now blow up the approximated family: it is defined over some etale nbhd of $S[X]$ and gives rise (for a properly chosen ν) to a deformation in $D_{E_0,X_0}(S)$ which is (ν_2)-equivalent with $(\bar{X} \to S)$ (generalization of a theorem of Hironaka and Rossi [7], Theorem 2).

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ON CLASSES OF ALGEBRAIC SYSTEMS CLOSED WITH RESPECT TO QUOTIENTS*

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Algebraic systems are understood in the sense of Malcev [9], p. 32, Malcev [8]. (They are called models in Chang–Keisler [7]; cf. also Monk [10], Def. 11.1.) A quotient of an algebraic system $\mathfrak{A}$ is its quotient $\mathfrak{A}/\Theta$ taken by some congruence Θ of $\mathfrak{A}$, cf. Malcev [9], p. 45.

Here we give syntactic characterizations of classes (of systems) closed w.r.t. quotients, subsystems, and ultraproducts, and of those closed w.r.t. quotients, subsystems, and products. The latter kind is called "strong variety".

By characterizing strong varieties a problem of Malcev is also answered, cf. Malcev [8], p. 328, Problems 1 and 2.

An *algebraic system* is a sequence $\mathfrak{A} = \langle A; R_i, F_j\rangle_{i\in I, j\in J}$ where R_i's are relations on A and $\langle A; F_j\rangle_{j\in J}$ is an algebra, cf. Malcev [9], p. 32. The following definition can be found on p. 45 in the same book.

A relation $\Theta \subseteq (A\times A)$ is a *congruence* of the above system iff Θ is a congruence of $\langle A; F_j\rangle_{j\in J}$. Let Θ be a congruence.

The *quotient* $\mathfrak{A}/\Theta = \langle A/\Theta; R_i/\Theta, F_j/\Theta\rangle_{i\in I, j\in J}$ is defined by fixing that $\langle A/\Theta; F_j/\Theta\rangle_{j\in J}$ is the usual factor algebra (or quotient algebra), and for any $b_1, \ldots, b_n \in A/\Theta$ we define $\langle b_1, \ldots, b_n\rangle \in R_i/\Theta$ to hold if and only if $(\exists a_1 \in b_1) \ldots (\exists a_n \in b_n)\langle a_1, \ldots, a_n\rangle \in R_i$.

The quotients of $\mathfrak{A}$ are also called *strong homorphic images* of $\mathfrak{A}$ by Malcev [9], p. 45, [8], p. 315, 328; Chang–Keisler [7]. (Strong homomorphic images are the same as regular quotients in terms of category theory.)

A system $\mathfrak{B} = \langle B; R'_i, F'_j\rangle_{i\in I, j\in J}$ is a *strong subsystem* of the above-mentioned system iff $\langle B; F'_j\rangle_{j\in J}$ is a subalgebra of $\langle A; F_j\rangle_{j\in J}$ and R'_i is the restriction of the relation R_i to B, for every $i \in I$. (I.e. $R'_i = R_i \cap {}^nB$ for some natural number n.) Cf. Malcev [9], p. 37.

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The language or signature of the above algebraic systems consists of relation symbols P_i $(i \in I)$ and function symbols G_j $(j \in J)$. Cf. Malcev [9], p. 210.

Proposition 1. *A class of algebraic systems is closed w.r.t. strong homomorphic images (quotients), strong subsystems, and ultraproducts iff it is axiomatizable by formulas of the shape*

$$[P_{i_1}(x_{11}, \ldots, x_{1m}) \wedge \ldots \wedge P_{i_n}(x_{n1}, \ldots, x_{nr})] \to [\pi_1 \vee \ldots \vee \pi_k],$$

where the variable symbols on the left of "→" are all distinct, and $\pi_1, \ldots, \pi_k$ *are arbitrary atomic formulas (i.e. prime formulas).*

Proof. First we note that an atomic formula π_i is of the form $P(\tau_1, \ldots, \tau_n)$, where P is either equality or some other relation symbol (P_i) and $\tau_1, \ldots, \tau_n$ are terms.

In Andréka–Németi [2] (cf. also Andréka–Németi [3], Németi–Sain [11]) it was proved that a subcategory (of any category) is closed w.r.t. regular quotients, strong subobjects, and ultraproducts iff it consists of the class of all objects injective w.r.t. a class Ax of finite epi-cones such that all arrows in Ax have regular-projective domains plus s. small domains and codomains. (An object a is *s. small* or "*ω-presentable*" iff Hom$(a, -)$ preserves direct limits.) I.e., if $(a \overset{f_i}{\twoheadrightarrow} b_i)_{1 \leqslant i \leqslant n}$ is in Ax, then a is regular-projective and $a, b_1, \ldots, b_n$ are s. small.

