We define here the category $qB$ of quotient Banach spaces and study its linear properties. This is an exact category whose definition can be motivated by functional analytic considerations. It is to be expected that $qB$ will be useful where functional analysis and homological algebra interact. The multilinear properties of $qB$, and the J. L. Taylor spectrum and quotient Banach spaces, will be considered elsewhere.

The category $q$ of quotients of complete bornological spaces can also be useful when functional analysis and homological algebra intermingle. It contains $qB$. More applications can be attained with $q$ than with $qB$. But $qB$ is a subcategory of $E$, the category of vector spaces, $q$ is not. Its properties can be explained in a much simpler language than the properties of $q$. It is the simpler language that will be used here.

I met the quotients of complete bornological spaces for the first time in 1960 [5], without defining their category, without showing that my constructions were natural in $q$. I withheld systematic research and publication when I realized that there would be more trivialities than applications in the incipient stages of the theory.

G. Noël gave a definition of $q$, and obtained some results on tensor products ([1], [2], [3], [4]). I do not like much his definition. It is too categorical, and not functional analytic enough. But it is not worse than the definition I had at the time and did not publish.

This paper, and the next two will stress the functional analytic aspects of the category $qB$. After all, it is the functional analysts who are expected to use this category. And the fact that $qB$ is (isomorphic with) a subcategory of $E$ may make the pill easier to swallow for functional analysts who do not have much intuition of what homological algebra is about.

The results contained in this paper and the properties of quotients of complete bornological spaces have already been announced [6], [7], [8], [9].

Let $E$ be a Banach space. A Banach subspace $F$ of $E$ is a vector subspace on which a Banach space norm exists which defines on $F$ a stronger topology than the topology induced by $E$. Applying the closed graph theorem, we observe that two such
Banach space norms on \( F \) are equivalent, i.e. that the Banach space structure of \( F \) is determined by that of \( E \) and the set \( F \).

**Definition 1.** A Banach quotient \( E/F \) is the quotient of Banach space \( E \) by a Banach subspace \( F \). Let \( E/F \) and \( E'/F' \) be two Banach quotients. A strict morphism \( u: E/F \to E'/F' \) is a linear mapping \( u: x + F \to u_1(x) + F' \) where \( u_1: E \to E' \) is a bounded linear mapping such that \( u_1(F) \subseteq F' \).

We shall not analyse further the Banach quotient "structure" of \( E/F \), but shall consider that it is given by its realisation as the quotient of a Banach space and a Banach subspace.

If \( E/F \) and \( E'/F' \) are two Banach quotients, if \( u_1: E \to E' \) is a bounded linear mapping such that \( u_1(F) \subseteq F' \), we shall say that \( u_1 \) induces \( u \) where
\[
u(x + F) = u_1(x) + F'.
\]

Note that \( u_1: F \to F' \) is a continuous linear mapping when we place on each of \( F \) and \( F' \) its own Banach space topology. If \( u = 0 \), \( u_1 \) is continuous and again \( u_1: E \to F' \) is continuous.

The composition of strict morphisms is obviously a strict morphism. The class of Banach quotients and strict morphisms is a subcategory of E.V., where E.V. is the category of vector spaces and linear maps.

**Definition 2.** This subcategory will be called \( \mathcal{QB} \).

\( \mathcal{QB} \) is a good category to work with, because it is usually easy to show that a notion is "natural" in \( \mathcal{QB} \). Unfortunately it does not only have good properties.

If \( E \) is a Banach space, and \( F \) a closed subspace, \( E_1 = E/F \) is a Banach space. It would be very nice if the Banach quotient \( E/F \) were isomorphic with \( E_1/F \). Unfortunately, this is not the case in \( \mathcal{QB} \) unless \( F \) is complemented.

We are led to enlarging \( \mathcal{QB} \).

**Definition 3.** Let \( E/F \) and \( E'/F' \) be two Banach quotients. A pseudo-isomorphism \( s: E/F \to E'/F' \) is a strict morphism induced by a surjective bounded linear \( s_1: E \to E' \) such that \( s_1(F) = F' \). The category \( \mathcal{QB} \) is the subcategory of E.V. generated by \( \mathcal{QB} \) and the inverses of the pseudo-isomorphisms.

