$y \in \{x \wedge b\}^\delta(a, b)$ and assume that y does not belong to the set $X^\delta(a, b)$. Then $y \wedge x = u > a$, $u \leqslant x \wedge b$, and hence $u \wedge y = 0$, which is a contradiction. Thus (1) is valid.

By using Lemma 9 and the same methods as in the investigation concerning the strong projectability (with the distinction that we always assume card $X = 1$), we can verify that the following statements are valid:

THEOREM 3. *The $\mathscr{K}_4$-kernels do exist.*

PROPOSITION 2. *$\mathscr{K}_4$ is a radical class.*

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TWO CLOSURE OPERATORS WHICH PRESERVE m-COMPACTICITY*

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In this paper we shall investigate some properties of closure operators studied in [3] and [4]. The investigated problems are exactly formulated in Section 1.4.

1. Introductory remarks

1.1. Throughout this paper L will denote a given complete lattice with an ordering denoted by $\leqslant$. Further, m and n will denote infinite cardinals.

1.2. A subset X of L is called *m-directed in L* if for every $Y \subseteq X$, $|Y| < m$, there exists $x \in X$ such that for every $y \in Y$ we have $y \leqslant x$. (See [4], Definition 5.) A closure operator u (abbreviation: CO) on L is called *m-algebraic* (abbreviation: m-ACO) if for every non-empty m-directed subset Y of $u(L)$ there is $\bigvee_L Y = \bigvee_{u(L)} Y$. (See [3], Definition 1.3.) An element $c \in L$ is called *m-compact in L* if for every $X \subseteq L$ such that $c \leqslant \bigvee_L X$, there exists $Y \subseteq X$ with $|Y| < m$ and $c \leqslant \bigvee_L Y$. A lattice L is called *m-algebraic* if every element x of L can be written as the join of some set of m-compact elements in L. (See [2], p. 32.) The following assertion is proved in [3]:

(1) *Let m be regular and let u be an m-ACO on L. If c is n-compact in L for some infinite cardinal $m \leqslant n$, then $u(c)$ is n-compact in $u(L)$. If L is n-algebraic, then also $u(L)$ is n-algebraic.* (See [3], Theorem 2.1. For irregular cardinals m is the guess of the compacticity of $u(c)$ more complicated, as shows the same Theorem 2.1 of [3].)

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1.3. In paper [4], the following class of closure operators is defined (we shall call them *m-Tulipani's closure operators*, also m-TCO): a CO u on L is called m-TCO if for every m-compact element $c \in L$ and for every $x \in L$ such that $c \leqslant u(x)$, there exists an m-compact element $d \in L$ such that $c \leqslant u(d)$ and $d \leqslant x$. In [4] are proved the following assertions:

(2) *Let u be an m-TCO in L and let c be m-compact in L. Then $u(c)$ is m-compact in $u(L)$. If L is an m-algebraic lattice, then also $u(L)$ is m-algebraic.* (See [4], Theorem 1. The author of paper [4] supposes that m is regular throughout his paper, but the regularity of m does not intervene in the proof of (2).)

(3) *Let L be an m-algebraic lattice for a regular cardinal m and let u be a CO on L. Then u is m-TCO iff it is m-ACO.* (See [4], Theorem 2.)

1.4. Comparing results (1)–(3), it is natural to formulate the following problems:

(a) *Let m be regular and let u be an m-TCO on L. If c is an n-compact element in L for some $m \leqslant n$, does it follows from here the n-compacticity of $u(c)$ in $u(L)$?* (A negative solution of this problem is given in Section 2 of the present paper.)

(b) *Characterize those complete lattices L which satisfy one of the following conditions:*

(b.1) *Let u be a CO on L. Then u is an m-ACO implies u is an m-TCO;*

(b.2) *Let u be a CO on L. Then u is an m-ACO on L iff u is an m-TCO;*

(b.3) *Let u be a CO on L. Then u is m-TCO implies that u is an m-ACO.*

We succeded to found the characterization of L for regular cardinal m and for (b.1) (see Section 3) and (b.2) (see Section 4). We did not this one for (b.3). In Section 5.3 we have formulated some other open problems.

1.5. Obviously, the following lemma holds (see also [3], Lemma 1.6). *Let u be a closure operator on L. Then for every $X \subseteq L$ there is*

$$V_{u(L)} u(X) = u(V_L X).$$

2. Concerning problem (a)

Let $\alpha \geqslant 1$ be an ordinal number such that $\aleph_\alpha$ is regular and let ω_α denote the smallest ordinal of the cardinality $\aleph_\alpha$. Let $[-2, -1]$ be the interval in the set of all real numbers ordered as usually and let $a \notin \omega_\alpha \cup [-2, -1]$; we define

$$L = \{0\} \oplus \big(((\omega_\alpha - \{0\}) \oplus [-2, -1)) + \{a\}\big) \oplus \{-1\},$$

where $\oplus$ denotes the ordinal sum and $+$ the cardinal one and where all singletons are considered as trivially ordered sets; see also Fig. 1.