In the category of algebraic systems (of a fixed similarity type) a system $\mathfrak{A} = \langle A; R_i, F_j \rangle_{i \in I, j \in J}$ is *s. small* iff $\langle A; F_j \rangle_{j \in J}$ is finitely presented and all the relations together relate only finitely many tuples (i.e., $\bigcup_{i \in I} R_i$ is finite). This is easy to check, cf. Banaschewski–Herrlich [6] or Németi–Sain [11].

An algebraic system $\langle A; R_i, F_j \rangle_{i \in I, j \in J}$ is *regular-projective* iff $\langle A; F_j \rangle_{j \in J}$ is absolutely freely generated by some $X \subseteq A$ and for every $i \neq r \in I$; $\langle a_1, \ldots, a_n \rangle \in R_i$, $\langle b_1, \ldots, b_z \rangle \in R_r$ the following holds:

$$a_1, \ldots, a_n, b_1, \ldots, b_z \in X \quad \text{and} \quad \operatorname{card}\{a_1, \ldots, a_n, b_1, \ldots, b_z\} = n + z.$$

This was proved in Andreka–Németi [1], but it is also easy to check, cf. e.g. Németi–Sain [11].

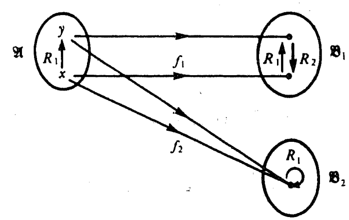
As it was pointed out in Banaschewski–Herrlich [6] and in Andréka–Németi [1], the algebraic system $\mathfrak{C}$ is injective w.r.t. an epimorphism $\mathfrak{A} \overset{f}{\twoheadrightarrow} \mathfrak{B}$ iff $\mathfrak{C} \models [D^+(\mathfrak{A}) \to D^+(\mathfrak{B})]$ where $D^+(\mathfrak{A})$ denotes the positive diagram of $\mathfrak{A}$ (cf. Chang–Keisler [7]). If $\mathfrak{B}$ is s. small, then $D^+(\mathfrak{B})$ is a finitary formula of the shape $\pi_1 \wedge \ldots \wedge \pi_n$ where π_i are prime formulas.

Similarly, $\mathfrak{C}$ is injective w.r.t. a cone $(\mathfrak{A} \overset{f_i}{\twoheadrightarrow} \mathfrak{B}_i)_{1 \leqslant i \leqslant n}$ iff

$$\mathfrak{C} \models D^+(\mathfrak{A}) \to [D^+(\mathfrak{B}_1) \vee \ldots \vee D^+(\mathfrak{B}_n)],$$

cf. Andréka–Németi [2], [4], [5].

To illustrate the way (the injectivity of) cones correspond to (the validity of) formulas, let $J = \emptyset$ and consider the cone



It corresponds to the formula

$$P_1(x, y) \to [P_2(y, x) \vee x = y],$$

since any model $\mathfrak{C}$ is injective w.r.t. the above cone $\overset{f_1}{\nearrow}\,\overset{}{\underset{f_2}{\searrow}}$ iff the above formula is valid in $\mathfrak{C}$. (Cf. Andréka–Németi [5].)

By the observations made earlier, $\mathfrak{A}$ is *regular-projective* iff its positive diagram $D^+(\mathfrak{A})$ is of the form

$$P_{i_1}(x_{11}, \ldots, x_{1m}) \wedge \ldots \wedge P_{i_n}(x_{n1}, \ldots, x_{nr}) \wedge \ldots$$

where all the variable symbols are distinct.

