References

[1] D. Dorninger, *Über die Anzahl von Polynomen und Polynomfunktionen auf endlichen Verbänden*, J. Reine Angew. Math. 273 (1975), 199–205.

[2] —, *Local polynomial functions on distributive lattices*, An. Acad. brasil. Ciênc., 50 (4) (1978), 433–437.

[3] D. Dorninger and W. Nöbauer, *Local polynomial functions on lattices and universal algebras*, Colloq. Math. 42 (1980), 83–93.

[4] G. Grätzer, *Boolean functions on distributive lattices*, Acta Math. Acad. Sci. Hung. 15 (1964), 195–201.

[5] —, *Lattice theory*, Freeman and Company, San Francisco 1971.

[6] H. Lausch and W. Nöbauer, *Algebra of polynomials*, North-Holland, Amsterdam–London 1973.

[7] —, —, *Funktionen auf endlichen Gruppen*, Publ. Math. Debrecen 23 (1973), 53–61.

[8] W. Nöbauer, *Über die affin vollständigen, endlich erzeugbaren Moduln*, Monatsh. Math. 82 (1976), 187–198.

[9] D. Schweigert, *Über endliche, ordnungspolynomvollständige Verbände*, Monatsh. Math. 78 (1974), 68–76.

[10] H. Werner, *Produkte von Kongruenzklassengeometrien universeller Algebren*, Math. Z. 121 (1971), 111–140.

[11] R. Wille, *Eine Charakterisierung endlicher, ordnungspolynomvollständiger Verbände*, Arch. Math. 28 (1977), 557–560.

[12] —, *Über endliche, ordnungsaffinvollständige Verbände*, Math. Z. 155 (1977), 103–107.

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PROJECTABLE KERNEL OF A LATTICE ORDERED GROUP

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Let $\mathcal{K}$ and $\mathcal{G}$ be non-empty classes of lattice ordered groups. Consider the following condition for $\mathcal{K}$ and $\mathcal{G}$:

(a) For each $G \in \mathcal{G}$ there exists a convex l-subgroup H of G such that (i) H belongs to $\mathcal{K}$, and (ii) whenever H_1 is a convex l-subgroup of G with $H_1 \in \mathcal{K}$, then $H_1 \subseteq H$.

If (a) is valid, then we express this fact by saying that $(\mathcal{K}, \mathcal{G})$-*kernels do exist*. Under the denotations as in (a), the lattice ordered group H is said to be the $(\mathcal{K}, \mathcal{G})$-*kernel* of G. Let $\mathcal{G}_1$ be the class of all lattice ordered groups; the $(\mathcal{K}, \mathcal{G}_1)$-kernels will be denoted as $\mathcal{K}$-kernels.

The existence of $(\mathcal{K}, \mathcal{G})$-kernels were investigated by several authors (cf. Byrd and Lloyd [3], Černák [4], Conrad [5], Gavalcová [6], Holland [7], Jakubík [8], [10], [11], [12], Kenny [14], Martinez [15], Redfield [16]). Let us mention the following typical results:

(i) Let $\mathcal{K}$ be a variety of lattice ordered groups. Then $\mathcal{K}$-kernels do exist. (Cf. Holland [7].)

(ii) Let $\mathcal{K}_1$ be the class of all archimedean lattice ordered groups. Then $\mathcal{K}_1$-kernels do exist. (Cf. Redfield [16].)

(iii) Let $\mathcal{K}_2$ be the class of all complete lattice ordered groups. Then $\mathcal{K}_2$-kernels do exist. (Cf. Jakubík [8].)

The following negative result is easy to verify (cf. Example 2 below):

(iv) Let $\mathcal{K}_0$ be the class of orthogonally complete lattice ordered groups. Then $\mathcal{K}_0$-kernels do not exist.

In this paper the following result will be established:

(v) Let $\mathcal{K}_3$ and $\mathcal{K}_4$ be the class of all strongly projectable or projectable lattice ordered groups, respectively. Then $\mathcal{K}_3$-kernels and $\mathcal{K}_4$-kernels do exist.

Let us remark that neither of the classes $\mathcal{K}_i$ $(i = 1, 2, 3, 4)$ is a variety.