For $x \in [-2, -1]$ define $u(x) = -1$, for $x \in L - [-2, -1]$ define $u(x) = x$. It is obvious that

L is a complete lattice;

u is a CO on L;

$x \in L$ is $\aleph_0$-compact in L iff $x \in \omega_\alpha$ and either $x = 0$ or x covers some $y \in \omega_\alpha$ (i.e. non $(\exists z \in L)(y < z < x)$);

a is $\aleph_1$-compact in L.

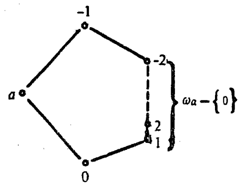

Fig. 1

From the characterization of $\aleph_0$-compact elements in L and from the definition of u it follows that u is $\aleph_0$-TCO. There is

$$u(L) = \{0\} \oplus \big((\omega_\alpha - \{0\}) + \{a\}\big) \oplus \{-1\};$$

since $\aleph_\alpha$ is regular, $a = u(a)$ must be $\aleph_{\alpha+1}$-compact in $u(L)$, but it is not k-compact for any cardinal $k \leqslant \aleph_\alpha$. We have supposed $\alpha \geqslant 1$; hence $\aleph_1 < \aleph_{\alpha+1}$ which gives a negative solution of problem (a).

3. Characterization of the lattices satisfying condition (b.1)

3.1. *Notation.* Denote by C the set of all m-compact elements in L. For $x \in L$ put

$$C(x) = \{y \in C;\ y \leqslant x\}, \quad (x] = \{y \in L;\ y \leqslant x\} \quad \text{and} \quad [x) = \{y \in L;\ x \leqslant y\}.$$

Denote by a the join of C in L and by 1 the greatest element of L.

3.2. Lemma. *Let L satisfy* (b.1). *If $V_L C(x) < x$, then $C(x) = C$, $a < x$, and the lattice $(a]$ is m-algebraic.*

Proof. The set $A = (V_L C(x)] \cup \{1\}$ is an m-algebraic closure system in L, thus the corresponding CO u is m-ACO. There is $u(z) = z$ for every $z \in A$ and $u(z) = 1$ for every $z \in L - A$. Especially $u(x) = 1$. Suppose that there exists $c \in C - C(x)$. Then $c \leqslant 1 = u(x)$. However, for every $d \in C(x)$, there is $u(d) = d$. Hence $c \nleqslant d = u(d)$ for any $d \in C(x)$, because $c \nleqslant x$

and $d \leqslant \bigvee_L C(x) < x$. This shows that the assumption $C - C(x) \neq \emptyset$ implies that u is not m-TCO. But u is m-ACO, i.e., L does not satisfy (b.1), a contradiction! Hence, by assumptions of the lemma, we have $C = C(x)$.

Take any $y \in (a]$. If $\bigvee_L C(y) < y$, then $C(y) = C$ and thus $y \notin (\bigvee_L C(y)] = (a]$. Therefore it must be $\bigvee_L C(y) = y$, and every element of $C(y)$ is m-compact in L (then it is also m-compact in $(a]$, of course).

3.3. Lemma. *Let L satisfy* (b.1). *Then the following condition holds:*

(4) *There is $L = (a] \cup [a)$, $(a]$ is an m-algebraic lattice and $[a)$ contains exactly one m-compact element (which is of course equal to a).*

Proof. Take any $x \in L - (a]$. Then by Lemma 3.2 we have $C = C(x)$ and therefore we have also $a = \bigvee_L C = \bigvee_L C(x) \leqslant x$, i.e., $x \in [a)$. This implies that $L = (a] \cup [a)$, where $(a]$ is — by Lemma 3.2 — an m-algebraic lattice. Since L is of the form $L = (a] \cup [a)$, the inclusion $C \subseteq (a]$ implies that only a is m-compact in $[a)$.

