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ON THE EQUATIONS $X = KXS$ AND $AX = XK$

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In [6] it was shown that all solutions of the first equation are finite-rank operators, and a simpler proof of this fact was given in [5]. Here we observe that this result can be derived as a corollary of Rosenblum's Theorem. We also obtain a similar theorem concerning the equation $AX = XK$ for suitable operators A.

All operators are bounded linear transformations on complex Banach spaces; in Corollary 1 we restrict our attention to operators on Hilbert space.

THEOREM 1 ([6]). *If K is compact and S is any operator, then all solutions X of the equation $X = KXS$ have finite rank.*

Proof. Suppose $X = KXS$. Choose a complex number λ such that $S+\lambda$ and $\frac{1}{\lambda}+K$ are both invertible. Then

$$KX(S+\lambda) = KXS+\lambda KX = (1+\lambda K)X,$$

so

$$X(S+\lambda)^{-1} = (1+\lambda K)^{-1}KX.$$

Choose a circle Γ about 0 which does not intersect the spectrum of $(1+\lambda K)^{-1}K$ and whose interior does not intersect the spectrum of $(S+\lambda)^{-1}$. Now let

$$P = \frac{1}{2\pi i}\int_{\Gamma}[z-(1+\lambda K)^{-1}K]^{-1}dz.$$

Then P is an idempotent which commutes with $(1+\lambda K)^{-1}K$, so

$$(PX)(S+\lambda)^{-1} = P(1+\lambda K)^{-1}K(PX).$$

The spectrum of $P(1+\lambda K)^{-1}K$ is inside Γ, and hence is disjoint from the spectrum of $(S+\lambda)^{-1}$. It follows from Rosenblum's theorem ([9]; [8], Corollary 0.13) that $PX = 0$. Thus $X = PX+(1-P)X = (1-P)X$. Since $1-P$ has finite rank, so does X.

The next corollary is implicit in [6] and [7] and explicit in [5] (Theorem 2), where it is proved for arbitrary Banach spaces.

COROLLARY 1 ([6], [7], [5]). *If K is a compact operator on a Hilbert space $\mathscr{H}$, if $\mathscr{M} \subset \mathscr{H}$ is the range of some operator mapping a Hilbert space $\mathscr{K}$ into $\mathscr{H}$, and if $\mathscr{M} \subset K\mathscr{M}$, then $\mathscr{M}$ is finite-dimensional.*

Proof ([5]). Let $X\colon \mathscr{K} \to \mathscr{H}$ have range $\mathscr{M}$. Then the range of X is contained in that of KX, so the theorem of Douglas and Halmos [2] implies that $X = KXS$ for some operator S. By Theorem 1, S has finite rank, so $\mathscr{M}$ has finite dimension.

COROLLARY 2. *If A is invertible and K is compact, then all the solutions X of $KX = XA$ have finite rank.*

Proof. From $XA = KX$ we get $X = KXA^{-1}$, so the result is an immediate consequence of Theorem 1.

COROLLARY 3. *If A is invertible and K is compact, then all the solutions X of $AX = XK$ have finite rank.*

Proof. Taking Banach space adjoints reduces this to Corollary 2.

By using a similar proof to that of Theorem 1 and quoting the extension of Rosenblum's Theorem given in [1], the following generalization of Corollary 3 can be given.

THEOREM 2. *If 0 is not in the approximate point spectrum of A and K is compact, then all the solutions X of $AX = XK$ have finite rank.*

Proof. The approximate point spectrum $\Pi(A)$ of the operator A is closed. Thus there exists a circle Γ about 0 whose interior does not intersect $\Pi(A)$; in addition, Γ can be chosen so that it does not intersect the spectrum of K. Let

$$P = \frac{1}{2\pi i}\int_{\Gamma}(z-K)^{-1}dz.$$

Then $A(XP) = XKP = (XP)KP$.

Since $\Pi(A)$ does not intersect the spectrum of KP, Theorem 4 of [1] implies that $XP = 0$, so $X = X(1-P)$ has finite rank.

COROLLARY 4. *If $0 \notin \Pi(A)$ and A has no point spectrum, and if K is a compact operator, then the only solution of $AX = XK$ is $X = 0$.*

Proof. Suppose $AX = XK$. By Theorem 2, the range of X is finite dimensional. The equation $AX = XK$ implies that the range of X is invariant under A. If the range of X was not $\{0\}$, then A would have a finite-dimensional invariant subspace and thus would have point spectrum. Therefore $X = 0$.

Remarks. (i) In [6] and [5], Theorem 1 is used to prove certain extensions of Lomonosov's Theorem on invariant subspaces.

(ii) Corollaries 2 and 3 above imply, in particular, that an invertible operator is not a quasi-affine transform (in the sense of [10]) of a compact operator. The fact that an invertible operator cannot be quasi-similar to a compact operator is a special case of the theorem of Fialkow [3] and Williams [11] that quasi-similar operators have intersecting essential spectra.

(iii) The full force of the compactness of K is not required in the above results. It suffices to assume that K is a Riesz operator; i.e., that K has at most countably many eigenvalues which accumulate (if at all) only at 0, that 0 is in the spectrum and that all other spectrum is point spectrum of finite multiplicity.

(iv) A number of interesting results on the equation $AX-XB = Y$ are given in [4].

Added in proof. S. Grabiner, in a paper *Spectral consequences of the existence of intertwining operators*, Comm. Math. 22 (1981), 227–238, has obtained results similar to Theorem 2.

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