We show this by an induction on the length of derivations from E' in $\mathfrak{A}$.

Suppose: if Bx is derivable from E' in $< \alpha$ steps, then $x \in I^\infty$.

Suppose: Ba is derivable from E' in α steps. We show $a \in I^\infty$.

Let $\mathscr{R}_A = \{x|\ Ax$ occurs before the last line in some (fixed) derivation from E' of $Ba\}$.

Let $\mathscr{R}_B = \{x|\ Bx$ occurs before the last line in this derivation$\}$.

Now, A occurs in the conclusion of only one axiom in E', namely $Bx \to Ax$. It follows that

$$\mathscr{R}_A \subseteq \mathscr{R}_B.$$

Also, by induction hypothesis, $\mathscr{R}_B \subseteq I^\infty$. Hence

$$(*) \qquad \mathscr{R}_A \subseteq I^\infty.$$

Finally, it should be clear that

$$E \vdash_{\langle \mathfrak{A}, \mathscr{R}_A \rangle} Ba.$$

This says

$$a \in I(\mathscr{R}_A).$$

Then by (*), since I is monotone,

$$a \in I(I^\infty) = I.$$

This concludes the proof.

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ON COMPATIBLE AND ORDER-PRESERVING FUNCTIONS ON LATTICES

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1

Let V be a lattice, k a positive integer and $F_k(V)$ the direct power $V^{(V^k)}$. A function $f \in F_k(V)$ is called *compatible* if for any congruence Θ on V and $(a_i, b_i) \in \Theta$, $i = 1, \ldots, k$, $(f(a_1, \ldots, a_k), f(b_1, \ldots, b_k)) \in \Theta$ holds, and f is called *order-preserving* if $a_i \leqslant b_i$, $i = 1, \ldots, k$, implies $f(a_1, \ldots, a_k) \leqslant f(b_1, \ldots, b_k)$. We denote by $C_k(V)$ the set of all k-place compatible functions on V and by $OF_k(V)$ the set of all k-place order-preserving functions on V. As it immediately follows by a result of Wille [12], $OF_k(V) \subseteq C_k(V)$ iff V is simple.

In the present paper we determine all distributive lattices V with $C_k(V) \subseteq OF_k(V)$, and we give necessary conditions for an arbitrary lattice V to satisfy $C_k(V) \subseteq OF_k(V)$. Thereby we obtain necessary conditions for a lattice to be (locally) k-affine complete and (locally) k-order affine complete resp. (for these concepts of completeness cf. Schweigert [9] and Wille [12]). Furthermore, we show that every distributive lattice is locally k-order affine complete (generalizing a result of Grätzer [4]) and that 1-affine completeness implies k-affine completeness in case of a distributive lattice.

Throughout this paper we adopt the following notational conventions: join, meet, inclusion, and proper inclusion in a lattice are denoted by $\cup$, $\cap$, $\leqslant$, and $<$, resp.; k always stands for a positive integer and V always denotes a lattice.

2

First we show that it is sufficient to consider the case $k = 1$ in order to answer the question whether $OF_k(V) \subseteq C_k(V)$ and whether $C_k(V) \subseteq OF_k(V)$, resp.

Let $\mathscr{R}$ be a set of binary relations on a lattice V and let $U(\mathscr{R})_k(V)$ be the set of all $f \in F_k(V)$ such that for any $R \in \mathscr{R}$ and $(x_i, y_i) \in R$, $i = 1, \ldots$ $\ldots, k$, $\big(f(x_1, \ldots, x_k), f(y_1, \ldots, y_k)\big) \in R$. (Cf. Lausch and Nöbauer [7].)

THEOREM 1. *Let $\mathscr{R}, \mathscr{S}$ be sets of binary relations on a lattice V. Then $U(\mathscr{R})_k(V) \subseteq U(\mathscr{S})_k(V)$ implies that $U(\mathscr{R})_m(V) \subseteq U(\mathscr{S})_m(V)$ for all positive integers $m \leqslant k$. If in addition every $R \in \mathscr{R}$ is reflexive and every $S \in \mathscr{S}$ is transitive, then $U(\mathscr{R})_1(V) \subseteq U(\mathscr{S})_1(V)$ implies that $U(\mathscr{R})_k(V) \subseteq U(\mathscr{S})_k(V)$ for any $k \geqslant 1$.*

Proof. Suppose $U(\mathscr{R})_k(V) \subseteq U(\mathscr{S})_k(V)$, $m \leqslant k$ and $f \in U(\mathscr{R})_m(V)$. If we define $\hat{f} \in F_k(V)$ by

$$\hat{f}(x_1, \ldots, x_k) := f(x_1, \ldots, x_m), \qquad (x_1, \ldots, x_k) \in V^k,$$

then clearly $\hat{f} \in U(\mathscr{R})_k(V)$, thus $\hat{f} \in U(\mathscr{S})_k(V)$, whence $f \in U(\mathscr{S})_m(V)$.

