UNIVERSAL ALGEBRA AND APPLICATIONS
BANACH CENTER PUBLICATIONS, VOLUME 9
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

NONFREE PROJECTIVES IN PRODUCTS OF GROUP VARIETIES

V. A. ARTAMONOV

Department of Mechanics and Mathematics, Moscow State University, 117234, Moscow, U.S.S.R.

The present paper continues the study of projectives in products of varieties of groups began in [1]–[4]. Throughout the paper by a *rank* of a projective group P we mean a rank of the Abelian group P/P'. Let A_n, $n \geqslant 0$, be a variety of all Abelian groups with identity $x^n = 1$. In particular, $A_0 = A$ is a variety of all Abelian groups. It was shown in [1] that all projectives in $A_n A$ of finite ranks are free. Moreover, in virtue of [2], if P is a retract of a $A_n A$-free group F of finite rank with a projection $f\colon F \to P$, then there exists in F a free generating set $z_1, \ldots, z_t$ such that $\operatorname{Ker} f$ as a normal subgroup is generating by $z_{d+1}, \ldots, z_t$ and

$$z_1^f, \ldots, z_d^f$$

is a free generating set for P. In [4] A. McIsaac proved that AA_2-projectives of rank 2 are free. On the other hand, for any pair of integers $r, n \geqslant 2$ with $r+n > 4$ in [3] it was constructed an example of a nonfree AA_n-projective group of rank r with $r+1$ generators. In this paper we show that for any locally finite variety of groups V and any integer $r \geqslant 2$ there exists a nonfree AV-projective of rank r with $r+1$ generators, except the case $r = 2$, $V = A_2$. The existence of these projectives has been conjectured by A. L. Smelkin.

Let V be a variety of groups in which a V-free group G of rank $r \geqslant 2$ is finite. Without loss of generality we can assume that V is nonabelian, and hence $\exp V = n \geqslant 3$. Let $d = |G|$ and let $X = \{x_1, \ldots, x_r\}$ be a free generating set for G. Consider the augmentation

$$\varepsilon\colon ZG \to Z, \qquad \varepsilon\Big(\sum_{g\in G} a_g g\Big) = \sum a_g.$$

Obviously, ε is a ring epimorphism and its kernel $\mathfrak{p}$ is called the *augmentation ideal*. Let

$$N = \sum_{g \in G} g \in \mathbf{Z}G.$$

It is easy to see that $\mathbf{Z}N$ is a trivial ideal of $\mathbf{Z}G$. The triviality means that $gN = N$ for all $g \in G$. The augmentation ε induces a ring homomorphism

$$\varepsilon'\colon \mathbf{Z}G/N \to \mathbf{Z}/d,$$

and therefore a homomorphism of groups of units

$$\varepsilon^*\colon (\mathbf{Z}G/N)^* \to (\mathbf{Z}/d)^*.$$

THEOREM 1. *If $r \geqslant 2$, $n = \exp V \geqslant 3$, then ε^* is not surjective.*

Proof. Put $H = G/G'$, $h = |H|$. Then H is a $\mathfrak{A}_n$-free group of rank r and therefore $h = n^r$. Note that each prime divisor of d divides n. Thus a ring homomorphism

$$\eta\colon \mathbf{Z}/d \to \mathbf{Z}/h$$

induces an epimorphism of groups of units

$$\eta^*\colon (\mathbf{Z}/d)^* \to (\mathbf{Z}/h)^*.$$

Let $\lambda\colon G \to H$ be a natural epimorphism. It determines a commutative diagram

$$\text{(1)} \qquad \begin{array}{ccc} (\mathbf{Z}G/N)^* & \xrightarrow{\varepsilon^*} & (\mathbf{Z}/d)^* \\ \downarrow{\scriptstyle \lambda^*} & & \downarrow{\scriptstyle \eta^*} \\ (\mathbf{Z}H/N_{r,n})^* & \xrightarrow[\varepsilon^*_{r,n}]{} & (\mathbf{Z}/h)^* \end{array}$$

where as in [3]

$$N_{r,n} = \sum_{y \in H} y \in \mathbf{Z}H,$$

and $\varepsilon^*_{r,n}$ for H is defined in the same way as ε^* for G. According to Theorem 1 in [3], $\varepsilon^*_{r,n}$ is not surjective. Since η^* is epimorphic by (1), the map ε^* is not surjective.