Projectable and strongly projectable lattice ordered groups and vector lattices have been dealt with in several papers (cf., e.g. Anderson, Conrad and Kenny [1], Bernau [2], Veksler [18], Jakubík [13]).

Let G be a lattice ordered group, $X \subseteq G$. The set

$$X^{\delta} = \{y \in G\colon\ |y| \wedge |x| = 0 \text{ for each } x \in X\}$$

is said to be a *polar* of G (cf. Šik [17]). If X is a one-element set, then X is called a *principal polar*. Each polar of G is a closed convex l-subgroup of G. The sets X^{δ} and $X^{\delta\delta}$ are said to be *complementary polars* of G.

Let $a, b \in G$, $a \leqslant b$, $X \subseteq [a, b]$, $Y_0 = \{g \in G\colon\ g \geqslant a\}$, $y \subseteq Y_0$. Put

$$X^{\delta}(a, b) = \{y \in [a, b]\colon\ y \wedge x = a \text{ for each } x \in X\},$$

$$X^{\delta\delta}(a, b) = Z^{\delta}(a, b) \quad \text{where} \quad Z = X^{\delta}(a, b),$$

$$Y^{\delta}(a) = \{t \in G\colon\ t \wedge y = a \text{ for each } y \in Y\},$$

$$Y^{\delta\delta}(a) = Z^{\delta}(a) \quad \text{where} \quad Z = Y^{\delta}(a).$$

The sets $Y^{\delta}(a)$ and $X^{\delta}(a, b)$ will be called *relative polars* and *relative bounded polars*, respectively. The sets $X^{\delta}(a, b)$ and $X^{\delta\delta}(a, b)$ are said to be *complementary relative polars* in the interval $[a, b]$. If $\operatorname{card} X = 1$, then $X^{\delta\delta}(a, b)$ is called a *principal relative polar*.

The lattice ordered group G is said to be *strongly projectable* (*projectable*) if each polar (each principal polar) of G is a direct factor of G. It is well known that a polar P of G is a direct factor of G if and only if the following condition is fulfilled:

(*) For each $0 \leqslant g \in G$, the set $P \cap [0, g]$ has the greatest element. Moreover, if a polar P is a direct factor of G, then P^{δ} is also a direct factor.

It will be shown that the strong projectability of G can be expressed by properties of bounded polars of G. Namely, the following result will be proved:

THEOREM 1. *The following conditions for a lattice ordered group G are equivalent*:

(i) *G is strongly projectable*;

(ii) *If $a, b \in G$, $a \leqslant b$, $X \subseteq [a, b]$, $x \in [a, b]$, then there are elements $y \in X^{\delta}(a, b)$, $z \in X^{\delta\delta}(a, b)$ such that $x = y \vee z$.*

We need some lemmas.

LEMMA 1. *Let $a, b \in G$, $a \leqslant b$, $X \subseteq [a, b]$. Then*

$$X^{\delta}(a, b) = a + \big((-a+X)^{\delta}(0, -a+b)\big).$$

This follows immediately from the fact that the mapping $\varphi(t) = -a+ +t$ $(t \in [a, b])$ is an isomorphism of the lattice $[a, b]$ onto the lattice $[0, -a+b]$.

From Lemma 1 we obtain:

LEMMA 2. *The condition* (ii) *of Theorem* 1 *is equivalent to the following condition*:

(iii) *If $c \in G$, $c > 0$, $X \subseteq [0, c]$, then there are elements $y \in X^{\delta}(0, c)$, $z \in X^{\delta\delta}(0, c)$ such that $c = y \vee z$.*

LEMMA 3. *Let $0 \leqslant c \in G$, $X \subseteq [0, c]$. Then*

$$X^{\delta}(0, c) = X^{\delta} \cap [0, c], \qquad X^{\delta\delta}(0, c) = X^{\delta\delta} \cap [0, c].$$

Proof. The first relation follows immediately from the defintion of $X^{\delta}(0, c)$. Write $X^{\delta}(0, c) = Y_0$. From $Y_0 \subseteq X^{\delta}$ we obtain $Y_0^{\delta} \supseteq X^{\delta\delta}$, and thus

$$Y_0^{\delta}(0, c) = Y_0^{\delta} \cap [0, c] \supseteq X^{\delta\delta} \cap [0, c].$$

