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THE CONCEPT OF (ν)-EQUIVALENCE IN ALGEBRAIC DEFORMATION THEORY

M. ROCZEN

Faculty of Mathematics, The Humboldt University, Berlin, G.D.R.

The theory of local moduli of algebraic schemes in the etale topology seems to be understood only in special cases like the following one: Take any germ of an algebraic scheme (X_0, x_0) with an isolated singularity at x_0 and identify two such pairs if they coincide in a strict etale neighbourhood of the marked points; so, by Elkik [4] there is an algebraic semiuniversal deformation of the equivalence class of (X_0, x_0).

Here we shall study deformations of objects related to the above-mentioned ones: Take the desingularization of an isolated singularity and consider the pair (X_0, E_0), where E_0 is the exceptional divisor of the nonsingular scheme X_0. Identify such pairs in the same way as above, i.e. if they coincide in strict etale nbhds of E_0. What is the deformation theory of this functor?

First, one has to remark that the connection with the deformation functor of the blowing down is not a trivial one; in general, it is not true that the fibres of families containing X_0 are contractible again. We shall see, for example, that if $\dim X_0 \geqslant 3$, E_0 is a normal intersection of non-singular components and the base field of complex numbers, then the divisor E_0 always extends uniquely to a relative divisor of any family containing X_0. We get something like an "exceptional divisor of the family" (the notation justified later) giving rise to infinitesimal "(ν)-nbhds" ($\nu \in \mathbf{N}$) in the family, and the induced map has the special fibre $X_0^{(\nu)} = \nu$th infinitesimal nbhd of E_0 in X_0. This map of deformation functors was first considered by Lieberman and Rossi [9] for analytic spaces, allowing a non-singular basis S. They showed by Grauert's method of normal projection that the map of deformation functors is injective for $\nu \gg 0$, and left as an open question whether it is surjective also. As we shall see here, the answer is positive in the algebraic case; the key point of this will be a more general kind of (ν)-equivalence, allowing also singular S.

This comes from a classical theorem on the smoothness of the Hilbert functor of the family which exists actually as an algebraic space by Artin, and we make essential use of Artin's theory of contractions and dilatations [1]. We shall show that: The deformation functor considered admits an algebraic semiuniversal deformation. One can get it by lifting the semiuniversal deformation of the "truncated" deformation functor.

This paper is written in an expository form and tries to give the main results and the ideas of their proofs. For a more detailed discussion we refer the reader to [12].

Throughout, we fix an algebraically closed ground field k, a smooth algebraic k-scheme X_0 and a proper divisor E_0 of X_0 with an ample conormal bundle $N_{E_0,X_0} = (J_0/J_0^2)^{-1}$ (J_0 = ideal sheaf of E_0) such that E_0 is a reduced normal intersection of nonsingular components, $\dim E_0 \geqslant 1$ and

$$H^1(E_0, N_{E_0,X_0}) = 0$$

(by Kodaira's vanishing theorem the last conditions is automatic for $\operatorname{char} k = 0$ and $\dim E_0 > 1$ [9]).

1

Let S be a scheme, and X and X' two S-schemes with fixed closed subschemes E, resp. E'. A (ν)-equivalence $(X, E) \sim (X', E')$ is an isomorphism of the νth infinitesimal nbhds of the subschemes (= closed subschemes, defined by the νth powers of the ideals J, resp. J' of E, resp. E') over S. This notion was originally used to study complex-analytic nbhds of exceptional divisors by their infinitesimal nbhds (compare [11], [7]).

Let $S \in C$ be the category of spectra of local k-algebras, which are Henselizations of k-algebras of finite type at a closed point; suppose that $X \to S$, $X' \to S$ are flat, separated and of finite type, having the special fibre X_0 at the closed point of S.

THEOREM 1.1. *The imbedding $E_0 \hookrightarrow X_0$ extends uniquely to a proper relative divisor $E \hookrightarrow X$ of X over S.*

This follows from the cohomological condition for the normal bundle of E_0, proving the smoothness of the algebraic space $\mathrm{Hilb}_{X/S}$ (which represents the Hilbert functor of $X \to S$) at the closed point E_0, and is nothing else than a version of a theorem of Kodaira, Severi, Spencer (compare [10], Lect. 23).