If, in addition, $\mathfrak{A}, \mathfrak{B}_1, \ldots, \mathfrak{B}_n$ are s. small, then the formula $D^+(\mathfrak{A}) \to [D^+(\mathfrak{B}_1) \vee \ldots \vee D^+(\mathfrak{B}_n)]$ is of the form

$$[P_{i_1}(x_{11}, \ldots) \wedge \ldots \wedge P_{i_m}(x_{m1}, \ldots)] \to [(\pi_{11} \wedge \ldots \wedge \pi_{1k}) \vee \ldots \vee (\pi_{n1} \wedge \ldots \wedge \pi_{nh})].$$

But, by propositional logic, this is equivalent to a set of formulas of the shape

$$[P_{i_1}(x_{11}, \ldots) \wedge \ldots \wedge P_{i_m}(x_{m1}, \ldots)] \to [\pi_1 \vee \ldots \vee \pi_l].$$

We have seen that the quoted kind of cones correspond to formulas of the above shape. Therefore the quoted axiomatizability theorem implies the present one. ∎

By a *strongly primitive class* we understand a class of algebraic systems closed w.r.t. strong homomorphic images, strong subsystems, and direct products (HSP).

Corollary. (i) *If there are only finitely many relation symbols, then the strongly primitive classes are exactly the ones axiomatizable by formulas* *of the form*

$$[P_{i_1}(x_{11}, \ldots) \wedge \ldots \wedge P_{i_m}(x_{m1}, \ldots)] \to \pi,$$

where all the variables x_{ij} are distinct and π is an atomic formula.

(ii) *In the case of arbitrarily many relation symbols the above characterization gives exactly those classes which are strongly primitive and in addition closed w.r.t. ultraproducts.*

(iii) *The strongly primitive classes are exactly those which are axiomatizable by formulas*

$$[\bigwedge_{\varrho<a} P_{i_\varrho}(x_{\varrho 1}, \ldots)] \to \pi,$$

where a is smaller than the least regular strict upper bound of the number of relation symbols (and π, x_{ij} are as in (i)).

The proof is straightforward on the basis of that of Proposition 1 and Andréka–Németi [2], where it is proved that if the class is closed w.r.t. products then it is enough to consider one-member cones.

The algebraic system $\langle B; R_i', F_j'\rangle_{i\in I, j\in J}$ is a *weak subsystem* of $\langle A; R_i, F_j\rangle_{i\in I, j\in J}$ iff $\langle B; F_j'\rangle_{j\in J}$ is a subalgebra of $\langle A; F_j\rangle_{j\in J}$ and in addition $R_i' \subseteq R_i$ for every $i \in I$; i.e., $\langle b_1, \ldots, b_n\rangle \in R_i'$ implies $\langle b_1, \ldots, b_n\rangle \in R_i$.

PROPOSITION 2. (i) *A class of algebraic systems is closed w.r.t. strong homomorphic images, weak subsystems, and ultraproducts iff it can be axiomatized by formulas of the shape*

$$[P_{i_1}(x_{11}, \ldots) \wedge \ldots \wedge P_{i_n}(x_{n1}, \ldots)] \to [e_1 \vee \ldots \vee e_m].$$

where $e_1, \ldots, e_m$ are equations (*and all the variable symbols x_{ij} on the left are distinct.*)

(ii) *A class of algebraic systems is closed w.r.t. strong homomorphic images, weak subsystems, and reduced products iff it can be axiomatized by formulas of the shape*

$$[P_{i_1}(x_{11}, \ldots) \wedge \ldots \wedge P_{i_n}(x_{n1}, \ldots)] \to e,$$

where e is an equation (*and the x_{ij}'s are distinct*).

Sketch of the proof. The proof applies the same theorem from Andréka–Németi [2] as the proof of Proposition 1 and proceeds in exactly the same way. The only difference is that here we substitute "weak subsystems" in place of the variable S in the quoted theorem. Since the factorization pair of "weak subsystems" is "regular epimorphisms", the cones will consist of regular (i.e. strong) epimorphisms, and therefore on the right side only equations will appear. To save space we do not go into details. ■

Propositions 1 and 2 answer Problems 1 and 2 in Malcev [8], p. 328.

Remark. By a similar use of injective subcategories as above, it can be shown that a $\forall\exists$-axiomatizable class is closed w.r.t. strong homomorphic images and products iff it is axiomatizable by formulas of the shape

$$[P_{i_1}(x_{11}, \ldots) \wedge \ldots \wedge P_{i_n}(x_{n1}, \ldots)] \to \exists \bar{y}(\pi_1 \wedge \ldots \wedge \pi_k),$$

where $\bar{y}$ is an arbitrary sequence of variables, π_i are atomic formulas and the left side is as in Proposition 1. (If the class is closed w.r.t. quotients but not products, then on the right we allow

$$\ldots \to [\exists \bar{y}(\pi_1 \wedge \ldots \wedge \pi_k) \vee \ldots \vee \exists \bar{y}(\pi_z \wedge \ldots \wedge \pi_r)].)$$

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