\( \mathcal{QB} \) has as objects the Banach quotients and as morphisms all compositions of strict morphisms and inverses of pseudo-isomorphisms. This makes sense because pseudo-isomorphisms are bijective linear maps.

A morphism of Banach quotients will simply be a morphism in the category \( \mathcal{QB} \).

**Definition 4.** A Banach space is free if it is isomorphic to \( l_1(X) \) for some \( X \). A Banach quotient \( E/F \) is standard if \( E \) is free.

**Proposition 1.** It is possible to associate to every Banach quotient \( E/F \) a standard quotient \( E_1/F_1 \) and a pseudo-isomorphism \( s: E_1/F_1 \to E/F \). In particular, every Banach quotient is isomorphic to a standard one.

Every Banach space is of course isomorphic (as a Banach space) to the quotient of a free Banach space by a closed subspace. Let \( E_1 \) be free and \( s_1: E_1 \to E \) be a continuous linear surjection. Let \( F = s_1^{-1}(F) \). Then \( E_1/F_1 \) is standard, and \( s_1: E_1 \to E \) induces a pseudo-isomorphism: \( s_1: E_1/F_1 \to E/F \).

**Proposition 2.** Every morphism from a standard Banach quotient is strict.

By induction, it is sufficient to show that \( x^{-1} \circ u \) is strict when \( u: E_1/F_1 \to E/F \) is a strict morphism, when \( E_1 \) is free, and when \( x: E'/F' \to E/F \) is a pseudo-isomorphism. Let \( u_1: E_1 \to E \) induce \( u \) and \( s_1: E' \to E \) induce \( s \) and be surjective.

It is known that a bounded linear \( v: E_1 \to E' \) exists, such that \( u_1 = s_1 \circ v \).

Since \( u_1: E_1 \to E \) and \( v_1: F_1 \to F \) are linear, \( x \circ u_1 = s_1 = v_1 \).

Since \( s_1 \) is surjective, \( x \circ v_1 \) is surjective. Hence \( x \circ v_1: E' \to E \) is surjective.

The existence of \( v_1 \) follows from the fact that \( E_1 \) is free and \( s_1 \) is surjective.

**Proposition 3.** Every morphism \( u \) of \( \mathcal{QB} \) can be expressed \( u = u_1 \circ s^{-1} \) where \( s \) is a standard pseudo-isomorphism from a standard quotient, and \( u_1 \) is a strict morphism from the same quotient.

Just combine Propositions 1 and 2.

Note. \( \mathcal{QB} \) is "equivalent" to the category of standard quotients and strict morphisms of such standard quotients. Restriction of \( \mathcal{QB} \) to this subcategory is inessential. Some mathematicians may consider that this restriction leads to a more elegant presentation of the theory. But the functional analyst who wants to consider a specific \( E/F \), e.g. \( E_1(\mathcal{L}_n) \), may object to replacing it by a pseudo-isomorphic \( l_1(F_1) \).

**Proposition 4.** A functor from the strict category of Banach quotients extends to the full category when it maps pseudo-isomorphisms on isomorphisms. The extension is unique.

**Proposition 5.** Let \( K \) be a category, and \( F, G: \mathcal{QB} \to K \) be two functors which extended to \( \mathcal{QB} \). Let \( F_1, G_1 \) be their extensions. Let \( H: F \to G \) be a functor homomorphism. Then \( H \) is a homomorphism of \( F_1 \) to \( G_1 \).

These propositions mean that a construction is natural in \( \mathcal{QB} \) if it behaves as it should under the action of a strict morphism and if the situation does not change essentially when we replace a problem by a pseudo-isomorphic one (when we replace the given Banach quotients by new, pseudo-isomorphic Banach quotients). In practice, the verification of each condition is within the realm of functional analysis.

The uniqueness of the extension of a functor is immediate. So is Proposition 5. Remember that each morphism\( u \) factors \( u = s^{-1} \circ u_1 \) strict, \( s \) being a pseudo-isomorphism. If \( F_1 \) is an extension of \( F \) to \( \mathcal{QB} \),
\[
F_1(u) = F_1(s^{-1}) \circ F_1(u_1) = F_1(s^{-1}) \circ F_1(u_1) = F(s^{-1}) \circ F(u_1).
\]
Let $F_1$ and $G_1$ be extensions of $F$ and $G$, and $H$: $F \rightarrow G$ a functor homomorphism. Let $u$ be a morphism $U \rightarrow U_1$, and $s$ a pseudo-isomorphism $U \rightarrow U'/V$. We must show that

\[ H(U_1/V)F_1(u) = G_1(u)H(U/V) \]

and assume that

\[ H(U/V)F(u) = G_1(u)H(U'/V'), \quad H(U_1/V)F(u) = G_1(u)H(U'/V'). \]

Proposition 5 follows.