Let $y \in Y_0^{\delta} \cap [0, c]$ and suppose that y does not belong to the set $X^{\delta\delta} \cap \cap [0, c]$. Then y does not belong to $X^{\delta\delta}$. Hence there is $0 < z \in X^{\delta}$ with $y \wedge z > 0$. Moreover, $y \wedge z \in Y_0$ and $y \wedge (y \wedge z) = y \wedge z > 0$, thus y does not belong to Y^{δ}, which is a contradiction. Hence $X^{\delta\delta}(0, c) = Y_0^{\delta}(0, c) = X^{\delta\delta} \cap [0, c]$.

LEMMA 4. *Let $0 < c \in G$, $X \subseteq G$. Put $Y = X^{\delta\delta} \cap [0, c]$, $Z = X^{\delta} \cap \cap [0, c]$. Then $Y^{\delta\delta}(0, c) = Y$ and $Z^{\delta\delta}(0, c) = Z$. Moreover, we have*

$$Z^{\delta}(0, c) = Y, \qquad Y^{\delta}(0, c) = Z.$$

Proof. Since $Y \subseteq [0, c]$, according to Lemma 3 we have

$$Y^{\delta\delta}(0, c) = Y^{\delta\delta} \cap [0, c].$$

Since $Y \subseteq X^{\delta\delta}$, we infer that

$$Y^{\delta\delta} \subseteq (X^{\delta\delta})^{\delta\delta} = X^{\delta\delta},$$

and hence

$$Y^{\delta\delta}(0, c) \subseteq X^{\delta\delta} \cap [0, c] = Y.$$

From the definition of $Y^{\delta\delta}(0, c)$ it follows immediately that $Y \subseteq Y^{\delta\delta}(0, c)$. Thus $Y = Y^{\delta\delta}(0, c)$. The relation $Z = Z^{\delta\delta}(0, c)$ can be verified analogously.

Let $t \in [0, c]$, $t \wedge z = 0$ for each $z \in Z$. Suppose that t does not belong to the set Y. Hence t cannot belong to $X^{\delta\delta}$. Thus there is $u \in X^{\delta}$ with $t_1 = t \wedge u > 0$. We have $t_1 \wedge z = 0$ for each $z \in Z$. On the other hand, $t_1 \in [0, c] \cap X^{\delta} = Z$, and hence $t_1 \wedge t_1 = 0$, a contradiction. Therefore, $Z^{\delta}(0, c) \subseteq Y$. Obviously, $Y \subseteq Z^{\delta}(0, c)$ and so $Z^{\delta}(0, c) = Y$. Analogously we can prove the relation $Y^{\delta}(0, c) = Z$.

Proof of Theorem 1. According to Lemma 2 it suffices to verify that the conditions (i) and (iii) are equivalent. Assume that (i) is valid and let

$0 \leqslant c \in G$, $X \subseteq [0, c]$. According to (i), X^δ and $X^{\delta\delta}$ are direct factors of G; since $X^\delta \cap X^{\delta\delta} = \{0\}$, we have

$$G = X^\delta \times X^{\delta\delta},$$

(the symbol $\times$ denotes the operation of the direct product). Let $c_1 = c(X^\delta)$ and $c_2 = c(X^{\delta\delta})$ be the corresponding components of the element c. We have $c = c_1 + c_2$, $c_1 \geqslant 0$, $c_2 \geqslant 0$, $c_1 \wedge c_2 = 0$, thus $c = c_1 \vee c_2$. Hence, in view of Lemma 3, (iii) holds.

Conversely, suppose that (iii) is valid. Let $X_1 \subseteq G$. We have to show that X_1^δ is a direct factor of G. Let $0 \leqslant c \in G$. Write $X = X_1^\delta \cap [0, c]$. From Lemma 4 it follows

$$X^{\delta\delta}(0, c) = X.$$

According to (iii) there are elements $y \in X^\delta(0, c)$, $z \in Z$ with $c = y \vee z$. Let $z_1 \in X_1^\delta \cap [0, c]$. Clearly, $z_1 \wedge y = 0$. Then

$$z_1 = z_1 \wedge c = z_1 \wedge (y \vee z) = (z_1 \wedge y) \vee (z_1 \wedge z) = z_1 \wedge z.$$

Hence z is the greatest element of $X_1^\delta \cap [0, c]$. In view of $(*)$, X_1^δ is a direct factor of G.