Now let every $R \in \mathscr{R}$ be reflexive and every $S \in \mathscr{S}$ be transitive and assume that $U(\mathscr{R})_1(V) \subseteq U(\mathscr{S})_1(V)$. Proceding by induction on k suppose that $U(\mathscr{R})_n(V) \subseteq U(\mathscr{S})_n(V)$ for all positive integers $n \leqslant k-1$ $(k \geqslant 2)$ and let $f \in U(\mathscr{R})_k(V)$. For $a \in V$ define $f^{(a)} \in F_{k-1}(V)$ by

$$f^{(a)}(x_1, \ldots, x_{k-1}) := f(x_1, \ldots, x_{k-1}, a), \qquad (x_1, \ldots, x_{k-1}) \in V^{k-1},$$

and for $(a_1, \ldots, a_{k-1}) \in V^{k-1}$ define $f_{a_1, \ldots, a_{k-1}} \in F_1(V)$ by

$$f_{a_1, \ldots, a_{k-1}}(x) := f(a_1, \ldots, a_{k-1}, x), \qquad x \in V.$$

Because of the reflexivity of the $R \in \mathscr{R}$ it follows that $f^{(a)} \in U(\mathscr{R})_{k-1}(V)$ and $f_{a_1, \ldots, a_{k-1}} \in U(\mathscr{R})_1(V)$, hence by induction assumption

$$f^{(a)} \in U(\mathscr{S})_{k-1}(V) \quad \text{and} \quad f_{a_1, \ldots, a_{k-1}} \in U(\mathscr{S})_1(V).$$

Now let a_i, b_i, $i = 1, \ldots, k$, be elements of V such that $(a_i, b_i) \in S$, where $S \in \mathscr{S}$. Then

$$\big(f(a_1, \ldots, a_k), f^{(a_k)}(b_1, \ldots, b_{k-1})\big) = \big(f^{(a_k)}(a_1, \ldots, a_{k-1}), f^{(a_k)}(b_1, \ldots, b_{k-1})\big) \in S$$

and

$$\big(f^{(a_k)}(b_1, \ldots, b_{k-1}), f(b_1, \ldots, b_k)\big) = \big(f_{b_1, \ldots, b_{k-1}}(a_k), f_{b_1, \ldots, b_{k-1}}(b_k)\big) \in S.$$

Since the relation S is transitive, this means $\big(f(a_1, \ldots, a_k), f(b_1, \ldots, b_k)\big) \in S$. Therefore, f belongs to $U(\mathscr{S})_k(V)$.

Remark. Obviously the lattice structure on V is not needed in the proof of Theorem 1.

COROLLARY. *$OF_k(V) \subseteq C_k(V)$ if and only if $OF_1(V) \subseteq C_1(V)$ and $C_k(V) \subseteq OF_k(V)$ if and only if $C_1(V) \subseteq OF_1(V)$.*

THEOREM 2 (cf. Wille [12]). *Let V be an arbitrary lattice. Then $OF_k(V) \subseteq C_k(V)$ if and only if V is simple.*

Proof. If V is simple, obviously $OF_k(V) \subseteq C_k(V)$. If V is not simple, then $OF_1(V) \nsubseteq C_1(V)$ as was shown by Wille [12] (proof of Hilfssatz 4). Hence by the Corollary of Theorem 1, $OF_k(V) \nsubseteq C_k(V)$.

Let $P_k(V)$ be the sublattice of $F_k(V)$ generated by the constant functions and the k projections from V^k to V, and $LP_k(V)$ be the set of all $f \in F_k(V)$ such that for any finite subset M of V^k there exists a $g \in P_k(V)$ (depending on M) such that $f(\mathfrak{x}) = g(\mathfrak{x})$ for all $\mathfrak{x} \in M$. $P_k(V)$ is called the *lattice of k-place polynomial functions on V and $LP_k(V)$ the set of k-place local polynomial functions on V*. (Cf Lausch and Nöbauer [6].) Clearly, $LP_k(V)$ gives rise to a sublattice of $F_k(V)$ and $P_k(V) \subseteq LP_k(V) \subseteq OC_k(V)$ $:= C_k(V) \cap OF_k(V)$.