THEOREM 2. *Let V be a variety of groups of exponent $n \geqslant 3$ and suppose that a V-free group G of rank $r \geqslant 2$ is finite. Then there exists $\mathfrak{A}V$-projective nonfree group of rank r with $r+1$ generators.*

Proof. Following Theorem 1 there exists an integer k such that

$$\text{(2)} \qquad k \in (\mathbf{Z}/d)^* \setminus \operatorname{Im} \varepsilon^* \quad \text{and even} \quad k \in (\mathbf{Z}/h)^* \setminus \operatorname{Im} \varepsilon^*_{r,n}.$$

In this case, by [6] the left ideal I generated in $\mathbf{Z}G$ by k and N is a projective nonfree left $\mathbf{Z}G$-module. Put

$$T = I \oplus \mathbf{Z}Gf_2 \oplus \ldots \oplus \mathbf{Z}Gf_r.$$

LEMMA 1. *T is a projective nonfree $\mathbf{Z}G$-module.*

Proof. Suppose $T \simeq (\mathbf{Z}G)^{r+1}$. Then

$$(\mathbf{Z}H)^{r+1} \simeq \mathbf{Z}H \underset{\mathbf{Z}G}{\otimes} T \simeq (\mathbf{Z}H \underset{\mathbf{Z}G}{\otimes} I) \oplus \mathbf{Z}Hf_2 \oplus \ldots \oplus \mathbf{Z}Hf_r,$$

and by Lemma 7 of [3]

$$\mathbf{Z}H \underset{\mathbf{Z}G}{\otimes} I \simeq \mathbf{Z}H.$$

Let now $\mu\colon I \to \mathbf{Z}G$ be a natural embedding, $\operatorname{Im}\mu$ a left ideal generated by k and N. Then we have a homomorphism of left $\mathbf{Z}H$-modules

$$\zeta = 1 \otimes \mu\colon \mathbf{Z}H = \mathbf{Z}H \underset{\mathbf{Z}G}{\otimes} I \to \mathbf{Z}H \underset{\mathbf{Z}G}{\otimes} \mathbf{Z}G = \mathbf{Z}H.$$

The image of ζ is a left ideal in $\mathbf{Z}H$ generated by k and $|G'|N_{r,n} = dh^{-1}N_{r,n}$. Since $(k, d) = 1$, the image of ζ is generated by k and $N_{r,n}$. Hence, by (2) and [6], $\operatorname{Im}\zeta$ is a nonfree projective $\mathbf{Z}H$-module of rank 1, and therefore it is a direct summand of $\mathbf{Z}H$. But this is impossible since $\mathbf{Z}H$ has the same rank 1 and $\operatorname{Im}\zeta$ is nonfree. The proof of lemma is complete.

Since $(k, d) = 1$, there exist integers k', d' such that

$$\text{(3)} \qquad kk' = 1 + d^2 d'.$$

Let J be a left ideal in $\mathbf{Z}G$ generated by k' and N. By [3], p. 107–108, there is an isomorphism of $\mathbf{Z}G$-modules,

$$f\colon (\mathbf{Z}G)^2 \to J \oplus I,$$

for which the basis $e_0 = (1, 0)$, $e_1 = (0, 1) \in (\mathbf{Z}G)^2$ maps onto

$$\text{(4)} \qquad f_0 = f(e_0) = (u', dd'v), \quad f_1 = f(e_1) = (ku' - dd'v, u),$$

where $u' = k'$, $v' = N \in J$; $u = k$, $v = N \in I$. A direct calculation shows that the projection ϱ of

$$W = J \oplus T = \bigoplus_{i=0}^{r} \mathbf{Z}Gf_i$$

onto T with the kernel J maps

$$f_0^\varrho = (0, dd'v) = -dd'Nf_0 + dd'k'Nf_1,$$

$$f_1^\varrho = (0, u) = -kf_0 + (1 + dd'N)f_1,$$

$$f_i^\varrho = f_i, \quad i \geqslant 2.$$

Using (3), we can choose in W a new base

$$u_1 = -kf_0+f_1,$$
$$u_i = f_i, \quad i \neq 1.$$

An easy computation shows that modulo dN

$$\begin{bmatrix}1 & k\\ 0 & 1\end{bmatrix}\begin{bmatrix}-dd'N & -k\\ dd'k'N & 1+dd'N\end{bmatrix}\begin{bmatrix}1 & -k\\ 0 & 1\end{bmatrix} \equiv \begin{bmatrix}0 & 0\\ 0 & 1\end{bmatrix}.$$

