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ON THE WREATH PRODUCT OF MONOIDS

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A *monoid* is a semigroup with identity 1. A set A is called a *left act over the monoid* R or *left* R*-monoid* if $\lambda a \in A$ and $(\lambda\mu)a = \lambda(\mu a)$ and $1a = a$ for each $\lambda, \mu \in R$ and $a \in A$. A monoid $\mathfrak{R}$ is called a *wreath product of the monoid* R *with the monoid* U *by the left* R*-act* A if $\mathfrak{R}$ is a set $R \times F(A, U)$, where $F(A, U)$ is the set of all maps from A into U and the multiplication is defined by

$$(\lambda, f)(\mu, g) = (\lambda\mu, f_\mu g),$$

where $f_\mu(x) = f(\mu x)$ and $fg(x) = f(x)g(x)$ for $(\lambda, f), (\mu, g) \in \mathfrak{R}$ and $x \in A$. We shall writte $\mathfrak{R} = (R \text{ wr } U|A)$

A left R-act A is said to be *admitted* if the following conditions are valid:

(1) $|A| \geqslant 2$;

(2) If $\lambda x = x$ for each $x \in A$, then $\lambda = 1$;

(3) For each $a \in A$ there exists a unique $\lambda \in R$ such that $\lambda x = a$ for every $x \in A$ (this λ is denoted by ν_a);

(4) If $\lambda, \mu \in R$, $a, b \in A$ and $a \neq b$, then there exists $\varrho \in R$ such that $\varrho a = \lambda a$ and $\varrho b = \mu b$.

THEOREM. *If A and B are admitted left acts over monoids R and S, respectively, U and V are monoids and*

$$(R \text{ wr } U|A) \cong (S \text{ wr } V|B),$$

then $|A| = |B|$ *and* $U \cong V$.

Proof. Let $\mathfrak{R} = (R \text{ wr } U|A)$, $H_a = \{(\nu_a, f) \mid f \in F(A, U)\}$ for $a \in A$ and $H = \bigcup_{a \in A} H_a$.

LEMMA 1. $\nu_a\lambda = \nu_a$ *and* $\lambda\nu_a = \nu_{\lambda a}$ *for each* $\lambda \in R$ *and* $a \in A$.

Evident.

LEMMA 2. (a) $H_a = (\nu_a, 1)\mathfrak{R}$; (b) *If* $a \neq b$, *then* $H_a \cap H_b = \emptyset$; (c) H *is a two-sided ideal of* $\mathfrak{R}$.

In fact, (b) is evident. Moreover,

$$(\nu_a, 1)(\lambda, f) = (\nu_a\lambda, f) = (\nu_a, f)$$

and

$$(\mu, g)(\nu_a, f) = (\mu\nu_a, g_{\nu_a}f) = (\nu_{\mu a}, g_{\nu_a}f) \in H_{\mu a} \subseteq H$$

for every (λ, f), $(\mu, g) \in \mathfrak{R}$ by Lemma 1.

LEMMA 3. *If* (λ, f), $(\mu, g) \in \mathfrak{R}$, a, $b \in A$, $\lambda^2 a = \lambda a = \mu b = \mu^2 b$ *and* $f(\lambda a) = g(\mu b)$, *then*

$$(\lambda, f)\mathfrak{R} \cap (\mu, g)\mathfrak{R} \neq \emptyset.$$

In fact,

$$(\lambda, f)(\nu_{\lambda a}, 1) = (\nu_{\lambda^2 a}, f_{\nu_{\lambda a}}) \in (\lambda, f)\mathfrak{R}$$

and

$$(\mu, g)(\nu_{\mu b}, 1) = (\nu_{\mu^2 b}, g_{\nu_{\mu b}}) \in (\mu, g)\mathfrak{R}$$

by Lemma 1. But

$$f_{\nu_{\lambda a}}(x) = f(\nu_{\lambda a}x) = f(\lambda a) = g(\mu b) = g(\nu_{\mu b}x) = g_{\nu_{\mu b}}(x)$$

for every $x \in A$, i.e. $f_{\nu_{\lambda a}} = g_{\nu_{\mu b}}$.

LEMMA 4. *If* $(\nu_a, f)^2 = (\nu_a, f) \in \mathfrak{R}$, *then we have* $(\nu_a, f)\mathfrak{R} = (\nu_a, 1)\mathfrak{R}$ *if and only if* $f(a) = 1$.

Furthermore, if

$$(*) \qquad (\nu_a, f)(\nu_b, g) = (\nu_a, f)$$

for each $a, b \in A$, *then* $f(b)g(a) = 1$.