V is called *(locally) k-order polynomially complete* iff $P_k(V) = OF_k(V)$ $(LP_k(V) = OF_k(V))$ (cf. Schweigert [9], Wille [11]); V is called *(locally) k-affine complete* iff $P_k(V) = C_k(V)$ $(LP_k(V) = C_k(V))$ (cf. Werner [10]), and V is said to be *(locally) k-order affine complete* iff $P_k(V) = OC_k(V)$ $(LP_k(V) = OC_k(V))$ (cf. Wille [12]).

COROLLARY 1. *If a lattice V is locally k-order polynomially complete, then V is simple. In case V is distributive the converse is also true.* (Cf. Wille [11], [12].)

Proof. The first claim is obvious from Theorem 2, the second one follows from the fact that a distributive lattice V is simple iff $|V| \leqslant 2$.

COROLLARY 2. *$C_k(V) \neq OF_k(V)$ for any lattice V with $|V| > 1$.*

Proof. Suppose $C_k(V) = OF_k(V)$, then V is simple, hence $F_k(V)$ $= OF_k(V)$ and therefore $|V|$ must be 1.

THEOREM 3. *Let V be an arbitrary lattice. Then $C_k(V) \nsubseteq OF_k(V)$ if and only if there exists a sublattice U of V such that $C_1(U) \nsubseteq OF_1(U)$ and a $\psi \in C_1(V)$ such that $\psi(V) = U$ and $\psi^2 = \psi$.*

Proof. According to the Corollary of Theorem 1 the condition of the theorem is obviously necessary. Therefore, assume that there exist a U and a ψ as supposed in the theorem. Then there is a function $f \in C_1(U)$ such that $f \notin OF_1(U)$. We define a mapping $g \in F_1(V)$ by $g(x) := f(\psi(x))$, $x \in V$. As it is easy to check, $g \in C_1(V)$, but since $\psi(x) = x$ for all $x \in U$, $g \notin OF_1(V)$. By the Corollary of Theorem 1 this implies $C_k(V) \nsubseteq OF_k(V)$.

COROLLARY 1. *Let V be an arbitrary lattice. If there exists an interval $[b, a]$ of V such that $C_1([b, a]) \nsubseteq OF_1([b, a])$, then $C_k(V) \nsubseteq OF_k(V)$.*

Proof. Take $U = [b, a]$ and define ψ by

$$\psi(x) := (a \cap x) \cup b, \quad x \in V;$$

then the statement follows from Theorem 3.

In the following an interval $[b, a]$ of V with $b < a$ is called a *proper interval.*

COROLLARY 2. *Let V be an arbitrary lattice. If V contains a proper interval which is a Boolean lattice, then $C_k(V) \nsubseteq OF_k(V)$.*

Proof. Let U be a proper interval of V which is a Boolean lattice, then $C_1(U) \nsubseteq OF_1(U)$ since for the mapping $f \in F_1(U)$ defined by $f(x) := x^*$, $x \in U$, where x^* denotes the relative complement of x in U, $f \in C_1(U)$, but $f \notin OF_1(U)$, as one can easily see. So the statement of Corollary 2 follows from Corollary 1.

COROLLARY 3. *If a lattice V contains a proper subdirectly irreducible interval, then $C_k(V) \nsubseteq OF_k(V)$.*

Proof. This is a consequence of Corollary 1 and the remark after Theorem 8 in Dorninger and Nöbauer [3].

Let U be a bounded lattice (with bounds 0 and 1) and U_1, U_2 sublattices of U such that $U_1 \cap U_2 = \{0, 1\}$, $U_1 \cup U_2 = U$, and $x_1 \cap x_2 = 0$, $x_1 \cup x_2 = 1$, for all $x_1 \in U_1 - \{0, 1\}$, $x_2 \in U_2 - \{0, 1\}$. Then U will be called the *disjoint sum of* U_1, U_2.