Let $a, b \in G$, $a \leqslant b$. Consider the following condition for $[a, b]$:

(α) For each $X \subseteq [a, b]$ there are elements $y \in X^\delta(a, b)$ and $z \in X^{\delta\delta}(a, b)$ such that $b = y \vee z$.

Lemma 5. *Let $a \in G$, $X \subseteq G$, $x \geqslant a$ for each $x \in X$, $y_i \geqslant 0$, $a + y_i \in X^\delta(a)$ $(i = 1, 2)$. Then $a + y_1 + y_2 \in X^\delta(a)$.*

Proof. We have

$$-a + X^\delta(a) = (-a + X)^\delta(0).$$

Since each polar of G is an l-subgroup of G, the set $(-a+X)^\delta(0)$ (being the positive cone of $(-a+X)^\delta$) is a subsemigroup of G. Since $y_1, y_2 \in (-a+X)^\delta(0)$, we have $y_1 + y_2 \in (-a+X)^\delta(0)$ and thus $a + y_1 + y_2 \in X^\delta(a)$.

Lemma 6. *Let $a, b, c \in G$, $a \leqslant b \leqslant c$. Assume that both $[a, b]$ and $[b, c]$ fulfil condition (α). Then $[a, c]$ also fulfils condition (α).*

Proof. Let $X \subseteq [a, c]$. Put

$$Y = X^\delta(a, c), \quad Z = X^{\delta\delta}(a, c),$$
$$Y_1 = Y \cap [a, b], \quad Z_1 = Z \cap [a, b].$$

By using the translation $\varphi(g) = g + a$ $(g \in G)$ it follows from Lemma 4 that

$$Y_1^{\delta\delta}(a, b) = Y_1, \quad Z_1^{\delta\delta}(a, b) = Z_1,$$
$$Y_1^\delta(a, b) = Z_1, \quad Z_1^\delta(a, b) = Y_1.$$

Since $[a, b]$ fulfils (α), there are elements $b_1 \in Y_1$, $b_2 \in Z_1$ with

$$b = b_1 \vee b_2.$$

Clearly, $b_1 \wedge b_2 = a$.

Write

$$Y' = b - a + Y, \quad Z' = b - a + Z.$$

Then Y' and Z' are complementary relative polars in the interval $[b, b - a + c] \supseteq [b, c]$. Put

$$Y_2 = Y' \cap [b, c], \quad Z_2 = Z' \cap [b, c].$$

Again, by translation and Lemma 4, we get

$$Y_2^{\delta\delta}(b, c) = Y_2, \quad Z_2^{\delta\delta}(b, c) = Z_2,$$
$$Y_2^\delta(b, c) = Z_2, \quad Z_2^\delta(b, c) = Y_2.$$

Because $[b, c]$ fulfils (α) there are elements $c_1 \in Y_2$, $c_2 \in Z_2$ with

$$c = c_1 \vee c_2.$$

Moreover, $c_1 \wedge c_2 = b$.

Put $b_{01} = -a + b_1$, $b_{02} = -a + b_2$, $c_{01} = -b + c_1$, $c_{02} = -b + c_2$. From the definition of the sets Y' and Z' we obtain

$$a + c_{01} \in Y, \quad a + c_{02} \in Z.$$

Thus, in view of Lemma 5,

$$a + b_{01} + c_{01} \in X^\delta(a), \quad a + b_{02} + c_{02} \in X^{\delta\delta}(a).$$

Hence

$$(a + b_{01} + c_{01}) \wedge (a + b_{01} + c_{02}) = a,$$
$$(b_{01} + c_{01}) \wedge (b_{02} + c_{02}) = 0,$$
$$(b_{01} + c_{01}) \vee (b_{02} + c_{02}) = (b_{01} + c_{01}) + (b_{02} + c_{02}).$$