In fact, $(\nu_a, f) \in (\nu_a, 1)\mathfrak{R}$ by Lemma 2 (a). If $(\nu_a, 1) \in (\nu_a, f)\mathfrak{R}$, then $(\nu_a, f)(\xi, v) = (\nu_a, 1)$ for some $(\xi, v) \in \mathfrak{R}$, whence $f(\xi x)v(x) = 1$ for each $x \in A$. Besides we have $f_{\nu_a}f = f$. Then

$$f(a) = f(a)f(\xi a)v(a) = f_{\nu_a}f(\xi a)v(a) = 1.$$

Now, if $f(a) = 1$, then $(\nu_a, f)(\nu_a, 1) = (\nu_a, f_{\nu_a})$ by Lemma 1. But $f_{\nu_a}(x) = f(a) = 1$ for each $x \in A$, i.e. $f_{\nu_a} = 1$ and $(\nu_a, 1) \in (\nu_a, f)\mathfrak{R}$. Further, from $(*)$ it follows that $f(b)g(a) = f_{\nu_b}g(a) = f(a) = 1$.

LEMMA 5. *If* I *is a two-sided ideal of* $\mathfrak{R}$, $I = \bigcup_{\mathfrak{a} \in \Xi} \mathfrak{a}\mathfrak{R}$, $\mathfrak{a}\mathfrak{b} = \mathfrak{a}$ *for every* $\mathfrak{a}, \mathfrak{b} \in \Xi$, $|\Xi| \geqslant 2$ *and* $\mathfrak{a}\mathfrak{R} \cap \mathfrak{b}\mathfrak{R} = \emptyset$ *for* $\mathfrak{a} \neq \mathfrak{b}$, *then* $I \subseteq H$.

We prove this lemma in a few steps.

(a) If (λ, f), $(\mu, g) \in \Xi$, then $\lambda\mu = \lambda$ and $f_\mu g = f$.
Evident.

(b) If $\mathfrak{a} = (\lambda, f) \in \Xi$, $\varrho \in R$, $w \in F(A, U)$, $a \in A$, $\varrho\lambda a = \lambda a$ and $w(\lambda a) = 1$, then

$$\mathfrak{i} = (\varrho, w)\mathfrak{a} \in \mathfrak{a}\mathfrak{R}.$$

In fact, $\mathfrak{i} = (\varrho\lambda, w_\lambda f)$. By (a) we have

$$(\varrho\lambda)^2 a = (\varrho\lambda)a = \lambda a = \lambda^2 a$$

and

$$w_\lambda f(\varrho\lambda a) = w_\lambda f(\lambda a) = w(\lambda^2 a)f(\lambda a) = f(\lambda a).$$

By Lemma 3 $\mathfrak{i}\mathfrak{R} \cap \mathfrak{a}\mathfrak{R} \neq \emptyset$. But $\mathfrak{i} \in I$, i.e. $\mathfrak{i} \in \mathfrak{c}\mathfrak{R}$ for a certain $\mathfrak{c} \in \Xi$. Therefore

$$\mathfrak{c}\mathfrak{R} \cap \mathfrak{a}\mathfrak{R} \supseteq \mathfrak{i}\mathfrak{R} \cap \mathfrak{a}\mathfrak{R} \neq \emptyset,$$

whence $\mathfrak{c} = \mathfrak{a}$, i.e. $\mathfrak{i} \in \mathfrak{a}\mathfrak{R}$.

(c) If $|\Xi| \geqslant 2$ and $\mathfrak{a} = (\lambda, f) \in \Xi$, then $\lambda = \nu_c$ for a certain $c \in A$.

In fact, otherwise we have $\lambda a \neq \lambda b$ for some $a, b \in A$. Further, there exists an element $\mathfrak{b} = (\mu, g) \in \Xi$ such that $\mathfrak{a} \neq \mathfrak{b}$. Since A is admitted, there exists $\varrho \in R$ such that $\varrho(\lambda a) = \lambda a$ and $\varrho(\lambda b) = \mu(\lambda b)$. Let $w \in F(A, U)$ such that $w(\lambda a) = 1$ and $w(\lambda b) = g(\mu b)$. We take $\mathfrak{i} = (\varrho\lambda, w_\lambda f)$ and $\mathfrak{w} = (\mu, gf)$. But $\mathfrak{i} = (\varrho, w)\mathfrak{a}$, $\varrho\lambda a = \lambda a$ and $w(\lambda a) = 1$. Therefore, $\mathfrak{i} \in \mathfrak{a}\mathfrak{R}$ by (b). Further, $\mathfrak{w} = \mathfrak{b}(1, f) \in \mathfrak{b}\mathfrak{R}$. Moreover,

$$(\varrho\lambda)^2(\mu b) = (\varrho\lambda)(\varrho\lambda b) = \varrho\lambda\mu\lambda b = (\varrho\lambda)(\mu b)$$

by (a). Hence,

$$(\varrho\lambda)^2(\mu b) = (\varrho\lambda)(\mu b) = \varrho(\lambda b) = \mu(\lambda b) = \mu b = \mu^2 b$$

and

$$w_\lambda f\big((\varrho\lambda)b\big) = w_\lambda f(\mu b) = w(\lambda b)f(\mu b) = g(\mu b)f(\mu b) = gf(\mu b)$$

by (a) also. Therefore,

$$\mathfrak{a}\mathfrak{R} \cap \mathfrak{b}\mathfrak{R} \supseteq \mathfrak{i}\mathfrak{R} \cap \mathfrak{w}\mathfrak{R} \neq \emptyset$$

by Lemma 3. This is a contradiction.