The main trouble with the notion of (ν)-equivalence is the following: If (X, E) and (X', E') are given and X, X' are nice (for example normal), each (ν)-equivalence induces a compatible system of (μ)-equivalences, $1 \leqslant \mu \leqslant \nu$. But in deformation theory we should allow also 0-divisors in the basis, and easy examples show that the above assertion is false without restrictions on the cohomology of N_{E_0,X_0}. It remains valid in our case because of

THEOREM 1.2. *Let $Y \to S$ be flat, separated and of finite type with special fibre $X_0^{(\nu)}$, $\nu \geqslant 2$. There is a uniquely determined closed subscheme $E \hookrightarrow Y$, flat and proper over S and inducing at the closed point $0 \in S$ the imbedding $E_0 \hookrightarrow X_0$. Moreover, the ideal sheaf of E in Y is locally generated by a single element.*

Idea of proof. Restricting ourselves to points of $\mathrm{Hilb}_{Y/S}$ whose ideal sheaves are locally generated by one element ("quasidivisors"), we get an open subspace of $\mathrm{Hilb}_{Y/S}$. We show its smoothness at the closed point E_0 in the same way as for the theorem above.

The existence assertion of the theorem we shall use later.

2

Take another algebraic k-scheme X_0' with a closed immersion $E_0' \hookrightarrow X_0'$. The two couples (X_0, E_0) and (X_0', E_0') we call "etale equivalent" if they coincide in strict etale nbhds of E_0, resp. E_0', i.e. if we have a commutative diagramm

$$\begin{array}{ccccc} X_0 & \leftarrow & X_0'' & \to & X_0' \\ \uparrow & & \uparrow & & \uparrow \\ E_0 & \leftarrow & E_0'' & \to & E_0' \end{array}$$

with both squares Cartesian, $X_0'' \to X_0$ and $X_0'' \to X_0'$ etale, $E_0'' \to E_0$ and $E_0'' \to E_0'$ isomorphisms; in a similar way we define the etale equivalence of (X, E) with a pair (X', E').

Now we define our deformation functors over C, resp. $\bar{C}$ (= category of spectra of complete local rings, formally of finite type over k):

$$D_{X_0}^{E_0}(S) = \left\{\begin{array}{l} X \to S, \text{ which are flat, separated, of} \\ \text{finite type together with a fixed etale equiv-} \\ \text{alence of the special fibre with } (X_0, E_0) \end{array}\right\} /\mathrm{mod} \sim$$

(where "$\sim$" denotes etale equivalence in strict etale nbhds of $E_0 \hookrightarrow X$ over S).

$$D_{E_0,X_0}(S) = \left\{\begin{array}{l} (X \to S, E \hookrightarrow X),\ X \to S \text{ as above, } E \hookrightarrow X \\ \text{proper relative divisor of } X \to S, \text{ to-} \\ \text{gether with a fixed etale equivalence of} \\ \text{the induced imbedding of the special} \\ \text{fibre with } (X_0, E_0) \end{array}\right\} /\mathrm{mod} \sim$$

(where "$\sim$" denotes etale equivalence in strict etale nbhds of E or, equivalently, of E_0).

Furthermore, of technical interest is the functor

$$\bar{D}_{E_0,X_0}(S) = \left\{\begin{array}{l}(\bar{X} \to S, \quad E \hookrightarrow \bar{X}), \quad \bar{X} \text{ a separated} \\ \text{Noetherian formal } S\text{-scheme, flat, and} \\ E \text{ a defining subscheme, flat and proper} \\ \text{over } S\text{, together with a fixed isomorp-} \\ \text{hism of the special fibre with the comple-} \\ \text{tion } X_{0\hat{E}_0} \text{ of } X_0 \text{ along } E_0 \end{array}\right\} \Big/ \begin{array}{l}\text{mod} \\ \text{isomorph.}\end{array}$$

Finally, we consider the usual deformation functors of the proper k-schemes $X_0^{(\nu)}$, $\nu \in \boldsymbol{N}$. From Theorem 1.1, we get

THEOREM 2.1. *The functor morphism*

$$\text{“forget } E\text{”}: D_{E_0,X_0} \to D_{X_0}^{E_0}$$

is an isomorphism over the category C.