3.4. Lemma. *Let m be regular and let L satisfy condition* (4). *Then L satisfies* (b.1).

Proof. Suppose that $u\colon L \to L$ is m-ACO. Take any $c \in C$, $x \in L$ such that $c \leqslant u(x)$. Then $c \leqslant a = \bigvee_L C$ (because $L = (a] \cup [a)$ and $[a)$ contains only one m-compact element which is a; i.e., $C \subseteq (a]$). If $a \leqslant x$, we can choose $d = c$; then $c \leqslant u(d)$ and $d \leqslant a \leqslant x$. If $x \leqslant a$, then $x = \bigvee_L C(x)$. The set $C(x)$ is m-direct (from the regularity of m and the definition of m-compact elements it follows that $C(x)$ is even a join m-subsemilattice of L, interpreted as a join m-semilattice), u is m-ACO and thus

$$u(x) = u\big(\bigvee_L C(x)\big) = \bigvee_{u(L)} u\big(C(x)\big) = \bigvee_L u\big(C(x)\big),$$

where the second equality follows from Section 1.5 and the third one from the fact that u is m-ACO and that also $u\big(C(X)\big)$ is m-directed. We have obtained that

$$c \leqslant u(x) \leqslant \bigvee_L u\big(C(x)\big)$$

and since c is m-compact in L, there exists $Y \subseteq C(x)$ such that $|Y| < m$ and $c \leqslant \bigvee_L u(Y)$. Thus

$$c \leqslant \bigvee_L u(Y) \leqslant \bigvee_{u(L)} u(Y) = u(\bigvee_L Y).$$

(The last equality follows from Section 1.5 again.) Moreover, $Y \subseteq C(x) \subseteq C$, $|Y| < m$, C is a join m-subsemilattice of L (recall m is regular); hence $d = \bigvee_L Y \in C$. The inclusion $Y \subseteq C(x)$ implies that $d \leqslant \bigvee_L C(x) = x$ and, following the preceding, there is $c \leqslant u(\bigvee_L Y) = u(d)$. Since $d \in C$, u is m-TCO.

3.5. Theorem. *Let m be a regular cardinal. Then L satisfies condition* (b.1) *iff it satisfies condition* (4).

Proof follows immediately from Lemmas 3.3 and 3.4.

4. Characterization of lattices satisfying condition (b.2)

4.1. *Notation.* We use the notation of Section 3.1. An element $x \in L$ is called *join-m-inaccessible* if for every non-empty m-directed set $M \subseteq L$, the equality $x = \bigvee_L M$ implies that $x \in M$ (for $m = \aleph_0$ this notion is defined in [1], for example). Suppose that L satisfies condition (4). Then we consider in the lattice L the following condition (recall that $a = \bigvee_L C$):

(5) *If $a \neq 1$, then a is join-m-inaccessible in $(a]$ and every element of $[a)$ different from 1 is join-m-inaccessible.*

4.2. Lemma. *Let u be an m-TCO on L and let M be a non-empty m-directed subset in $u(L)$. Then $C(\bigvee_L M) = C\big(u(\bigvee_L M)\big)$.*

Proof. Write $s = \bigvee_L M$. Since $s \leqslant u(s)$, we have $C(s) \subseteq C\big(u(s)\big)$. Take any $c \in C\big(u(s)\big)$. Then $c \leqslant u(s)$; u being by assumption m-TCO, there exists $d \in C(s)$ such that $c \leqslant u(d)$. Since $d \in C(s)$ where $s = \bigvee_L M$, there exists $X \subseteq M$ such that $|X| < m$ and $d \leqslant \bigvee_L X$. Further, since M is m-directed, there exists $y \in M$ such that for every $x \in X$ we have $x \leqslant y$; especially $d \leqslant \bigvee_L X \leqslant y$. Following this inequality we obtain $u(d) \leqslant u(y) = y$ (recall that $y \in M \subseteq u(L)$); since $y \leqslant s$, there is $c \leqslant s$, i.e., $c \in C(s)$ what we had to prove.

4.3. Lemma. *If a lattice L satisfies* (b.2), *then it satisfies* (4) *and* (5).

Proof. If L satisfies (b.2), then it satisfies also (b.1) and by Lemma 3.3 it satisfies (4). Suppose that $a < 1$ and that a is not join m-inaccessible in $(a]$. Put $A = \{x \in L;\ x < a\} \cup \{1\}$. This is obviously a closure system in L. Denote the corresponding closure operator by u and show that u is m-TCO. Take any $c \in C$, $x \in L$ such that $c \leqslant u(x)$. If $a \leqslant x$, then $u(x) = 1$ and we can set $d = c$, because $c \in C(x)$, $C(x) = C \subseteq (a]$ and $c \leqslant u(c) = u(d)$. Suppose then $x < a$ (by (4), L is of the form $L = (a] \cup [a)$ and thus every element of L is comparable with a). Then $x = u(x)$ and we can again put $d = c$. This proves that u is m-TCO. Further, we have supposed that a is not join-m-inaccessible, i.e., there exists $M \subseteq (a]$ such that $\bigvee_L M = a$ and $a \notin M$. Then $M = u(M)$ and $\bigvee_L M = a < 1 = \bigvee_A M$, a contradiction with condition (b.2). Thus, a must be join-m-inaccessible.