COROLLARY 4. *If a lattice V contains an interval $U = [b, a]$ which is the disjoint sum of two bounded lattices U_1, U_2 with $|U_1|, |U_2| \geqslant 3$, then $C_k(V) \nsubseteq OF_k(V)$.*

Proof. If $|U_1| = 3$, then $C_k(V) \nsubseteq OF_k(V)$ by Corollary 2, thus we may assume that $|U_1| \geqslant 4$. Let $a_1, b_1 \in U_1$ such that $b < a_1 < a$, $b < b_1 < a$ and $a_1 \nleqslant b_1$, and let $f \in F_1(U)$ be defined by

$$f(x) := \begin{cases} a_1, & \text{if } x = b, \\ b_1, & \text{otherwise}. \end{cases}$$

We show that $f \in C_1(U)$, i.e. $(x, y) \in \Theta$ implies $\big(f(x), f(y)\big) \in \Theta$ for any congruence Θ on U. If $x, y \neq b$ or $x = y = b$, this is obvious, thus by the symmetry of Θ it suffices to show that $(x, b) \in \Theta$ and $x \neq b$ implies $\big(f(x), f(b)\big) \in \Theta$, i.e. $(b_1, a_1) \in \Theta$. If there is an $x \neq b$ with $(x, b) \in \Theta$, a straightforward computation shows that $\Theta = U \times U$ or $\Theta = (U_1 - \{a\})^2 \cup \cup (U_2 - \{b\})^2$ or $\Theta = (U_1 - \{b\})^2 \cup (U_2 - \{a\})^2$, whence $(b_1, a_1) \in \Theta$. Therefore, $f \in C_1(U)$ but $f \notin OF_1(U)$. From this we can conclude our claim by Corollary 1.

3

Next we show that in case of distributive lattices the converse of Corollary 2 of Theorem 3 is also true.

THEOREM 4. *Let V be a distributive lattice. Then $C_k(V) \nsubseteq OF_k(V)$ if and only if V contains a proper interval which is a Boolean lattice.*

Proof. The "if-part" of our assertion follows from Corollary 2 of Theorem 3. Now suppose that $C_k(V) \nsubseteq OF_k(V)$. Then by the Corollary of Theorem 1 there exists an $f \in C_1(V)$ such that $f \notin OF_1(V)$. Let $a, b \in V$ such that $a > b$ and $f(a) \ngeqslant f(b)$. We define a function $g \in F_1(V)$ by

$$g(x) := \big(f(a) \cup f(x)\big) \cap \big(f(a) \cup f(b)\big), \quad x \in V.$$

Obviously, $g \in C_1(V)$ and $g(a) = f(a) < f(a) \cup f(b) = g(b)$.

Next we consider the function $h \in F_1([b, a])$ which is defined by

$$h(x) := \big(a \cap g(x)\big) \cup b, \quad x \in [b, a].$$

Since V satisfies the congruence extension property, $h \in C_1([b, a])$. We claim that $h(a) < h(b)$. Since $h(a) \leqslant h(b)$, it suffices to prove that $h(a) \neq h(b)$.

Suppose that $a \cup g(a) \neq a \cup g(b)$; then $a \cup g(a) < a \cup g(b)$, and therefore there exists a prime ideal P of V such that $a \cup g(a) \in P$ and $a \cup g(b) \notin P$. If Θ_P denotes the congruence on V induced by P, i.e. $\Theta_P = P^2 \cup (V - P)^2$, then $(a, b) \in \Theta_P$ and hence $\big(a \cup g(a), a \cup g(b)\big) \in \Theta_P$ since $g \in C_1(V)$, a contradiction. Therefore, $a \cup g(a) = a \cup g(b)$. Since $a \cup g(a) = a \cup g(b)$ and $a \cap g(a) = a \cap g(b)$ cannot hold simultaneously, $a \cap g(a) \neq a \cap g(b)$, whence $a \cap g(a) < a \cap g(b)$. Setting $g_1(x) := a \cap g(x)$ and applying the dual of the preceding arguments to $g_1(x)$ and b instead of $g(x)$ and a, it immediately follows that $h(a) \neq h(b)$. Therefore, h is a function such that $h \in C_1([b, a])$ but $h \notin OF_1([b, a])$. From this one can conclude, by Grätzer [4], that $[b, a]$ contains a proper interval U which is a Boolean lattice. Clearly, U is also an interval of V.

Theorem 4 in connection with the following Theorem 5 is a generalization of a result of Grätzer [4].

