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A PERTURBATION OF LOMONOSOV'S THEOREM

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One of the consequences of Lomonosov's remarkable work is the following: if A commutes with some non-scalar operator which commutes with a non-zero compact operator, then A has an invariant subspace (cf. [3], [5], [6]). In this talk I describe a generalization of this result obtained by the committee consisting of C. K. Fong, E. Nordgren, M. Radjabalipour, H. Radjavi, and the speaker; for details see [2] (which also generalized [4]). A consequence of this committee-work is: if A commutes with some non-scalar operator B which satisfies an equation of the form $BK = K\varphi(B)$, where K is a non-zero compact operator and φ is an analytic function mapping some bounded open set containing $\sigma(B)$ into itself, then A has an invariant subspace. The first sentence above is the case where $\varphi(z) = z$.

To describe the main theorem of [2] we need some notation. Let $\mathscr{X}$ be a complex Banach space and let $\mathscr{A}$ be an algebra of bounded operators on $\mathscr{X}$. We assume that $\mathscr{A}$ is an operator range, in the sense that there is a continuous linear mapping $\mathscr{I}$ from some Banach space $\mathscr{Y}$ into the space of bounded operators on $\mathscr{X}$ such that the range of $\mathscr{I}$ is $\mathscr{A}$. Suppose also that there exist non-zero compact operators K_1 and K_2 such that $\mathscr{A}K_1 \subset K_2\mathscr{A}$ (i.e., for each A in $\mathscr{A}$ there exists an A' in $\mathscr{A}$ such that $AK_1 = K_2A'$).

THEOREM ([2]). *Under the above hypotheses, there is a non-trivial subspace invariant under $\mathscr{A}$ and, unless $\mathscr{A}$ consists of multiples of the identity, there is also a non-trivial subspace invariant under the commutant of $\mathscr{A}$.*

Note that $\mathscr{A}$ can be non-commutative, in which case the two assertions of the theorem are quite distinct.

COROLLARY ([2]). *If $BK = K\varphi(B)$, where K is a non-zero compact operator and φ is an analytic function mapping a bounded open set containing $\sigma(B)$ into itself, then B has a hyperinvariant subspace unless B is a multiple of the identity.*

The Corollary follows from the Theorem by taking $\mathscr{A}$ to be

$$\{\Psi(B)\colon \Psi \text{ bounded and analytic on } \mathscr{D}\},$$

where $\mathscr{D}$ is the given open set.

The propositions of the first paragraph above are special cases of the Corollary. As has been pointed out by Pearcy and Shields [5], it is not known if there are any operators A which do not satisfy the hypotheses of the consequence of Lomonosov's work; (Cowen [1] has recently shown that the unilateral shift commutes with an operator which commutes with a compact operator). Thus we certainly do not know of any operators which fail to satisfy the more general criteria following from [2].

Added in proof. There are operators that do not satisfy Lomonosov's hypotheses; see the paper by Haduin, E. Nordgren, H. Radjavi and P. Rosenthal in J. Funct. Anal. 38 (1980), 410–415.

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SOME QUESTIONS IN OPERATOR THEORY AND APPLICATIONS IN ANALYSIS

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The questions fall into two areas. In both cases, full accounts will appear elsewhere in [1], [2].

1. Invariant subspaces of finite convolution operators

A finite convolution operator is an operator on $L^2(0, 1)$ having the form

$$T: f(x) \to \int_0^x k(x-t)f(t)\,dt,$$

where $k \in L^1(0, 1)$. By a symbol for such an operator we mean any function of the form

$$A(z) = \int_0^1 e^{itz}k(t)\,dt + e^{iz}G(z),$$

where $G(z)$ is analytic and bounded in some half-plane $y > \eta$, where η is a real number. A survey is made of recent progress on the problems of giving conditions on symbols for the unicellularity and similarity of such operators, and, more generally, of arbitrary operators in the commutant of the integration operator

$$J: f(x) \to \int_0^x f(t)\,dt$$

on $L^2(0, 1)$. A table of known examples is given. A section of the paper lists open problems in the theory.

2. Cayley inner functions and applications in analysis

A Cayley inner function is defined to be any analytic function $\xi(z)$ satisfying $\xi(z) = \xi(z^*)^*$ for $z \neq z^*$ (if $z = x+iy$, then $z^* = x-iy$), $\mathrm{Im}\,\xi(z) > 0$ for $\mathrm{Im}\,z > 0$, such that $\xi(x) = \xi(x+i0)$ is real a.e. on the real axis. We say that ξ maps a real