Moreover, $c_{01} \wedge b_{02} = 0$, from which we infer

$$c_{01} + b_{02} = b_{02} + c_{01}.$$

Therefore

$$\begin{aligned}(a + b_{01} + c_{01}) \vee (a + b_{02} + c_{02}) &= a + \big((b_{01} + c_{01}) \vee (b_{02} + c_{02})\big) \\ &= a + \big((b_{01} + c_{01}) + (b_{02} + c_{02})\big) \\ &= a + (b_{01} + b_{02}) + (c_{01} + c_{02}) \\ &= a + (b_{01} \vee b_{02}) + (c_{01} \vee c_{02}) \\ &= [(a + b_{01}) \vee (a + b_{02})] + (c_{01} \vee c_{02}) \\ &= [b_1 \vee b_2] + (c_{01} \vee c_{02}) = b + (c_{01} \vee c_{02}) \\ &= (b + c_{01}) \vee (b + c_{02}) = c_1 \vee c_2 = c.\end{aligned}$$

Thus both the elements $a+b_{01}+c_{01}$, $a+b_{02}+c_{02}$ belong to the interval $[a, c]$. From Lemma 3 we obtain (by using a translation)

$$Z = X^{\delta\delta}(a, c) = X^{\delta\delta}(a)\cap[a, c],$$

and obviously

$$Y = X^{\delta}(a, c) = X^{\delta}(a)\cap[a, c].$$

Hence $y = a+b_{01}+c_{01}\in Y$, $z = a+b_{02}+c_{02}\in Z$ and $y\vee z = c$. Therefore the interval $[a, c]$ fulfils condition (α).

LEMMA 7. *Let $0\leqslant g_1\in G$, $0\leqslant g_2\in G$ and suppose that both the intervals $[0, g_1]$ and $[0, g_2]$ fulfil (α). Then $[0, g_1+g_2]$ also fulfils (α).*

Proof. The interval $[g_1, g_1+g_2]$ being isomorphic with $[0, g_2]$, the asertion follows from Lemma 6.

LEMMA 8. *Let $c_1, c\in G$, $0\leqslant c_1\leqslant c$. Suppose that $[0, c]$ fulfils condition (α). Then the interval $[0, c_1]$ also fulfils condition (α).*

Proof. Let $X\subseteq[0, c_1]$. Since $[0, c]$ fulfils (α), there are elements $y\in X^{\delta}(0, c)$, $z\in X^{\delta\delta}(0, c)$ with $c = y\vee z$. Write $y_1 = y\wedge c_1$, $z_1 = z\wedge c_1$. Then $c_1 = y_1\vee z_1$ and from Lemma 2 we easily obtain $y_1\in X^{\delta}(0, c_1)$, $z_1\in X^{\delta\delta}(0, c_1)$. Hence $[0, c_1]$ fulfils (α).

Let H_1 be the set of all elements $a\in G$, $a\geqslant 0$ such that $[0, a]$ fulfils (α). Clearly, $0\in H_1$. According to Lemma 7, H_1 is a subsemigroup of the semigroup G^+. Moreover, by Lemma 8, H_1 is a convex subset of G^+. Hence H_1 is a sublattice of G^+. From this it follows that the set $H = H_1-H_1$ is a convex l-subgroup of G.

THEOREM 2. *H is the strongly projectable kernel (i.e., the $\mathscr{K}_3$-kernel) of G.*

Proof. From the definition of H and from the fact that each interval of H is isomorphic with some interval of the form $[0, c]$ with $0\leqslant c\in H_1$ it follows that each interval of H fulfils condition (α). Hence, by Theorem 1, H is strongly projectable. Let A be a convex l-subgroup of G and suppose that A is strongly projectable. Let $0\leqslant a\in A$. Then, in view of Theorem 1, the interval $[0, a]$ fulfils (α). Thus we have $a\in H$. Therefore $A^+\subseteq G$ and from this it follows $A\subseteq G$, completing the proof.