We must still prove that the functor $F$: $\mathcal{B} \rightarrow K$ has an extension if it maps a pseudo-isomorphism onto an isomorphism. If it has an extension $F_1$, if $u$: $U \rightarrow V \rightarrow U_1/V_1$ is a general morphism which factors $u = x^{-1}u_1$, we know that $F_1(u) = (x^{-1})F(u_1)$.

We must therefore prove that $F(x^{-1})F(u_1)$ does not depend on the factorization $u = x^{-1}u_1$. This being done, we must show that the mapping

\[ u \rightarrow F_1(u) = F(x^{-1})F(u_1) \]

is a functor.

**Lemma 1.** Let $s'$: $U'/V' \rightarrow U/V$ and $s''$: $U''/V'' \rightarrow U'/V'$ be pseudo-isomorphisms. A Banach quotient $X/Y$ and pseudo-isomorphisms' $t'$: $X/Y \rightarrow U'/V'$, $t''$: $X/Y \rightarrow U''/V''$ can be found in such a way that $s' = s''$, $s''$.

We consider surjective maps $s_1$: $U \rightarrow U_1$, $s_1': U' \rightarrow U$ which induce $s'$ and $s''$, and define

\[ X = \{(x, x') \in U \times U_1 | x' = s_1(x) = s''(x') \} \]

with the norm

\[ \| (x', x')_r \| = \| x' \| + \| x'' \| \]

Then $X$ is a Banach space, $Y = V \times V$ is a Banach subspace of $X$.

Letting $t_1$: $X \rightarrow U'$ and $t_1'$: $X \rightarrow U''$ be defined by

\[ t_1(s_1(x, x')) = x', \quad t_1'(s_1(x, x')) = x'' \]

we see that $t_1, t_1'$ induce pseudo-isomorphisms $t'$: $X/Y \rightarrow U'/V'$, $t''$: $X/Y \rightarrow U''/V''$ and $s' = s''$, $s''$. Lemma 1 is proved.

We can now show that the relation $F_1(u_1)F_1(u_2) = (u_1 \circ u_2)$ defines a mapping. Let $s_1$: $U_1/V_1 \rightarrow U/V$ and $s_1$: $U_1/V_1 \rightarrow U/V$ be pseudo-isomorphisms, with $U_1/V_1$ and $U_2/V_2$ standard. Find $X/Y$ and pseudo-isomorphisms $t_1$: $X/Y \rightarrow U_1/V_1$, $t_2$: $X/Y \rightarrow U_1/V_1$ such that $s_1 = s_1 = s_2 = t_2$. We must show that $F_1(u_1)F_1(u_2) = F_1(u_1)F_1(u_2)$ if $u_1 \circ u_2 = u_1 \circ u_2$, but

\[ u_1 \circ u_2 = u \circ s_1 = s_1 \circ u = s_2 \circ t_2 = u_1 \circ t_2 \]

hence $F_1(u_1)F_1(u_2) = F_1(u_1)F_1(u_2)$. Multiply this on the right by the inverse of $F_1(t_1)F_1(t_2) = F_1(t_1)F_1(t_2)$. We obtain the required equality.

To prove that $F_1$ is a functor, we consider $u_1$: $U_1/V_1 \rightarrow U_1/V_1$, $u_2$: $U_1/V_1 \rightarrow U_1/V_1$ (i = 1, 2) and want to show that $F_1(u_1)F_1(u_2) = F_1(u_1 \circ u_2)$. Let $U_1/V'_1$ be standard Banach quotients, and $s_1$: $U_1/V'_1 \rightarrow U_1/V_1$ be pseudo-isomorphisms. Let $u_1 = u_1 \circ s_1$, and $s_1 = u_1 \circ s_1$. The morphisms $u_1, s_1$ are strict and

\[ F_1(u_1) = F_1(u_1)F_1(s_1)^{-1}, \quad F_1(u_1 \circ u_2) = F_1(u_1 \circ u_2)F_1(s_1)^{-1}. \]

