NONFREE PROJECTIVES IN PRODUCTS OF GROUP VARIETIES

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The present paper continues the study of projectives in products of varieties of groups begun in [1]–[4]. Throughout the paper by a rank of a projective group \( P \) we mean a rank of the Abelian group \( F/P' \). Let \( A_\lambda, \lambda > 0 \), be a variety of all Abelian groups with identity \( \varepsilon^n = 1 \). In particular, \( A_0 = A \) is a variety of all Abelian groups. It was shown in [1] that all projectives in \( A_\lambda A \) of finite ranks are free. Moreover, in virtue of [2], if \( P \) is a retract of a \( A_\lambda A \)-free group \( F \) of finite rank with a projection \( f : F \to P \), then there exists in \( F \) a free generating set \( \xi_1, \ldots, \xi_l \) such that \( K \sigma f \) as a normal subgroup is generating by \( \varepsilon_1^{\xi_1}, \ldots, \xi_l \) and

\[ \varepsilon_1^{\xi_1}, \ldots, \varepsilon_l^{\xi_l} \]

is a free generating set for \( P \). In [4] A. McIsaac proved that \( A_\lambda A \)-projectives of rank 2 are free. On the other hand, for any pair of integers \( r, n > 2 \) with \( r + n > 4 \) in [3] it was constructed an example of a nonfree \( A_\lambda A \)-projective group of rank \( r \) with \( r + 1 \) generators. In this paper we show that for any locally finite variety of groups \( V \) and any integer \( r \geq 2 \) there exists a nonfree \( A_\lambda V \)-projective group of rank \( r \) with \( r + 1 \) generators, except the case \( r = 2 \), \( V = A_\lambda \). The existence of these projectives has been conjectured by A. L. Smel'kin.

Let \( V \) be a variety of groups in which a \( V \)-free group \( G \) of rank \( r \geq 2 \) is finite. Without loss of generality we can assume that \( V \) is nonabelian, and hence \( \exp V = n > 3 \). Let \( V = [G] \) and let \( X = \{\xi_1, \ldots, \xi_l\} \) be a free generating set for \( G \). Consider the augmentation

\[ \sigma : ZG \to Z, \quad \sigma \left( \sum_{i} \sigma_i \xi_i \right) = \sum_{i} \sigma_i. \]
Obviously, ε is a ring epimorphism and its kernel p is called the augmentation ideal. Let

$$N = \sum_{g \in \mathbb{Z}G} g \in \mathbb{Z}G,$$

It is easy to see that $\mathbb{Z}N$ is a trivial ideal of $\mathbb{Z}G$. The triviality means that $gN = N$ for all $g \in G$. The augmentation ε induces a ring homomorphism

$$\varepsilon^*: (\mathbb{Z}G/N)^* \to (\mathbb{Z}/d)^*,$$

and therefore a homomorphism of groups of units

$$\varepsilon^*: (\mathbb{Z}G/N)^* \to (\mathbb{Z}/d)^*.$$

**Theorem 1.** If $r \geq 2$, $n = \exp V \geq 3$, then $\varepsilon^*$ is not surjective.

**Proof.** Put $H = G/G'; h = [H]$. Then $H$ is a $\mathbb{Z}$-free group of rank $r$ and therefore $h = w$. Note that each prime divisor of $d$ divides $n$. Thus a ring homomorphism

$$\eta: \mathbb{Z}/d \to \mathbb{Z}/h$$

induces an epimorphism of groups of units

$$\eta^*: (\mathbb{Z}/d)^* \to (\mathbb{Z}/h)^*.$$

Let $\lambda: G \to H$ be a natural epimorphism. It determines a commutative diagram

$$\begin{array}{ccc}
(\mathbb{Z}G/N)^* & \to & (\mathbb{Z}/d)^* \\
| & | & | \\
(\mathbb{Z}H/N_r)^* & \to & (\mathbb{Z}/h)^* \\
| & | & | \\
N_{r, n} & \to & (\mathbb{Z}/h)^*
\end{array}$$

where as in [3]

$$N_{r, n} = \sum_{g \in \mathbb{Z}H} g \in \mathbb{Z}H,$$

and $\varepsilon^*$ for $H$ is defined in the same way as $\varepsilon$ for $G$. According to Theorem 1 in [3], $\varepsilon^*$ is not surjective. Since $\varepsilon^*$ is epimorphic by (1), the map $\varepsilon^*$ is not surjective.

**Theorem 2.** Let $V$ be a variety of groups of exponent $n \geq 3$ and suppose that a $V$-free group $G$ of rank $r \geq 2$ is finite. Then there exists a $V$-projective nonfree group of rank $r$ with $r + 1$ generators.