Remark 1. If $\mathscr{K}$ is a class of lattice ordered groups, and if H is a $\mathscr{K}$-kernel of a lattice ordered group G, then H is an l-ideal of G. In fact, for each $g\in G$, $-g+H+g$ is a convex l-subgroup of G isomorphic with H, whence $-g+H+g\subseteq H$.

Remark 2. For each lattice ordered group G, the archimedean kernel and the complete kernel (i.e., the $\mathscr{K}_1$-kernel and the $\mathscr{K}_2$-kernel) of G is a closed l-ideal of G. (Cf. [8] and [10].) The following example shows that the strongly projectable kernel of G need not be closed.

EXAMPLE 1. Let F be the set of all real functions defined on the set $\boldsymbol{R}$ of all reals. The group and lattice operations in F are defined componentwise. Let A be the set of all constant functions of F and let B be the set of all functions of F having a one-elements support. Further, let G and H be the subgroup of the group F generated by the set $A\cup B$ or B, respectively. Then G and H are l-subgroups of F, and H is the $\mathscr{K}_3$-kernel of G. The lattice ordered group H fails to be closed in G (in fact, G is the smalest closed l-subgroup of G containing H as a subset, and $G\neq H$).

The following example shows that orthogonally complete kernels (i.e., $\mathscr{K}_0$-kernels, cf. the Introduction) need not exist.

EXAMPLE 2. Let G, H and B be as in Example 1. For each $b\in B$ let G_b be the convex l-subgroup of G generated by the element b. Then each G_b is linearly ordered and hence it is orthogonally complete. Clearly,

$$\bigvee_{b\in B} G_b = H,$$

and if H_1 is a convex l-subgroup of G with $H\subseteq H_1$, then H_1 fails to be orthogonally complete.

EXAMPLE 3. Let F be as in Example 1 and F_c be the set of all $f\in F$ that are continuous. Then F_c is an l-subgroup of F, F is strongly projectable and F_c fails to be projectable. Hence neither $\mathscr{K}_3$ nor $\mathscr{K}_4$ is a variety.

A class $\mathscr{K}\neq\emptyset$ of lattice ordered groups is said to be a *radical class* [9] if it fulfils the following conditions:

(a) $\mathscr{K}$ is closed with respect to isomorphisms.

(b) Whenever $G\in\mathscr{K}$ and G_1 is a convex l-subgroup of G, then $G_1\in\mathscr{K}$.

(c) If G is a lattice ordered group and if $\{G_i\}$ is a system of convex l-subgroups of G such that each G_i belongs to $\mathscr{K}$, then $\bigvee G_i$ belongs to $\mathscr{K}$ as well.

PROPOSITION 1. *$\mathscr{K}_3$ is a radical class.*

Proof. Obviously, $\mathscr{K}_3$ fulfils (a). From Theorem 2 it follows that $\mathscr{K}_3$ satisfies (c). By using Lemma 8 we obtain that condition (b) holds for $\mathscr{K}_3$.

Until now we have been dealing with strong projectability and hence the power of the set X in the lemmas above might be arbitrary.

If we consider the projectability, then we have to investigate the case where X is a one-element set. We need the following lemma.

LEMMA 9. *Let $a, b, x\in G$, $a\leqslant b$, $a\leqslant x$, $X = \{x\}$. Then*

$$X^{\delta}(a, b) = \{x\wedge b\}^{\delta}(a, b). \tag{1}$$

Proof. If $y\in X^{\delta}(a, b)$, then $y\in[a, b]$ and $y\wedge x = 0$; hence $0 = (y\wedge b)\wedge x = y\wedge(b\wedge x)$ and thus $y\in\{x\wedge b\}^{\delta}(a, b)$. Conversely, let

$y \in \{x \wedge b\}^{\delta}(a, b)$ and assume that y does not belong to the set $X^{\delta}(a, b)$. Then $y \wedge x = u > a$, $u \leqslant x \wedge b$, and hence $u \wedge y = 0$, which is a contradiction. Thus (1) is valid.