We must show that

\[ F_1(u_1 \circ u_2) = F_1(u_1)F_1(s_1)^{-1}F_1(u_2). \]

This is the case, $s_2^{-1} \circ u_1$ is strict because $U_1/V'_1$ is standard. The relation $s_2 \circ s_2^{-1} \circ u_1 = u_1$ shows that $F_1(s_2^{-1} \circ u_1) = F_1(s_2^{-1} \circ u_1)$ hence

\[ F_1(u_1 \circ u_2)F_1(s_1)^{-1}F_1(u_2) = F_1(u_1)F_1(s_1)^{-1}F_1(u_2) = F_1(u_1 \circ u_2) \]

which is the desired relation.

5

**PROPOSITION 6.** $\mathcal{B}$ is an exact category. A sequence of morphisms of $\mathcal{B}$ is exact when it is exact in the category $E/V$ of vector spaces.

An exact category is one in which every mapping has a kernel, a cokernel, and in which the natural mapping of the cokernel of the kernel mapping in the kernel of the cokernel mapping is an isomorphism.

**DEFINITION 5.** Let $E/F$ be a Banach quotient. Let $E'$ be a Banach subspace of $E$ which contains $F$. Then $E'/F$ is a subobject or subquotient of $E/F$, and $E'/E$ is a quotient object. The map $E'/F \rightarrow E/F$ induced by inclusion $E' \rightarrow E$ is the inclusion map, or the canonical injection. The mapping $E'/F \rightarrow E/F$ induced by the identity $E \rightarrow E$ is the quotient map or the canonical surjection.

It is standard, in a vector category, to say that a morphism $f$ is monic if $u = 0$ whenever $i = u = 0$, and that $i$ is epic if $i = 0$ when $u \circ i = 0$, where we assume that $i \circ u$ and $i \circ u$ exist. Canonical injections are clearly monic, canonical surjections are epic (they are monic, or epic in $E/V$).

**LEMMA 2.** A bijective morphism of $\mathcal{B}$ is an isomorphism. Every Banach quotient is isomorphic to a standard one. Every morphism from a standard quotient is strict. It will be sufficient to show that a strict bijective morphism is an isomorphism. Let $u$: $E/F \rightarrow E'/F'$ be a bijective strict morphism, induced by $u_1$: $E \rightarrow E'$.

Bijection of $u$ implies that $u_1(E + F) = E'$ and $F = u_1^{-1}F$. Consider $E_1 = E \oplus F$, $F_1 = F \oplus F_1$. The linear mapping $E_1/F_1 \rightarrow E'/F'$ induced by $x \oplus y$ is a pseudo-isomorphism. The mapping $E_1/F_1 \rightarrow E/F_1$ induced by $x \oplus y$ is an isomorphism of the strict category. Its inverse is induced by the projection mapping $x \oplus y \rightarrow x$.

Let $u$ be the composition of the two isomorphisms above, it is an isomorphism.

**LEMMA 3.** Every monomorphism $u$: $E_1/F_1 \rightarrow E/F$ of $\mathcal{B}$ factors in a unique way $u = i \circ u_1$, where $u_1$: $E_1/F_1 \rightarrow E'/F'$ is an isomorphism, where $E'/F$ is a subquotient of $E/F$ and where $i$: $E/F \rightarrow E/F$ is the inclusion map.
Replacing $E_i/F_i$, eventually by an isomorphic standard quotient, we may assume that $u$ is strict. Assume that $u_1$ induces $u$.

If $u$ is monic, $F_i = u_1^*F_i$. Otherwise $u_1^*F_i/F_i$ would be a non-null subquotient of $E_i/F_i$. The composition of the canonical injection $u_1^*F_i/F_i \to E_i/F_i$ with $u_1$ would be zero, but the canonical injection would not be zero.

Let next $E' = u_1 E_i + F_i$. This is a Banach subspace of $E$. The mapping $u_1: E_1 \to E'$ induces a bijective morphism $u': E_1/F_1 \to E'/F_i$ and $u$ is the composition of the isomorphism $u'$ and the canonical injection $E'/F_i \to E/F_i$.

To show that $E'$ is determined by $u$, we assume that $u = i_1 u_2 u_3$, where $u_2$ is an isomorphism and $i_1$ a canonical injection $E_1/F_1 \to E/F_i$. Then $E_1/F_i$ is the range of the linear mapping $u$, and this implies that $E_2 = E'$.