**Proof.** Following Theorem 1 there exists an integer $k$ such that

$$k \in (\mathbb{Z}/d)^* \setminus \text{Im} \varepsilon^*$$

and even $k \in (\mathbb{Z}/h)^* \setminus \text{Im} \varepsilon^*$. In this case, by [6] the left ideal $I$ generated in $\mathbb{Z}G$ by $k$ and $N$ is a projective nonfree $\mathbb{Z}G$-module. Put

$$T = I \oplus \mathbb{Z}G_1 \oplus \cdots \oplus \mathbb{Z}G_k.$$

**Lemma 1.** $T$ is a projective nonfree $\mathbb{Z}G$-module.

**Proof.** Suppose $T \cong (\mathbb{Z}G)^{k+1}$. Then

$$(\mathbb{Z}H)^{k+1} \cong \mathbb{Z}H \otimes T \cong (\mathbb{Z}H \otimes I) \oplus \mathbb{Z}H_1 \oplus \cdots \oplus \mathbb{Z}H_k,$$

and by Lemma 7 of [3]

$$\mathbb{Z}H \otimes I \cong \mathbb{Z}H.$$

Let now $\mu: I \to \mathbb{Z}G$ be a natural embedding, $\text{Im} \mu$ a left ideal generated by $k$ and $N$. Then we have a homomorphism of left $\mathbb{Z}H$-modules

$$\zeta = 1 \otimes \mu: \mathbb{Z}H \otimes I \to \mathbb{Z}H \otimes \mathbb{Z}G = \mathbb{Z}H.$$

The image of $\zeta$ is a left ideal in $\mathbb{Z}H$ generated by $k$ and $[G] \mathbb{Z}N_{r, n} = dh^{-1}N_{r, n}$. Since $(k, d) = 1$, the image of $\zeta$ is generated by $k$ and $N_{r, n}$. Hence, by (2) and [5], $\text{Im} \zeta$ is a nonfree projective $\mathbb{Z}H$-module of rank 1, and therefore it is a direct summand of $\mathbb{Z}H$. But this is impossible since $\mathbb{Z}H$ has the same rank 1 and $\text{Im} \zeta$ is nonfree. The proof of lemma is complete.

Since $(k, d) = 1$, there exist integers $h', d'$ such that

$$h' = 1 + kd'.$$

Let $J$ be a left ideal in $\mathbb{Z}G$ generated by $k$ and $N$. By [3], p. 107–108, there is an isomorphism of $\mathbb{Z}G$-modules,

$$f_0: (\mathbb{Z}G)^k \to J \oplus I,$$

for which the basis $e_0 = (1, 0)$, $e_1 = (0, 1) \in (\mathbb{Z}G)^k$ maps onto

$$f_0 = f_0(e_0) = (u', dd'v), \quad f_1 = f_0(e_1) = (ku' - dd'v, w),$$

where $u' = k$, $v' = N \in G$, $u = h$, $v = N \in I$. A direct calculation shows that the projection $e$ of

$$W = J \oplus T \to \mathbb{Z}G_1$$

onto $T$ with the kernel $J$ maps

$$f_0 = (0, dd'v) \to -dd'N_f + dd'k'N_f,$$

$$f_1 = (0, w) \to -hf_0 + (1 + dd'N)f_1,$$

$$f_i = f_1, \quad i \geq 2.$$
Using (3), we can choose in $W$ a new base
\[
\begin{align*}
u_1 &= -b_k^2 + f_1, \\
u_i &= f_i, \quad i \neq 1.
\end{align*}
\]
An easy computation shows that modulo $dN$
\[
\begin{bmatrix}
1 & k \\
0 & 1 + dN
\end{bmatrix}
\begin{bmatrix}
1 & -k \\
dN & 1 + dN
\end{bmatrix}
\begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}
= \begin{bmatrix}
0 & 0 \\
0 & 1
\end{bmatrix}.
\]
This consideration shows that in the new base $u_1, \ldots, u_r$, the projection $\varphi$ has the form
\[
\begin{align*}
u_i &= \alpha a N u_i + b N u_i, \\
u_i &= \gamma d u_i + (1 + pN) u_i, \\
u_i &= u_i, \quad i \geq 2,
\end{align*}
\]
where $a, b, c, p \in \mathbb{Z} + dN$.

Let now $G_i$ be a $V$-free group with free generators $x_0, \ldots, x_r$. Denote by $\theta: G_i \to G$ a natural projection which sends $x_0 \to 1$ and $x_i \to x_i$ for $i \geq 1$. We shall also denote by $\theta$ its extension to a ring homomorphism $\theta: ZG_i \to ZG$. Put
\[
M = \bigoplus_{i \in \mathbb{Z}} ZG_i \otimes \mathbb{Z}
\]
and denote by $l: M \to \mathbb{Z}$ the augmentation ideal of $ZG_i$, a homomorphism of left $ZG_i$-modules for which $l(u_i) = x_i - 1, 0 \leq i \leq r$. In virtue of [5], a $AV$-free group $F$ of rank $r + 1$ is a group of all matrices
\[
\begin{bmatrix}
g & v \\
0 & 1
\end{bmatrix}, \quad g \in G, \ v \in M, \ l(v) = a - 1
\]
with the usual matrix multiplication. A free generating set in $F$ can be chosen in the form
\[
\begin{align*}
y_i &= \begin{bmatrix}
x_i & u_i \\
0 & 1
\end{bmatrix}, \quad 0 \leq i \leq r.
\end{align*}
\]
Consider now a $\theta$-semilinear endomorphism $\tau$ of $M$ for which $u_i'$ are defined by columns of the matrix
\[
\begin{bmatrix}
N d(x_0) & \sigma d(x_i) \\
N & 1 + pN
\end{bmatrix},
\]
where $\sigma d(x_0) = 1 + x_0 + \ldots + x_i^{r-1}$.