LEMMA 6. *If* $\Xi = \{(\nu_a, f^a) | \ a \in A\}$ *has the properties of Lemma* 5, *then*

$$\mathfrak{z}(\Xi) = \{\mathfrak{z} | \ \mathfrak{z} \in \mathfrak{R}, \mathfrak{z}(\nu_a, f^a) = (\nu_a, f^a)\mathfrak{z} \text{ for each } a \in A\}$$
$$= \{(1, z) | \ z \in F(A, U),\ z(b) = f^b(a)z(a)f^a(b) \text{ for each } a, b \in A\}.$$

In fact, if z has the described property, then

$$(**) \qquad f^a z = z_{\nu_a} f^a$$

for each $a \in A$, since using $(*)$ we have

$$f^a z(x) = f^a(x)z(x) = f^a(x)f^x(a)z(a)f^a(x) = z(a)f^a(x) = z_{\nu_a}f^a(x)$$

for each $x \in A$. The converse implication is proved similarly. But $(**)$ is equivalent to

$$(\nu_a, f^a)(1, z) = (1, z)(\nu_a, f^a),$$

since

$$(\nu_a, f^a)(1, z) = (\nu_a, f^a z) \quad \text{and} \quad (\nu_a, z_{\nu_a} f^a) = (1, z)(\nu_a, f^a).$$

So, the right-hand side belongs to $\mathfrak{z}(\Xi)$. Now, if $(\xi, z) \in \mathfrak{z}(\Xi)$, then by Lemma 1 $\nu_{\xi a} = \xi\nu_a = \nu_a\xi = \nu_a$ for each $a \in A$. Therefore, $\xi a = a$ for each $a \in A$, whence $\xi = 1$ by condition (2). So, $\mathfrak{z}(\Xi)$ belongs to the right-hand side.

LEMMA 7. *If* $\Xi = \{(\nu_a, f^a) \mid a \in A\}$ *has the properties of Lemma* 5, *then* $\mathfrak{z}(\Xi) \cong U$.

In fact, we take a certain $a \in A$ and in view of Lemma 6 we define the map $\Psi: \mathfrak{z}(\Xi) \to U$ by $\Psi(1, z) = z(a)$. Evidently, Ψ is a homomorphism. If $(1, z') \in \mathfrak{z}(\Xi)$ and $z(a) = z'(a)$, then

$$z(x) = f^x(a) z(a) f^a(x) = f^x(a) z'(a) f^a(x) = z'(x)$$

for each $x \in A$ by Lemma 6. So, $(1, z) = (1, z')$, i.e. Ψ is an injection. If $u \in U$, then we put $z(x) = f^x(a) u f^a(x)$. By $(*)$

$$f^b(a) z(a) f^a(b) = f^b(a) f^a(a) u f^a(a) f^a(b) = f^b(a) u f^a(b) = z(b),$$

i.e. $(1, z) \in \mathfrak{z}(\Xi)$ by Lemma 6. Moreover,

$$\Psi(1, z) = z(a) = f^a(a) u f^a(a) = u$$

because $(\nu_a, f^a)\big(\nu_b, f^b(a)\big) = (\nu_a, 1)$ and $f^a(a) = 1$ by Lemma 4. So Ψ is a surjection.

Proof of the theorem. Let $\mathfrak{S} = (S \operatorname{wr} V | B)$ and let Φ be an isomorphism of $\mathfrak{R}$ onto $\mathfrak{S}$. Let

$$K_b = \{(\nu_b, f) \mid f \in F(B, V)\},$$

where $b \in B$, and $K = \bigcup_{b \in B} K_b$. From Lemmas 2 and 5 it follows that $\Phi(H) = K$. Further, $|A| = |B|$. Moreover,

$$U \cong \mathfrak{z}\{(\nu_a, 1) \mid a \in A\} \cong \mathfrak{z}\{\Phi(\nu_a, 1) \mid a \in A\} \cong V$$

by Lemmas 5 and 7.

Remark. If F is a free right R-act with the bases X and $P(X)$ is the monoid of all maps from X into X, then

$$\operatorname{End} F = \big(P(X) \operatorname{wr} R | X\big) \quad ([1], \text{ Remark } 3).$$

Then the theorem gives a generalization of Fleisher's theorem ([2], Corollary 2; cf. [3] also).

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