By using Lemma 9 and the same methods as in the investigation concerning the strong projectability (with the distinction that we always assume card $X = 1$), we can verify that the following statements are valid:

THEOREM 3. *The $\mathscr{K}_4$-kernels do exist.*

PROPOSITION 2. *$\mathscr{K}_4$ is a radical class.*

References

[1] M. Anderson, P. Conrad, O. Kenny, *Splitting properties in archimedean l-groups*, J. Austral. Math. Soc. 23 (1977), 247–256.
[2] S. Bernau, *Orthocompletion of lattice groups*, Proc. London Math. Soc. 16 (1966), 107–130.
[3] R. D. Byrd, J. T. Lloyd, *Kernels in lattice-ordered groups*, Proc. Amer. Math. Soc. 57 (1976), 16–18.
[4] Š. Černák, *On some types of maximal l-subgroups of a lattice ordered group*, Math. Slovaca (to appear).
[5] P. Conrad, *Epi-archimedean lattice ordered groups*, Czech. Math. J. 24 (1974), 192–218.
[6] T. Gavalcová, *α-kompaktnost' v l-gruppach*, Matem. časopis 24 (1974), 21–30.
[7] W. Ch. Holland, *Varieties of l-groups are torsion classes*, to appear.
[8] J. Jakubík, *Conditionally α-complete sublattices of a distributive lattice*, Algebra Univ. 2 (1972), 255–261.
[9] —, *Radical classes and radical mappings of lattice ordered groups*, Symposia Math. 31 (1977), 1–27.
[10] —, *Archimedean kernel of a lattice ordered group*, Czech. Math. J. 28 (1978), 140–154.
[11] —, *Generalized Dedekind completion of a lattice ordered group*, Czech. Math. J. (to appear).
[12] —, *Maximal Dedekind completion of a lattice ordered group*, ibid., to appear.
[13] —, *Orthogonal hull of a strongly projectable lattice ordered group*, ibid., to appear.
[14] G. O. Kenny, *Archimedean kernel of a representable l-group*, Notices Amer. Math. Soc. 21 (1974), A 590–591.
[15] J. Martinez, *Archimedean-like classes of lattice ordered groups*, Trans. Amer. Math. Soc. 186 (1973), 33–49.
[16] R. Redfield, *Archimedean and basic elements in completely distributive lattice ordered groups*, Pacific. J. Math. 63 (1976), 247–254.
[17] F. Šik, *K teorii strukturno uporjadočennych grupp*, Czech. Math. J. 6 (1965), 1–25.
[18] A. I. Veksler, *Projekcionnye svojstva vektornych rešetok i teorema Frejdental'a*, Math. Nachr. 74 (1976), 7–25.

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TWO CLOSURE OPERATORS WHICH PRESERVE m-COMPACTICITY*

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In this paper we shall investigate some properties of closure operators studied in [3] and [4]. The investigated problems are exactly formulated in Section 1.4.

1. Introductory remarks

1.1. Throughout this paper L will denote a given complete lattice with an ordering denoted by $\leqslant$. Further, m and n will denote infinite cardinals.

1.2. A subset X of L is called *m-directed in L* if for every $Y \subseteq X$, $|Y| < m$, there exists $x \in X$ such that for every $y \in Y$ we have $y \leqslant x$. (See [4], Definition 5.) A closure operator u (abbreviation: CO) on L is called *m-algebraic* (abbreviation: m-ACO) if for every non-empty m-directed subset Y of $u(L)$ there is $\bigvee_L Y = \bigvee_{u(L)} Y$. (See [3], Definition 1.3.) An element $c \in L$ is called *m-compact in L* if for every $X \subseteq L$ such that $c \leqslant \bigvee_L X$, there exists $Y \subseteq X$ with $|Y| < m$ and $c \leqslant \bigvee_L Y$. A lattice L is called *m-algebraic* if every element x of L can be written as the join of some set of m-compact elements in L. (See [2], p. 32.) The following assertion is proved in [3]:

(1) *Let m be regular and let u be an m-ACO on L. If c is n-compact in L for some infinite cardinal $m \leqslant n$, then $u(c)$ is n-compact in $u(L)$. If L is n-algebraic, then also $u(L)$ is n-algebraic.* (See [3], Theorem 2.1. For irregular cardinals m is the guess of the compacticity of $u(c)$ more complicated, as shows the same Theorem 2.1 of [3].)

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