**Lemma 4.** Every epimorphism $u: E/F \to E_0/F_0$, factors in a unique way $u = u' s$, where $u'$ is an isomorphism $E/E' \to E_1/F_1$, where $E'/F_i$ is a quotient object of $E/F$, and where $s: E/F \to E/E'$ is the canonical surjection.

Assume first that $u$ is strict (this will be the case when $E/F$ is standard). Let $u_1: E \to E_i$ induce $u$. We have the equality $u_1 E_i + F_i = E_i$, otherwise the quotient map $E_1/F_1 \to E_1(u_1 E_i + F_i)$ would not be zero, but its composition with $u_1$ would be, and $u$ would not be epic.

Let $E' = u_1^*F_i$. Then $u_1: E \to E_i$ induces a bijective morphism $E'/E' \to E_1/F_1$ and the composition of this bijective morphism and the quotient mapping $E/F \to E/E'$ is equal to $u$.

$E'$ is determined by $u$. Just as in the proof of Lemma 3, we observe that the quotient $E/E'$ is determined by the linear mapping $u$, and $E/E'$ determines $E'$.

If the epimorphism $u$ is not strict, we consider a pseudo-isomorphism $\sigma: U/V \to E/F_i$, with $U/V$ standard, then $u = u \circ \sigma$ is strict. Let $v = u \circ \sigma$. We find a quotient object $U'/U'$, and an isomorphism $U'/U' \to E_i/F_i$ in such a way that $v$ is the composition of the quotient map and the isomorphism.

The pseudo-isomorphism $\sigma$ is induced by a bounded surjection $\sigma_1: U \to E$. Let $E' = \sigma_1 U'$, then $\sigma_1$ induces a pseudo-isomorphism $U/U' \to E/E'$ and $u$ is the composition of the quotient surjection $E/(E \to E')$, the inverse of the pseudo-isomorphism $U/ U' \to E/E'$, and the isomorphism $U'/ U' \to E_i/F_i$.

**Lemma 5.** Every morphism of $qB$ has a kernel and a cokernel.

A kernel of $u$: $E_1/F_1 \to E_2/F_2$ is a monomorphism $i: U/V = E_i/F_i$, such that $u i = 0$, and such that vector factors $v = i_1 v_1$, if $u v = 0$, Applying Lemma 3, we see that we may take $U/V$ to be a subobject of $E_1/F_i$, i.e. $U = E_1 \subseteq E_i$, $V = F_i$ with $F_i \subseteq E_i \subseteq E$, and $i$ to be the canonical injection. It is standard practise to say that $E_i/F_i$ is the kernel of $u$.

If $u$ is strict, induced by $u_1: E_1 \to E_2$, its kernel is $E_1/F_1$, where $E_1 = u_1^*F_1$. If $u$ is not strict, we find a pseudo-isomorphism $\sigma: U/V \to E_i/F_i$, induced by a surjection $\sigma_1: U \to E_i$, in such a way that $u = \sigma \circ \sigma$ is strict. The morphism $v$ has a kernel $U_{1/V}$. Letting $E_1 = \sigma_1 U_{1/V}$, we see that $E_1/F_i$ is a kernel of $u$.

A cokernel of $u$: $E_1/F_1 \to E_2/F_2$ is an epimorphism $s: E_2/F_2 \to U/V$ such that $s u = 0$ and such that vector factors $v = v_1 s$, where $u v = 0$. Applying Lemma 4, we see that we may take $s' \to s$ to be a canonical surjection $s: E_2/F_2 \to E_i/F_i$. And again, it is standard practise to say that $E_2/F_2$ is the cokernel of $u$.

We may assume that $u$ is strict, replacing eventually $E_i/F_i$ by a pseudo-isomorphic standard object. Let $u_1$ induce $u$. We see that $E_1/(u_1 E_i + F_i)$ is a cokernel.

The proof of Proposition 6 is now straightforward.

6

The following two definitions are convenient but do not follow from general categorial principles. The category $B$ can be embedded in $qB$, mapping every Banach space $E$ onto the quotient $E/J$. The “exact sequences” of $B$ are the sequences of mappings of $B$ which are exact in $qB$. The definitions of exact and right-exact functors from $B$ are made to order, in order that Proposition 7 hold.