This consideration shows that in the new base $u_0, \ldots, u_r$ the projection ϱ has the form

$$(5)\qquad \begin{aligned} u_0^\varrho &= daNu_0+bNu_1,\\ u_1^\varrho &= dcu_0+(1+pN)u_1,\\ u_i^\varrho &= u_i, \quad i \geqslant 2, \end{aligned}$$

where $a, b, c, p \in \mathbf{Z}+\mathbf{Z}N$.

Let now G_0 be a V-free group with free generators $x_0, \ldots, x_r$. Denote by $\theta\colon G_0 \to G$ a natural projection which sends $x_0 \to 1$ and $x_i \to x_i$ for $i \geqslant 1$. We shall also denote by θ its extension to a ring homomorphism $\theta\colon \mathbf{Z}G_0 \to \mathbf{Z}G$. Put

$$M = \mathbf{Z}G_0 \underset{\mathbf{Z}G}{\otimes} W = \bigoplus_{i=0}^{r} \mathbf{Z}G_0 u_i$$

and denote by $l\colon M \to \mathfrak{m}$, $\mathfrak{m}$ the augmentation ideal of $\mathbf{Z}G_0$, a homomorphism of left $\mathbf{Z}G_0$-modules for which $l(u_i) = x_i-1$, $0 \leqslant i \leqslant r$. In virtue of [5], a $\mathfrak{A}V$-free group F of rank $r+1$ is a group of all matrices

$$(6)\qquad \begin{bmatrix}g & v\\ 0 & 1\end{bmatrix}, \quad g \in G_0,\ v \in M,\ l(v) = a-1$$

with the usual matrix multiplication. A free generating set in F can be chosen in the form

$$(6')\qquad y_i = \begin{bmatrix}x_i & u_i\\ 0 & 1\end{bmatrix}, \quad 0 \leqslant i \leqslant r.$$

Consider now a θ-semilinear endomorphism τ of M for which u_i^τ are defined by columns of the matrix

$$\left[\begin{array}{cc|ccc} Nad(x_0) & cd(x_0) & & & \\ bN & 1+pN & & 0 & \\ \hline & & 1 & & \\ & 0 & & \ddots & \\ & & & & 1 \end{array}\right], \quad d(x_0) = 1+x_0+\ldots+x_0^{d-1}.$$

Note that this $(r+1)\times(r+1)$ matrix is congruent of the matrix of ϱ modulo x_0-1.

LEMMA 2. *A map* φ

$$\begin{bmatrix}a & v\\ 0 & 1\end{bmatrix}^\varphi = \begin{bmatrix}a^\theta & v^\tau\\ 0 & 1\end{bmatrix}$$

is an endomorphism of the group F.

Proof. We need only to show that the images of free generators y_i belong to F. In fact,

$$y_0^\varphi = \begin{bmatrix}1 & aNd(x_0)u_0+bNu_1\\ 0 & 1\end{bmatrix},$$

and $y_0^\varphi \in F$, because following (6)

$$l\big(aNd(x_0)u_0+bNu_1\big) = aNd(x_0)(x_0-1)+bN(x_1-1) = 0.$$

Similarly, $y_i^\varphi \in F$. The proof is complete.

Since ϱ in (5) is idempotent, it follows that $\pi = \varphi^2$, is idempotent, that is, $P = \operatorname{Im}\pi$ is a $\mathfrak{A}V$-projective group with $r+1$ generators. The rank of P equals r since $P/V(P) \simeq G$. Thus we have only to prove

LEMMA 3. *P is not free in* $\mathfrak{A}V$.