Note that this $(r+1) \times (r+1)$ matrix is congruent of the matrix of $\varphi$ modulo $x_i - 1$.

**Lemma 2.** A map $\varphi$
\[
\begin{bmatrix}
a & v \\
0 & 1
\end{bmatrix}
\]
is an endomorphism of the group $F$.

**Proof.** We need only to show that the images of free generators $y_i$ belong to $F$. In fact,
\[
y_i' = \begin{bmatrix}
a N d(x_0) u_i + b N u_i \\
0
\end{bmatrix},
\]
and $y_i' \in F$, because following (6)
\[
l(n N d(x_0) u_i + b N u_i) = a N d(x_0)(x_0 - 1) + b N(x_0 - 1) = 0.
\]
Similarly, $y_i' \in F$. The proof is complete.

Since $\varphi$ in (5) is idempotent, it follows that $\pi = \varphi^t$, is idempotent, that is, $F = Im \varphi$ is a $AV$-projective group with $r+1$ generators. The rank of $F$ equals $r$ since $F/\langle \varphi \rangle = G$. Thus we have only to prove

**Lemma 3.** $F$ is not free in $AV$.

**Proof.** Suppose that $F$ is free in $AV$ with free generating set
\[
x_i = \begin{bmatrix}
g_i & h_i \\
0 & 1
\end{bmatrix}, \quad g_i \in G, \ h_i \in M, \ l(h_i) = g_i - 1, \quad i = 1, \ldots, r.
\]
In this case, $g_1, \ldots, g_r$ is a free generating set for $G$. By Lemma 2 we have
\[
x_i - x_i' = \begin{bmatrix}
g_i & h_i' \\
0 & 1
\end{bmatrix}.
\]
Suppose that
\[
h_i = \sum_{j} h_{ij}(x_0, \ldots, x_r) y_j.
\]
Then by $\theta$-linearity
\[
h_i = \sum_{j} h_{ij}(x_0, \ldots, x_r) y_j = h_i + \sum_{j} h_{ij} y_j
\]
where $h_{ij}$ belong to the ideal in $ZG_i$ generated by $x_i - 1$. Here $\varphi$ is a $\theta$-semilinear endomorphism of $M$ with the matrix $\varphi$ from (5). Note that $h_i', \ldots, h_i' \in T \subset W \subset M$. Let $B$ be a $ZG$-submodule in $T$ generated by these
r elements. If \( T = B \), the \( T \) has \( r \) generators which is not the case since \( T \) is nonfree. Thus to prove this lemma we have to show that \( T = B \).

Elements \( g_i - 1 \) generate an augmentation ideal \( p \) in \( \mathbb{Z}G \) as a left ideal. Since \( g_i - 1 = l(g_i) \), we need to verify that \( B \) contains each element \( q e T \) with \( l(q) \) belonging to the left ideal generated by \( x_n - 1 \). Let

\[
q = \sum_{i=1}^{r} g_i(x_1, \ldots, x_n) u_i \in T, \quad \sum_{i=1}^{r} g_i(x_n - 1) = 0.
\]

By (6),

\[
es = \begin{bmatrix} 1 & \sum_{i=1}^{r} q_i u_i \\ 0 & 1 \end{bmatrix} \in F,
\]

and therefore by Lemma 2,

\[
es^* = \begin{bmatrix} 1 & \sum_{i=1}^{r} q_i u_i^2 \\ 0 & 1 \end{bmatrix} = \prod \begin{bmatrix} g_i & h_i \\ 0 & 1 \end{bmatrix}^{\delta_i}.
\]

This implies that

\[
\sum_{i=1}^{r} q_i u_i^2 \in B.
\]

So without loss of generality we can assume that

\[
\sum_{i=1}^{r} q_i u_i^2 = 0,
\]

that is, \( q_2 = \ldots = q_r = 0 \). Hence, again by (5), we have

\[
q_2 u_1 = 0 \quad \text{by (5).}
\]

Note that \( N \) belongs to the center of \( \mathbb{Z}G \) and \( N_{G} = N_{g}(1, \ldots, 1) \). Thus in (7)

\[
g_0 = N_{G} d_{G}(1, \ldots, 1) = N d_{G}(1, \ldots, 1),
\]

and since \( d > 1 \), we have \( g_0(1, \ldots, 1) = g_0 = 0 \). Hence, in (7) we have \( q = 0 \), \( T = B \) which is impossible.

References