**Definition 6.** A sequence $(u, v, w): E \to E'$, $E' \to E''$ of morphisms in the category $B$ of Banach spaces is exact if it is exact in the category $E, V$ of vector spaces.

**Definition 7.** A functor $\Phi$ from $B$ into an exact category $K$ is right-exact if it maps a short exact sequence $0 \to E \to E' \to E'' \to 0$ onto a right-exact sequence $\Phi(E) \to \Phi(E') \to \Phi(E'') \to 0$. It is exact if $\Phi$ is right-exact and maps a monomorphism on a monomorphism.

**Proposition 7.** Every right-exact functor $\Phi: B \to K$ extends to a right-exact functor $\Phi: qB \to K$.

Two different extensions of $\Phi$ are isomorphic. If $\Phi$ and $\Phi'$ are right-exact $B \to K$ and if $\Phi_1$, $\Phi_2$, are right-exensions of $\Phi$ and $\Phi'$, every functor homomorphism $H: \Phi \to \Phi'$ has a unique extension $H_1: \Phi_1 \to \Phi_2$.

The right-extension of an exact functor $B \to K$ is exact $qB \to K$.

The proof is easy diagram chasing. We identify systematically a Banach space $E$ and the quotient $E/J$.

$E/F$ is the cokernel of inclusion $i: F \to E$. We define $\Phi_i(E/F) = coker \Phi(i)$.

If $u: E/F \to E'/F'$ is a strict morphism, induced by $u_1: E \to E'$, $E_i$, $F_i$ is the restriction of $u_1$, $\Phi(u) = (\Phi_1(E)) \to \Phi_2(E')$ making the following diagram commutative:

\[
\begin{align*}
\Phi(E) & \to \Phi(E') & \to \Phi(E'') \to 0 \\
\Phi(F) & \to \Phi(F') & \to \Phi(F'') \to 0
\end{align*}
\]

(where $\Phi(u_1)$, $\Phi(u_2)$ respectively map $\Phi(E) \to \Phi(E')$, $\Phi(F') \to \Phi(F'')$). It is straightforward that $\Phi(u)$ does not depend on the choice of $u_1$ inducing $u$, also that $\Phi(u \circ v) = \Phi(u) \circ \Phi(v)$. In other words, $\Phi$ is a functor $qB \to K$.

To show that $\Phi$ extends to $qB$, we must prove that it maps a pseudo-isomorphism onto an isomorphism. Let $s: E/F \to E'/F'$ be such a pseudo-isomorphism, let $s_1$ induce $s$ and be surjective, let $s_1$ be the restriction of $s_1$ to $F$, considered as a mapping from $F$ to $F'$. Then $s_1$ is surjective, and has the same kernel $K$ as $s_1$. 


We have, in \( qB \), the following commutative diagram with exact rows and columns:

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & \rightarrow & k & \rightarrow & F & \rightarrow & 0 \\
0 & \rightarrow & k & \rightarrow & E & \rightarrow & 0 \\
0 & \rightarrow & E_1/F_1 & \rightarrow & E/F & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

Apply the functor \( \Phi \) to the two upper rows of this diagram, \( \Phi_1 \) to the third row, map \( \Phi(E_1) \rightarrow \Phi_1(E_1/F_1) \), \( \Phi(E) \rightarrow \Phi_1(E/F) \) by the cokernel mappings. We obtain a commutative diagram in which the two upper rows and the three columns are right-exact.

\[
\begin{array}{cccc}
\Phi(K) & \rightarrow & \Phi(F_1) & \rightarrow & \Phi(F) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Phi(K) & \rightarrow & \Phi(E_1) & \rightarrow & \Phi(E) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & \Phi_1(E_1/F_1) & \rightarrow & \Phi_1(E/F) & \rightarrow & 0
\end{array}
\]

It is well known that the exactness of the third row follows, i.e. \( \Phi_1(E_1/F_1) \rightarrow \Phi_1(E/F) \) is an isomorphism.

