Now $U$ has a spectral representation $U \sim \sum U(\mu) \otimes \sum U^2(\mu)$. But since $L^2(\mu) \sim L^2(\mu) \otimes L^2(\mu)$ we can just as well write for a spectral representation of $U$, $U \sim \sum L^2(\mu) \otimes \sum L^2(\mu) \otimes L^2(\mu)$. Letting $M$ be the subspace $M = \sum L^2(\mu)$ and $N$ the subspace $N = \sum L^2(\mu) \otimes \sum L^2(\mu)$, we are in a position where Proposition 3 is applicable; every operator that commutes with $U$ has $M$ and $N$ as reducing subspaces.

Let $A$, $B$, and $W$ be the following operators: $A$ is the backward shift on $M = \sum L^2(\mu)$, i.e. $A: (f_1, f_2, \ldots) \mapsto (f_2, f_3, \ldots)$. The (representation of elements of $\sum L^2(\mu)$ as sequences) are self-explanatory. On $N$, define $A$ to be zero. On $M$ let $B$ be the operator $B: (f_1, f_2, f_3, \ldots) \mapsto (0, f_2, f_3, \ldots)$, and on $N$ let $B$ equal zero. ($B$ is an orthogonal projection.) And let $W = A$. Finally, let the role of $H$ in Theorem 2" be played here by the subspace which is the range of $B$.

It is straightforward to check that $A = WB$ and that all conditions of the factorization of Theorem 2" are met. But can there be an invertible operator $D$ that commutes with $U$ and maps $AH$ into $H$? From Proposition 3 we have seen that such an operator $D$ would have to map $M$ one-to-one onto $M$. But $AH = M$ whereas $H$ is a proper subspace of $M$. Thus $D$ could not map $AH$ into $H$.

References


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DISTRIBUTION OF EIGENVALUES AND NUCLEARITY

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In this paper we shall use the terminology introduced in [6]. In particular, $\Omega(E, F)$ denotes the set of all (boundedly linear) operators from the Banach space $E$ into the Banach space $F$. Since we are concerned with spectral properties of operators, all Banach spaces under consideration are supposed to be complex.

1. $\mathfrak{S}^1_{\Omega}$-operators

Let $S \in \Omega(E, F)$ and put

$$N(\lambda, S) := \bigcup_{n=1}^{\infty} \{ x \in E : (I_x - S^n)x = 0 \}.$$ 

Here $I_x$ denotes the identity map of $E$. If $N(\lambda, S) \neq \emptyset$, then $\lambda \in \mathbb{C}$ (complex field) is called an eigenvalue of $S$ and

$$a(\lambda, S) := \dim(N(\lambda, S))$$

is said to be its algebraic multiplicity.

Let $0 < p < \infty$. An operator $S \in \Omega(E, F)$ is of Riesz type $I_p$ if

$$\sum_{\mathfrak{t} \in \mathbb{T}} a(\lambda(S), \mathfrak{t}) \lambda^p < \infty$$

for all $L \in \Omega(F, E)$.

The class of these operators will be denoted by $\mathfrak{S}^1_{\Omega}$.

Remark. If $S \in \mathfrak{S}^1_{\Omega}(E, F)$, then we have

$$\sum_{\mathfrak{t} \in \mathbb{T}} a(\lambda(S), \mathfrak{t}) \lambda^p = \sum_{\mathfrak{t} \in \mathbb{T}} \overline{a(\lambda, S)} \lambda^p,$$

where $(\lambda(S), i \in \mathbb{R}^*)$ is the (countable) family of all eigenvalues $\lambda \neq 0$ repeated according to their (finite) algebraic multiplicities.

In order to check the following result we need an elementary consequence of the spectral mapping theorem; [1], VII.3.19.