Obviously, we have a commutative diagramm of natural maps

$$\begin{array}{ccc} D_{E_0,X_0} & \xrightarrow{\gamma_\nu} & D_{X_0^{(\nu)}} \\ & {\scriptstyle\alpha}\searrow \quad \nearrow{\scriptstyle\beta_\nu} & \\ & \bar{D}_{E_0,X_0} & \end{array}$$

and our original intention to compare the “Correct” deformation functor of X_0 in nbhds of E_0 with that of $X_0^{(\nu)}$ reduces by 2.1 to the study of the properties of γ_ν.

THEOREM 2.2. *For*

$$\nu \geqslant \nu_1(X_0, E_0) = \text{minimum } \{\mu \in \boldsymbol{N},\ \mu \geqslant 2,\ H^1(E_0, \Theta_{X_0} \otimes (J_0/J_0^2)^{\mu'}) = 0 \quad \textit{for all } \mu' \geqslant \mu\}$$

the map γ_ν is injective over C.

Idea of proof. First one shows that β_ν is injective on $\operatorname{im} \alpha$, using the local methods of [8], 7.4.3.2, and patching the local solutions (the obstruction lies in $H^1(\Theta_{X/S} \otimes (J/J^2)^\nu) = 0$). Now one can see (by replacing the divisor $E \hookrightarrow X$ by a multiple) that $E \hookrightarrow X$ is contractible into the base S (as an algebraic space, compare Artin [1]). By a consequence of Artin's approximation theorem the local ring of the contraction in $0 \in S$ is uniquely determined by its completion, whence follows Theorem 2.2.

3

Now we want to study the surjectivity of γ_ν. For this purpose we first show the surjectivity of β_ν for artinian $S \in C$. We start with the local solution: Define

$$\nu_2 = \nu_2(X_0, E_0) = \text{minimum } \{\nu,\ \nu \geqslant 2\nu_1(X_0, E_0),\ \nu \text{ power of 2 such that } H^2(\Theta_{X_0} \otimes (J_0/J_0^2)^{\nu'}) = 0 \text{ for } \nu' \geqslant \nu\}.$$

Choose any $(Y \to S) \in D_{X_0^{(\nu)}}(S)$, $\nu \geqslant \nu_2$. One shows that the induced deformation in $D_{X_0^{(\nu/2)}}(S)$ is locally (!) given by an imbedding in $X_{0\hat{E}_0} \times \times S$ by the $(\nu/2)$-th power of a suitable ideal inducing J_0. This follows from a global (!) statement similar to Theorem 1.2, and the obstruction to lifting the local solution to deformations in $D_{X_0^{(\nu')}}$, $\nu' = \nu/2 + i$ ($i = 1, 2, 3, \ldots$) vanishes because of $H^2(\Theta_{X_0} \otimes (J_0/J_0^2)^{\nu'}) = 0$. By Grothendieck's “théorème de existence” ([6], III, 5.4.5) we get

THEOREM 3.1. *For $s \in \bar{C}$, $\nu \geqslant \nu_2(X_0, E_0)$ the map*

$$\beta_\nu(S): \bar{D}_{E_0,X_0}(S) \to D_{X_0^{(\nu)}}(S)$$

is bijective.

By Schlessinger [13] the functor $D_{X_0^{(\nu)}}$ has a formal semiuniversal deformation; this is effective by Grothendieck's theorem. If we show

THEOREM 3.2. *The map*

$$\alpha(S): D_{E_0,X_0}(S) \to \bar{D}_{E_0,X_0}(S)$$

is surjective for $S \in \bar{C}$;

then follows the existence and effectivity of the formal semiuniversal deformation of D_{E_0,X_0}, and from the injectivity of γ_ν over C we deduce

COROLLARY 3.3. *D_{E_0,X_0} has an algebraic semiuniversal deformation, i.e. there is a $T \in C$, $(Z \to T) \in D_{E_0,X_0}(T)$ such that for any $S \in C$ and $(X \to S) \in D_{E_0,X_0}(S)$ there is a morphism $S \to T$ in C, inducing $(X \to S)$ from $(Z \to T)$, which is uniquely determined on the tangent spaces.*

COROLLARY 3.4. *For $\nu \geqslant \nu_2(X_0, E_0)$ the functor morphism*

$$\gamma_\nu: D_{E_0,X_0} \to D_{X_0^{(\nu)}}$$

is an isomorphism over C.