The next item on the agenda is a proof that \( \Phi_1 \) is right-exact, and that \( \Phi_1 \) is exact when \( \Phi \) is exact. Every monomorphism of \( qB \) is equivalent to canonical injection, every epimorphism is equivalent to a canonical surjection (Lemmas 3 and 4). It will be sufficient to show that \( \Phi_1 \) maps a short exact sequence such as

\[
\begin{array}{cccc}
0 & \rightarrow & F & \rightarrow & E & \rightarrow & 0 \\
0 & \rightarrow & F & \rightarrow & E & \rightarrow & 0
\end{array}
\]

onto a right-exact sequence, even onto an exact sequence when \( \Phi \) is exact.

We consider the commutative diagram with exact rows and columns in \( qB \)

\[
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
0 & \rightarrow & F & \rightarrow & E & \rightarrow & 0 \\
0 & \rightarrow & E & \rightarrow & E' & \rightarrow & 0 \\
0 & \rightarrow & E/F & \rightarrow & E'/F' & \rightarrow & 0 \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

and deduce from this a commutative diagram right in \( K \) whose upper two rows and whose three columns are right-exact in \( K \) (when \( \Phi \) is right-exact).

\[
\begin{array}{cccc}
\Phi(F) & \rightarrow & \Phi(F) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow \\
\Phi(E) & \rightarrow & \Phi(E) & \rightarrow & \Phi(E') & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\Phi_1(E/F) & \rightarrow & \Phi_1(E/F) & \rightarrow & \Phi_1(E/E') & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \rightarrow & 0 & \rightarrow & 0 & \rightarrow & 0
\end{array}
\]

The mappings of this diagram are, respectively the images by \( \Phi \) or \( \Phi_1 \) of the corresponding mappings in \( B \) or \( qB \), or are suitable cokernel mappings. The upper two rows of the diagram, and all three columns are right-exact. The third row is therefore right-exact.

If the functor \( \Phi \) is exact, the morphisms \( \Phi(F) \rightarrow \Phi(E), \Phi(F) \rightarrow \Phi(E), \Phi(E') \rightarrow \Phi(E), \Phi(E) \rightarrow \Phi(E) \), and of course \( \Phi(F) \rightarrow \Phi(F) \) are monic. The mapping \( \Phi_1(E/F) \rightarrow \Phi_1(E/F) \) is then monic, \( \Phi_1 \) is exact.

We must still show that the extension \( \Phi_1 \) is unique modulo isomorphism, and that every functor homomorphism \( H : \Phi \rightarrow \mathcal{V} \) has a unique extension \( H_1 : \Phi_1 \rightarrow \mathcal{V} \). We will of course be sufficient to show that every functor homomorphism has a unique extension. If \( \Phi_1 \) and \( \Phi_2 \) are two extensions of the same functor \( \Phi \), the identity \( \Phi \rightarrow \Phi \) will extend to functor homomorphisms \( \Phi_1 \rightarrow \Phi_2, \Phi_2 \rightarrow \Phi_1 \).

The compositions of these homomorphisms are functor homomorphisms \( \Phi_1 \rightarrow \Phi_1, \Phi_2 \rightarrow \Phi_2 \), extending the identity \( \Phi \rightarrow \Phi \), these compositions are the identity homomorphisms \( \Phi_1 \) and \( \Phi_2 \) are isomorphic functors.

Let \( \Phi_1 \) and \( \mathcal{V} \) be right-exact functors \( qB \rightarrow K \) extending \( \Phi \) and \( \mathcal{V} \), and let \( H : \Phi \rightarrow \mathcal{V} \) be a functor homomorphism. There cannot be more than one extension \( H_1 : \Phi_1 \rightarrow \mathcal{V} \) of \( H \). Consider a \( qB \) space \( E/F \). We have the right-exact sequences \( \Phi(F) \rightarrow \Phi(E) \rightarrow \Phi_1(E/F) \rightarrow 0, \mathcal{V}(F) \rightarrow \mathcal{V}(E) \rightarrow \mathcal{V}(E/F) \rightarrow 0 \); only one mapping \( \Phi_1(E/F) \rightarrow \mathcal{V}(E/F) \) makes the following diagram commutative:

\[
\begin{array}{cccc}
\Phi(F) & \rightarrow & \Phi(E) & \rightarrow & \Phi_1(E/F) & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{V}(F) & \rightarrow & \mathcal{V}(E) & \rightarrow & \mathcal{V}(E/F) & \rightarrow & 0
\end{array}
\]

(The mappings \( \Phi(F) \rightarrow \mathcal{V}(F), \Phi(E) \rightarrow \mathcal{V}(E) \) in this diagram are \( H(F) \) and \( H(E) \) respectively.) This proves that the extension is unique if it exists.

