsilon_1^{\alpha_1}\varepsilon_2^{\alpha_2}\ldots\varepsilon_t^{\alpha_t}a_1^{\beta_1}a_2^{\beta_2}\ldots a_j^{\beta_j} = b^2.$$

Then the class of the ideal $\mathfrak{a}_1^{\beta_1}\mathfrak{a}_2^{\beta_2}\ldots\mathfrak{a}_j^{\beta_j}(b)^{-1}$ belongs to $\mathrm{Cl}_2(F)$ and consequently $A_1^{\beta_1}A_2^{\beta_2}\ldots A_j^{\beta_j} = 1$. Hence $\beta_1 = \beta_2 = \ldots = \beta_j = 0$ and (11) takes the form

$$\zeta_m^{\alpha}\varepsilon_1^{\alpha_1}\varepsilon_2^{\alpha_2}\ldots\varepsilon_t^{\alpha_t} = b^2.$$

It follows that $\alpha = \alpha_1 = \alpha_2 = \ldots = \alpha_t = 0$.

Now we shall prove that the elements (10) generate Γ/F^{*2}. If $a \in \Gamma$, then $1 = (-1, a)_v \equiv (-1)^{v(a)} \pmod{v}$ for every finite place v. Thus $2 | v(a)$ for $v \notin S(2)$. Consequently $(a) = \mathfrak{a}^2\mathfrak{b}$, where the ideal $\mathfrak{b}$ satisfies $v(\mathfrak{b}) = 0$ for $v \notin S(2)$. Let $\mathfrak{A}$ be the class containing $\mathfrak{a}$, and let $A \in \mathrm{Cl}(F)/\mathrm{Cl}_2(F)$ be the element represented by $\mathfrak{A}$. Since $\mathfrak{A}^2 = [\mathfrak{b}]^{-1} \in \mathrm{Cl}_2(F)$ then A is a product of some A_i's. Let e.g. $A = A_1A_2\ldots A_h$. Then $\mathfrak{A} = \mathfrak{A}_1\mathfrak{A}_2\ldots\mathfrak{A}_h\cdot\mathfrak{B}$, where $\mathfrak{B} \in \mathrm{Cl}_2(F)$, and consequently $\mathfrak{a} = \mathfrak{a}_1\mathfrak{a}_2\ldots\mathfrak{a}_h\bar{\mathfrak{b}}(c)$ for some $c \in F^*$ and $\bar{\mathfrak{b}}$ satisfying $v(\bar{\mathfrak{b}}) = 0$ for $v \notin S(2)$.

Hence $(a) = \mathfrak{a}^2\mathfrak{b} = (a_1)(a_2)\ldots(a_h)\mathfrak{b}_1\mathfrak{b}_2\ldots\mathfrak{b}_h\bar{\mathfrak{b}}^2(c)^2$ and it follows that $a(a_1a_2\ldots a_hc^2)^{-1} \in U(2)$. Thus there exist $\gamma, \gamma_1, \gamma_2, \ldots, \gamma_t \in Z$ such that

$$a = a_1a_2\ldots a_h\zeta_m^{\gamma}\varepsilon_1^{\gamma_1}\varepsilon_2^{\gamma_2}\ldots\varepsilon_t^{\gamma_t}\cdot c^2.$$

COROLLARY 1. *For every number field F we have*

$$r_2(\mathfrak{R}_2F) \geqslant r_1 + r - 1 + j - r_2(K_2O_F/\mathfrak{R}_2F).$$

Proof. For the exact sequence $0 \to A \to B \to C \to 0$ of finite abelian groups we have evidently $r_2(B) \leqslant r_2(A) + r_2(C)$. Hence $r_2(K_2O_F) \leqslant r_2(\mathfrak{R}_2F) + r_2(K_2O_F/\mathfrak{R}_2F)$, and corollary follows.

COROLLARY 2. *If $F = Q(\sqrt{d})$, d squarefree, satisfy*

$$r_2\big(\mathrm{Cl}(F)\big) \geqslant \begin{cases} 1 & \text{if } d \equiv 5 \pmod 8, \\ 3 & \text{if } d \equiv 7 \pmod 8 \text{ and } d < 0, \\ 2 & \text{otherwise}, \end{cases}$$

then $r_2(\mathfrak{R}_2F) \geqslant 1$.

Proof. If $d \equiv 5 \pmod 8$, then 2 is inert and hence $\mathrm{Cl}_2(F) = 0$. Thus $j = r_2\big(\mathrm{Cl}(F)\big)$. From Table 1 we have $r_2(K_2O_F/\mathfrak{R}_2F) = r_1 + r - 1$, and the corollary follows from Corollary 1.

We proceed analogously in the remaining cases, due to the observation that $j \geqslant r_2\big(\mathrm{Cl}(F)\big) - 1$, if 2 splits or ramifies.