Suppose that $z \in [a)$, $z \neq 1$, is not join-m-inaccessible. Then $a < z$ and there exists an m-directed set $M \subseteq [a)$ such that $z \notin M$ and $\bigvee_L M = z$. Put $A = \big(\bigcup_{y \in M} (y]\big) \cup \{1\}$. Then A is obviously a closure system in L. Denoting by u the corresponding CO, we shal prove that u is m-TCO. Take any $c \in C$, $x \in L$ with $c \leqslant u(x)$. If $x \in A$, we can put $d = c$. If $x \in L - A$, then $u(x) = 1$ and we can put $d = c$ again (because $c \in C \subseteq (a]$, $a < z$ and thus, by the definition of A, we have $L - A \subseteq [a)$, i.e., $c \leqslant x$). Hence, u is m-TCO. On the other hand, since $z \notin M$, by the definition of u we have

$$\bigvee_L M = z < 1 = u(\bigvee_L M)$$

and following 1.5, since $M \subseteq A$, we have

$$u(V_L M) = V_A M.$$

Moreover, M is a non-empty m-directed subset of $u(L)$. This is a contradiction with (b.2) supposed to be true in L.

4.4. LEMMA. *Let m be regular and let L satisfy* (4) *and* (5). *Then L satisfies* (b.2).

Proof. L satisfies (b.1) by 3.4. Suppose that u is an m-TCO on L and let $M \subseteq u(L)$ be a non-empty m-directed subset. Put $s = V_L M$. If $M \cap [a) \neq \emptyset$, then it is a non-empty m-directed subset of $[a)$. Hence, by (5), we have

$$s = V_L(M \cap [a)) \in M \cup \{1\} \subseteq u(L)$$

which implies that $V_L M = V_{u(L)} M$. Suppose then that $M \cap [a) = \emptyset$. Then $M \subseteq (a] - \{a\}$ and $s \leqslant a$. We shall prove that $a \nleqslant u(s)$: by Section 4.2 we have $C(s) = C(u(s))$ and therefore the assumption $a \leqslant u(s)$ implies $C(s) = C$. Since $s \leqslant a$, we have $s = VC$, i.e., $s = a$. Hence, by (5), $a \in M$: a contradiction with the assumption $M \cap [a) = \emptyset$. We have proved that $u(s) < a$. (By (4), the case $u(s) \| a$ cannot occur.) Since the lattice $(a]$ is m-algebraic, we have that by Section 4.2 there is $C(s) = C(u(s))$; hence

$$V_L M = s = V_L C(s) = V_L C(u(s)) = u(s) = u(V_L M) = V_{u(L)} M.$$

This proves that u is m-ACO.

4.5. THEOREM. *Let m be regular. Then L satisfies* (b.2) *if and only if it satisfies* (4) *and* (5).

Proof follows immediately from Lemmas 4.3 and 4.4.

5. Final remarks

Sections 5.1, 5.2 contain some partial results concerning (b.3). In Section 5.3, we have formulated some open problems concerning the results of this paper and those of papers [3] and [4] and which we considered as interesting ones.

5.1. LEMMA. *Let u be an m-TCO and let L satisfy the following condition:*

(6) *If a non-empty $M \subseteq u(L)$ is m-directed, and if $V_L M \notin M$, then for every $x \in L$ there is*

$$(V_L M < x) \Rightarrow C(V_L M) \subsetneqq C(x).$$

Then u is m-ACO on L.

Proof. Let $\emptyset \neq M \subseteq u(L)$ be m-directed. Put $d = V_L M$. If $d \notin M$ and $d < u(d)$, then by (6) we have $C(d) \neq C(u(d))$, a contradiction with Lemma 4.2. Thus, for $d \notin M$, we have $d = u(d)$, i.e., $V_L M = u(V_L M) = V_{u(L)} M$. If $d \in M$, then obviously $V_L M = V_{u(L)} M$, because $M \subseteq u(L)$, which completes the proof.

5.2. COROLLARY. *Let L satisfy the following condition:*

(7) *For every $x, y \in L$ such that $x < y$ there is $C(x) \neq C(y)$ (i.e., $C(x) \subset C(y)$).*

Then L satisfies (b.3).

The proof follows immediately from Lemma 5.1.

5.3. Open problems:

(c) *Characterize those complete lattices which satisfy* (b.3).

(d) *What modifications of results of Sections 3 and 4 can arrise for irregular cardinal m?*

(e) *m-ACO and m-TCO are two examples of CO preserving the m-compacticity. Characterize all CO with this property* (Added in proof: This problem has already been solved, see [5].)

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