THEOREM 5. *$LP_k(V) = OC_k(V)$ for any distributive lattice V, i.e. all distributive lattices are locally k-order affine complete.*

Proof. Since $LP_k(V) \subseteq OC_k(V)$, it suffices to show that $OC_k(V) \subseteq LP_k(V)$. So let $f \in OC_k(V)$ and $\mathfrak{x}_1, \ldots, \mathfrak{x}_n \in V^k$ with n a positive integer. Choose $a, b \in V$ such that $\{\mathfrak{x}_1, \ldots, \mathfrak{x}_n\} \cup \{f(\mathfrak{x}_1), \ldots, f(\mathfrak{x}_n)\} \subseteq [b, a]$ and define $g \in F_k([b, a])$ by

$$g(\mathfrak{x}) := \big(a \cap f(\mathfrak{x})\big) \cup b, \quad \mathfrak{x} \in ([b, a])^k.$$

Then $g(\mathfrak{x}_i) = f(\mathfrak{x}_i)$, $i = 1, \ldots, n$, and, since V satisfies the congruence extension property, $g \in OC_k([b, a])$. Hence by Grätzer [4], $g \in P_k([b, a])$, i.e. $g(x_1, \ldots, x_k) = w(a_1, \ldots, a_m, x_1, \ldots, x_k)$ for all $(x_1, \ldots, x_k) \in ([b, a])^k$, where w is a word in $a_1, \ldots, a_m \in [b, a]$ and $x_1, \ldots, x_k$. Let $\hat{g} \in F_k(V)$ be defined by

$$\hat{g}(x_1, \ldots, x_k) = w(a_1, \ldots, a_m, x_1, \ldots, x_k), \qquad (x_1, \ldots, x_k) \in V^k,$$

then $\hat{g} \in P_k(V)$ and $\hat{g}(\mathfrak{x}) = g(\mathfrak{x})$ for all $\mathfrak{x} \in ([b, a])^k$, hence

$$\hat{g}(\mathfrak{x}_i) = g(\mathfrak{x}_i) = f(\mathfrak{x}_i), \qquad i = 1, \ldots, k.$$

From this we can conclude $f \in LP_k(V)$.

COROLLARY 1. *A distributive lattice V is locally k-affine complete if and only if V does not contain a proper interval which is a Boolean lattice.* (Cf. Grätzer [4].)

Proof. Follows from Theorem 4 and Theorem 5.

COROLLARY 2. *A countable distributive lattice V is locally k-affine complete if and only if it does not contain an interval which is prime or a free Boolean algebra with countably many free generators.*

Proof. The "only if-part" is clear by Theorem 4.

Suppose that $LP_k(V) \neq C_k(V)$; then by Theorem 5 $C_k(V) \nsubseteq OF_k(V)$. Theorem 4 implies that V must contain a proper interval U which is a finite or countable Boolean lattice. If U does not contain a prime interval, then U is a countable Boolean lattice without any atoms. It is well known (cf. e.g. Grätzer [5], p. 112) that up to isomorphisms there exists exactly one countable Boolean lattice with no atoms. Since a free Boolean algebra with countably many free generators also has no atoms, U is a free Boolean algebra with countably many free generators.

COROLLARY 3. *A distributive lattice V is k-affine complete if and only if it is 1-affine complete.*

Proof. If V is k-affine complete, then it is 1-affine complete (cf. Nöbauer [8]). If, on the other hand, V is 1-affine complete, then $P_1(V) = LP_1(V) = OC_1(V) = C_1(V)$. By Dorninger [2] (Theorem 1), $P_1(V) = LP_1(V)$ implies $P_k(V) = LP_k(V)$, by Theorem 5 $LP_k(V) = OC_k(V)$, and by the Corollary to Theorem 1, $OC_1(V) = C_1(V)$ implies $OC_k(V) = C_k(V)$. Thus we infer $P_k(V) = C_k(V)$.

COROLLARY 4. *A chain V is k-affine complete if and only if V does not contain a prime interval.* (Cf. Grätzer [4].)

Proof. The necessity follows from Theorem 4. If, on the other hand, V does not contain a prime interval, then by Dorninger and Nöbauer [3] (Theorem 9) V is 1-affine complete, whence by Corollary 3 V is k-affine complete.

4

Last we consider (finite) direct products: Let V be the direct product of two lattices U and W, then (by Nöbauer [8] and Dorninger and Nöbauer [3]) there exists an isomorphism μ from $C_k(V)$ onto $C_k(U) \times C_k(W)$ which assigns to a function $f \in C_k(V)$ a pair $(g, h) \in C_k(U) \times C_k(W)$ such that

$$f\big((x_1, y_1), \ldots, (x_k, y_k)\big) = \big(g(x_1, \ldots, x_k), h(y_1, \ldots, y_k)\big),$$

for all $(x_1, \ldots, x_k) \in U^k$, $(y_1, \ldots, y_k) \in W^k$. This isomorphism μ is called *decomposition isomorphism*. As it is easy to see, $f \in OC_k(V)$ if and only if $\mu(f) \in OC_k(U) \times OC_k(W)$. Therefore, we have