EXAMPLES. 1. (J. Tate [8]). Let $d = -35$ or more generally let $d \equiv 5 \pmod 8$ have at least two prime factors. Then the class number of the field $F = Q(\sqrt{d})$ is even and from Corollary 2 we obtain that $r_2(\mathfrak{R}_2F) \geqslant 1$.

2. Let $F = Q(\zeta_{29})$. It is known that 2 is inert in F and $\mathrm{Cl}(F) = (Z/2Z)^3$. Then $\mathrm{Cl}_2(F) = 0$ and from Corollary 1 and Table 1 we obtain $r_2(\mathfrak{R}_2F) \geqslant j = r_2\big(\mathrm{Cl}(F)\big) = 3$.

Table 1

	F	m	v	m_v	Nv	Relations	r_1+r-1	$K_2O_F/\mathfrak{R}_2F$
1.	Q	2	∞ 2	2 2	— 2	$g_\infty^2 = g_2^2 = 1$ $g_\infty \cdot g_2 = 1$	1	$Z/2Z$
2.	$Q(i)$	4	2	4	2	$g_2^4 = 1,\ g_2 = 1$	0	0
3.	$Q(\sqrt{-3})$	6	2 3	6 6	4 3	$g_2^2 = g_3^3 = 1$ $g_2g_3 = 1$	0	0
4.	$Q(\sqrt{d})$, $d<0$ $d \neq -1, -3$ $d \not\equiv -3 (9)$ $d \equiv 1\ (8)$	2	2_1 2_2	2 2	2 2	$g_{2_1}^2 = g_{2_2}^2 = 1$ $g_{2_1}g_{2_2} = 1$	1	$Z/2Z$
5.	$d \equiv 5\ (8)$	2	2	6	4	$g_2^2 = 1,\ g_2^3 = 1$	0	0
6.	$d \equiv 3\ (8)$ $d \equiv 2\ (4)$	2	2	2	2	$g_2^2 = 1,\ g_2 = 1$	0	0
7.	$d \equiv 7\ (8)$	2	2	4	2	$g_2^4 = 1,\ g_2^2 = 1$	0	$Z/2Z$

Table 1, cont.

	F	m	v	m_v	Nv	Relations	r_1+r -1	$K_2O_F/\mathfrak{K}_2F$
4′.	$Q(\sqrt{d})$, $d>0$ $d\not\equiv -3(9)$ $d\equiv 1\ (8)$	2	∞_1 ∞_2 2_1 2_2	2 2 2 2	— — 2 2	$g_{\infty_1}^2 = g_{\infty_2}^2 = 1$ $g_{2_1}^2 = g_{2_2}^2 = 1$ $g_{\infty_1}\cdot g_{\infty_2}\cdot g_{2_1}\cdot g_{2_2} = 1$	3	$(Z/2Z)^3$
5′.	$d\equiv 5\ (8)$	2	∞_1 ∞_2 2	2 2 6	— — 4	$g_{\infty_1}^2 = g_{\infty_2}^2 = 1$ $g_2^2 = 1$ $g_{\infty_1}\cdot g_{\infty_2}\cdot g_2^3 = 1$	2	$(Z/2Z)^2$
6′.	$d\equiv 3\ (8)$ $d\equiv 2\ (4)$	2	∞_1 ∞_2 2	2 2 2	— — 2	$g_{\infty_1}^2 = g_{\infty_2}^2 = 1$ $g_2^2 = 1$ $g_{\infty_1}\cdot g_{\infty_2}\cdot g_2 = 1$	2	$(Z/2Z)^2$
7′.	$d\equiv 7\ (8)$	2	∞_1 ∞_2 2	2 2 4	— — 2	$g_{\infty_1}^2 = g_{\infty_2}^2 = 1$ $g_2^4 = 1$ $g_{\infty_1}\cdot g_{\infty_2}\cdot g_2^2 = 1$	2	$(Z/2Z)\oplus$ $\oplus(Z/4Z)$
8.	$Q(\zeta_p)$ p odd prime	$2p$	2_1 2_2 ... 2_r p	$2(2^f-1)$ $2(2^f-1)$ ... $2(2^f-1)$ $p(p-1)$	2^f 2^f ... 2^f p	$g_{2_i}^2 = 1$ $g_p^p = 1$ $\prod_{i=1}^{r} g_{2_i}^{\frac{1}{p}(2^f-1)}\times$ $\times g_p^{\frac{1}{2}(p-1)} = 1$	$r-1$	$(Z/2Z)^{r-1}$

If $d\equiv -3$ (mod 9), then in the lines 4.–7. and 4′.–7′. of the table one should add the following line

			3	6	3	$g_3^3 = 1$		

and in the last column one should add the direct summand $Z/3Z$.

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