Idea of proof for 3.2. If $(\bar{X} \to S, E) \in \bar{D}_{E_0,X_0}(S)$ is any family, $S \in \bar{C}$, there is a formal contraction along E over S, finite over some $S[[X]]$ ($X = (X_1, \ldots, X_t)$ indeterminates) and a relative complete intersection outside $V(X)$. Approximate the contraction for $\nu \gg 0$ by a theorem of Elkik ([4], 2, Théorème 4) over the algebraic power series ring $S\langle X\rangle$ over S. One can see that $\bar{X}$ is essentially a blowing up of its contraction;

now blow up the approximated family: it is defined over some etale nbhd of $S[X]$ and gives rise (for a properly chosen ν) to a deformation in $D_{E_0,X_0}(S)$ which is (ν_2)-equivalent with $(\bar{X} \to S)$ (generalization of a theorem of Hironaka and Rossi [7], Theorem 2).

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ON CLASSES OF ALGEBRAIC SYSTEMS CLOSED WITH RESPECT TO QUOTIENTS*

I. SAIN

Institute for Co-ordination of Computer Techniques, Budapest, Hungary

Algebraic systems are understood in the sense of Malcev [9], p. 32, Malcev [8]. (They are called models in Chang–Keisler [7]; cf. also Monk [10], Def. 11.1.) A quotient of an algebraic system $\mathfrak{A}$ is its quotient $\mathfrak{A}/\Theta$ taken by some congruence Θ of $\mathfrak{A}$, cf. Malcev [9], p. 45.

Here we give syntactic characterizations of classes (of systems) closed w.r.t. quotients, subsystems, and ultraproducts, and of those closed w.r.t. quotients, subsystems, and products. The latter kind is called "strong variety".

By characterizing strong varieties a problem of Malcev is also answered, cf. Malcev [8], p. 328, Problems 1 and 2.

An *algebraic system* is a sequence $\mathfrak{A} = \langle A; R_i, F_j\rangle_{i\in I, j\in J}$ where R_i's are relations on A and $\langle A; F_j\rangle_{j\in J}$ is an algebra, cf. Malcev [9], p. 32. The following definition can be found on p. 45 in the same book.

A relation $\Theta \subseteq (A\times A)$ is a *congruence* of the above system iff Θ is a congruence of $\langle A; F_j\rangle_{j\in J}$. Let Θ be a congruence.

The *quotient* $\mathfrak{A}/\Theta = \langle A/\Theta; R_i/\Theta, F_j/\Theta\rangle_{i\in I, j\in J}$ is defined by fixing that $\langle A/\Theta; F_j/\Theta\rangle_{j\in J}$ is the usual factor algebra (or quotient algebra), and for any $b_1, \ldots, b_n \in A/\Theta$ we define $\langle b_1, \ldots, b_n\rangle \in R_i/\Theta$ to hold if and only if $(\exists a_1 \in b_1) \ldots (\exists a_n \in b_n)\langle a_1, \ldots, a_n\rangle \in R_i$.

The quotients of $\mathfrak{A}$ are also called *strong homorphic images* of $\mathfrak{A}$ by Malcev [9], p. 45, [8], p. 315, 328; Chang–Keisler [7]. (Strong homomorphic images are the same as regular quotients in terms of category theory.)

A system $\mathfrak{B} = \langle B; R_i', F_j'\rangle_{i\in I, j\in J}$ is a *strong subsystem* of the above-mentioned system iff $\langle B; F_j'\rangle_{j\in J}$ is a subalgebra of $\langle A; F_j\rangle_{j\in J}$ and R_i' is the restriction of the relation R_i to B, for every $i \in I$. (I.e. $R_i' = R_i \cap {}^nB$ for some natural number n.) Cf. Malcev [9], p. 37.

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