THEOREM 6. *Let U and W be lattices and $V = U \times W$. Then the decomposition isomorphism μ induces an isomorphism from $OC_k(V)$ onto $OC_k(U) \times \times OC_k(W)$.*

COROLLARY 1. *$V = U \times W$ is locally k-order affine complete if and only if U and W are, and V is k-order affine complete only if U and W are; in case that U, W are bounded the converse of the latter statement is also true.*

COROLLARY 2. *Let $V = U \times W$. Then $C_k(V) \subseteq OF_k(V)$ if and only if $C_k(U) \subseteq OF_k(U)$ and $C_k(W) \subseteq OF_k(W)$.*

The proofs of Corollary 1 and 2 can be given by similar arguments as in Nöbauer [8] (proof of Lemma 5) or Dorninger and Nöbauer [3] (proof of Corollary 1 of Theorem 2) using Theorem 6, Theorem 6 of Dorninger and Nöbauer [3] and Hilfssatz 4 of Dorninger [1].

THEOREM 7. *Let U, W be distributive lattices with $|U|, |W| > 1$, and $V = U \times W$. Then $P_k(V) = OC_k(V)$ if and only if U and W are bounded.*

Proof. The "if-part" follows from Grätzer [4] or from Theorem 5 and Dorninger and Nöbauer [3] (Theorem 7). Now suppose that $P_k(V) = OC_k(V)$. If μ denotes the decomposition isomorphism from $C_k(V)$ onto $C_k(U) \times C_k(W)$, then

$$\mu\big(P_k(U \times W)\big) \subseteq P_k(U) \times P_k(W) \subseteq OC_k(U) \times OC_k(W) = \mu\big(OC_k(U \times W)\big),$$

the last equality holding by Theorem 6. From this we can conclude that $\mu\big(P_k(U \times W)\big) = P_k(U) \times P_k(W)$, whence U and W are bounded (cf. Dorninger and Nöbauer [3], Theorem 6).

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PROJECTABLE KERNEL OF A LATTICE ORDERED GROUP

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Let $\mathscr{K}$ and $\mathscr{G}$ be non-empty classes of lattice ordered groups. Consider the following condition for $\mathscr{K}$ and $\mathscr{G}$:

(a) For each $G \in \mathscr{G}$ there exists a convex l-subgroup H of G such that (i) H belongs to $\mathscr{K}$, and (ii) whenever H_1 is a convex l-subgroup of G with $H_1 \in \mathscr{K}$, then $H_1 \subseteq H$.

If (a) is valid, then we express this fact by saying that $(\mathscr{K}, \mathscr{G})$-*kernels do exist*. Under the denotations as in (a), the lattice ordered group H is said to be the $(\mathscr{K}, \mathscr{G})$-*kernel* of G. Let $\mathscr{G}_1$ be the class of all lattice ordered groups; the $(\mathscr{K}, \mathscr{G}_1)$-kernels will be denoted as $\mathscr{K}$-kernels.

The existence of $(\mathscr{K}, \mathscr{G})$-kernels were investigated by several authors (cf. Byrd and Lloyd [3], Černák [4], Conrad [5], Gavalcová [6], Holland [7], Jakubík [8], [10], [11], [12], Kenny [14], Martinez [15], Redfield [16]). Let us mention the following typical results:

(i) Let $\mathscr{K}$ be a variety of lattice ordered groups. Then $\mathscr{K}$-kernels do exist. (Cf. Holland [7].)

(ii) Let $\mathscr{K}_1$ be the class of all archimedean lattice ordered groups. Then $\mathscr{K}_1$-kernels do exist. (Cf. Redfield [16].)

(iii) Let $\mathscr{K}_2$ be the class of all complete lattice ordered groups. Then $\mathscr{K}_2$-kernels do exist. (Cf. Jakubík [8].)

The following negative result is easy to verify (cf. Example 2 below):

(iv) Let $\mathscr{K}_0$ be the class of orthogonally complete lattice ordered groups. Then $\mathscr{K}_0$-kernels do not exist.

In this paper the following result will be established:

(v) Let $\mathscr{K}_3$ and $\mathscr{K}_4$ be the class of all strongly projectable or projectable lattice ordered groups, respectively. Then $\mathscr{K}_3$-kernels and $\mathscr{K}_4$-kernels do exist.

Let us remark that neither of the classes $\mathscr{K}_i$ $(i = 1, 2, 3, 4)$ is a variety.