Proof. Suppose that P is free in $\mathfrak{A}V$ with free generating set

$$z_i = \begin{bmatrix}g_i & h_i\\ 0 & 1\end{bmatrix}, \quad g_i \in G,\ h_i \in M,\ l(h_i) = g_i-1,$$

$i = 1, \ldots, r$. In this case, $g_1, \ldots, g_r$ is a free generating set for G. By Lemma 2 we have

$$z_i = z_i^\pi = \begin{bmatrix}g_i & h_i^{\tau^2}\\ 0 & 1\end{bmatrix}.$$

Suppose that

$$h_i = \sum_j h_{ij}(x_0, \ldots, x_r)u_j.$$

Then by θ-linearity

$$h_i = h_i^{\tau^2} = \sum h_{ij}(1, x_1, \ldots, x_r)u_j^{\tau^2} = h_i^\varrho + \sum h'_{ij}u_j$$

where h'_{ij} belong to the ideal in $\mathbf{Z}G_0$ generated by x_0-1. Here ϱ is a θ-semilinear endomorphism of M with the matrix ϱ from (5). Note that $h_1^\varrho, \ldots$ $\ldots, h_r^\varrho \in T \subset W \subset M$. Let B be a $\mathbf{Z}G$-submodule in T generated by these

r elements. If $T = B$, the T has r generators which is not the case since T is nonfree. Thus to prove this lemma we have to show that $T = B$.

Elements $g_i - 1$ generate an augmentation ideal $\mathfrak{p}$ in $\boldsymbol{Z}G$ as a left ideal. Since $g_i - 1 = l(h_i)$, we need to verify that B contains each element $q \in T$ with $l(q)$ belonging to the left ideal generated by $x_0 - 1$. Let

$$(7) \qquad q = \sum_{i=0}^{r} q_i(x_1, \ldots, x_r) u_i \in T, \qquad \sum_{i=1}^{r} q_i(x_i - 1) = 0.$$

By (6),

$$s = \begin{bmatrix} 1 & \sum_{i=1}^{r} q_i u_i \\ 0 & 1 \end{bmatrix} \in F,$$

and therefore by Lemma 2,

$$s^{\pi} = \begin{bmatrix} 1 & \sum_{i=1}^{r} q_i u_i^{\tau^2} \\ 0 & 1 \end{bmatrix} = \prod \begin{bmatrix} g_j & h_j \\ 0 & 1 \end{bmatrix}^{\pm 1}.$$

This implies that

$$\sum_{i=1}^{r} q_i u_i^{\varrho} \in B.$$

So without loss of generality we can assume that

$$\sum_{i=1}^{r} q_i u_i^{\varrho} = 0,$$

that is, $q_2 = \ldots = q_r = 0$, $q_1 u_1^{\varrho} = 0$ by (5). Hence, again by (5), we have

$$(7') \qquad q_0 u_0 + q_1 u_1 = q = q^{\varrho} = q_0 u_0^{\varrho} = dNq_0 a u_0 + Nq_0 b u_1.$$

Note that N belongs to the center of $\boldsymbol{Z}G$ and $Nq_0 = Nq_0(1, \ldots, 1)$. Thus in (7′)

$$q_0 = Na\, dq_0(1, \ldots, 1) = Na^2 d^3 q_0(1, \ldots, 1),$$

and since $d > 1$, we have $q_0(1, \ldots, 1) = q_0 = 0$. Hence, in (7) we have $q = 0$, $T = B$ which is impossible.

References

[1] V. A. Artamonov, *Projective metabelian groups and Lie algebras*, Izv. AN SSSR, ser. math., 42.2 (1978), 226–236.

[2] —, *The categories of free metabelian groups and Lie algebras*, Comment. Math. Univ. Carol. 18.1 (1977), 143–159.

[3] V. A. Artamonov, *Projective metabelian nonfree groups*, Bull. Austral. Math. Soc. 13.1 (1975), 101–115.

[4] A. J. Mc Isaac, *The freeness of some projective metabelian groups*, Bull. Austral. Math. Soc. 13.2 (1975), 161–168.

[5] V. N. Remeslennikov, B. G. Sokolov, *Some properties of Magnus embeddings*, Algebra i Logika 9.5 (1970), 566–578.

[6] R. G. Swan, *Periodic resolutions for finite groups*, Ann. of Math. 72.2 (1960), 267–291.

Presented to the Semester
Universal Algebra and Applications
(February 15 – June 9, 1978)