\( H_1(E/F) \) can be defined in this way. The left-hand square of the above commutative diagram can be constructed when only \( \Phi \) and \( H \) are given. \( H_1(E/F) \) is the morphism of the cokernel of \( \Phi(F) \rightarrow \Phi(E) \) into the cokernel of \( \mathcal{V}(F) \rightarrow \mathcal{V}(E) \) which makes the diagram commutative.

To prove that \( H_1 \) defined in this way is a functor homomorphism, it is sufficient (Proposition 5) to prove that it is such a homomorphism over \( qB \), i.e. to prove that

\[
\Phi_1(u) \circ H_1(E/F) = H_1(E/F') \circ \mathcal{V}(u)
\]

when \( u : E/F \rightarrow E'/F' \) is a strict morphism. We assume that \( u_1 : E \rightarrow E' \) induces \( u \) and consider the restriction \( u_1 : F \rightarrow F' \) of \( u_1 \).

We have the two commutative squares

\[
\begin{array}{cccc}
\Phi(F) & \rightarrow & \Phi(F') & \rightarrow & \Phi(E') & \rightarrow & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathcal{V}(F) & \rightarrow & \mathcal{V}(F') & \rightarrow & \mathcal{V}(E') & \rightarrow & 0
\end{array}
\]

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and mappings $\Phi(F) \to \Phi(E)$, $\Phi(F') \to \Phi(E')$, $\Psi(F) \to \Psi(E)$ and $\Psi(F') \to \Psi(E')$ building up this diagram to a commutative cube. The square

$$
\begin{align*}
\Phi_{1}(E/F) & \to \Phi_{1}(E'/F') \\
\Psi_{1}(E/F) & \to \Psi_{1}(E'/F')
\end{align*}
$$

is the cokernel of this mapping of commutative squares. It is known that this cokernel is again a commutative square. Proposition 7 is proved.


References


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**QUOTIENT BANACH SPACES; MULTILINEAR THEORY**

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The multilinear structure of the category $qB$ is defined by putting a $qB$-structure on the vector space $qB(E/F, E'/F')$. Multilinear mappings $E_1/F_1 \times \ldots \times E_n/F_n \to E'/F'$ are defined by induction.

Strict multilinear mappings are specially interesting. These are induced by bounded multilinear mappings $a_i: E_i \to E'$ such that $a(x_1, \ldots, x_n) \in E'$ as soon as one of the $x_i$ belongs to the corresponding $F_i$. All $qB$-multilinear maps $E_1/F_1 \times \ldots \times E_n/F_n \to E'/F'$ are strict if the $E_i/F_i$ are standard $qB$-spaces.

The tensor product which can be defined in $qB$ is a right-exact functor as it should be. It is unfortunately not an extension of the tensor product which is defined in the category of Banach spaces. If $F$ is the closure of $E$, $E/F$ is the “Banachization” of $E/F$. The projective tensor product of two Banach spaces is the Banachization of their $qB$-tensor product.

We are interested in $qB$-algebras. These are $qB$-spaces $A$ with a bilinear multiplication belonging to $q_{2}(A, A, A)$. The $qB$-algebra is strict if its multiplication is a strict bilinear mapping. It is commutative, or associative if its multiplication is commutative, or associative. The structure of a $qB$-subalgebra can be put on the center of a $qB$-algebra.

Every $qB$-algebra is isomorphic with a strict $qB$-algebra. A strict $qB$-algebra is the quotient of a Banach algebra by a two-sided Banach ideal. An associative $qB$-algebra is isomorphic with the quotient $A/x$ of an associative Banach algebra by a Banach ideal. The isomorphic $A/x$ can even be chosen in such a way that $Z(A/x) = Z(A)/x$, where $Z(A)$ and $Z(A/x)$ are the centers of $A$ and of $A/x$.

Every commutative and associative $qB$-algebra is isomorphic with the quotient of a commutative and associative Banach algebra by a Banach ideal.

This paper is a sequel of [2].

1

Let $E/F$ and $E'/F'$ be $qB$-spaces. Call $qB(E/F, E'/F')$ the space of bounded linear mappings $E \to E'$ which map $F$ into $F'$, and $qB(E/F, E'/F')$ the space of bounded