Lemma. Let $0 < p < \infty$ and $n = 1, 2, \ldots$ Then

$$\sum_{\mathfrak{t} \in \mathbb{T}} a(\lambda(S), \mathfrak{t}) \lambda^p = \sum_{\mathfrak{t} \in \mathbb{T}} a(\lambda(S), \mathfrak{t}) \lambda^p$$

for all $S \in \Omega(E, F)$. [361]
We are now prepared to prove

**Proposition 1.** Let \(0 < p < \infty\) and \(n = 1, 2, \ldots\). Then for every operator ideal \(\mathcal{U}\) the inclusions \(\mathcal{U} \subseteq \mathcal{U}^{\mathbb{N}}\) and \(\mathcal{U} \subseteq \mathcal{U}^{\mathbb{N}}\) are equivalent.

**Proof.** Suppose that \(\mathcal{U} \subseteq \mathcal{U}^{\mathbb{N}}\). If \(S \in \mathcal{U}(E, F)\) and \(L \in \mathcal{U}(F, E)\), then there exists a factorization

\[
LS: E \to M_1 \to \cdots \to M_k \to E
\]

such that \(T_i \in \mathcal{U}(M_{i+1}, M_i)\) for \(k = 1, \ldots, n\). Form the Cartesian product \(M := M_1 \times \cdots \times M_n\) equipped with any suitable norm. Then by

\[
\mathcal{Y}: (x_1, \ldots, x_n, \ldots) \to (T_1 x_1, T_2 x_2, \ldots)
\]

we define an operator \(T \in \mathcal{U}(M, M)\). Observe that \(E\) can be identified with the subspace \((0) \times \cdots \times (0) \times M_n\) of \(M\) which is invariant under \(T^*\). Moreover, the restriction of \(T^*\) to \(E\) coincides with \(LS = T_1 \cdots T_n\). So by the preceding lemma, we have

\[
\sum_{\ell \in \mathcal{C}} |a(\mu, LS)| \mu |\ell|^p \leq \sum_{\ell \in \mathcal{C}} |a(\lambda, T^*)| \lambda |\ell|^p = \sum_{\ell \in \mathcal{C}} |a(\lambda, T)| \lambda |\ell|^p < \infty.
\]

Therefore \(S \in \mathcal{U}(E, F)\). This proves that \(\mathcal{U} \subseteq \mathcal{U}^{\mathbb{N}}\). In order to check the converse implication we suppose that \(\mathcal{U} \subseteq \mathcal{U}^{\mathbb{N}}\). If \(S \in \mathcal{U}(E, F)\) and \(L \in \mathcal{U}(F, E)\), then \((LS)^* \in \mathcal{U}(E, F)\). Hence

\[
\sum_{\ell \in \mathcal{C}} |a(\lambda, LS)| \lambda |\ell|^p = \sum_{\ell \in \mathcal{C}} |a(\mu, (LS)^*)| \mu |\ell|^p < \infty.
\]

Therefore \(S \in \mathcal{U}(E, F)\). This proves that \(\mathcal{U} \subseteq \mathbb{U}^{\mathbb{N}}\).

**Proposition 2.** If \(X \in \mathcal{U}(E, F)\), \(S \in \mathcal{U}^{\mathbb{N}}(E, F)\), and \(B \in \mathcal{U}(F, F_0)\) then \(BSX \in \mathcal{U}^{\mathbb{N}}(E_0, F_0)\).

**Proof.** Let \(L_0 \in \mathcal{U}(F_0, F_0)\). Then the operators \(L_0 BSX\) and \(X_{L_0} BS\) are related; cf. [6], 273.1. Therefore we have

\[
\sum_{\ell \in \mathcal{C}} |a(\lambda, BSX)| \lambda |\ell|^p = \sum_{\ell \in \mathcal{C}} |a(\lambda, BSX)| \lambda |\ell|^p < \infty.
\]

This proves the assertion.

Next we show that \(\mathcal{U}^{\mathbb{N}}\) is not an operator ideal. This yields a negative answer to a problem which has been posed in 1969; cf. [5].

**Proposition 3.** Let \(0 < p < \infty\) and \(n = 1, 2, \ldots\). Then there are a Banach space \(E\) as well as operators \(S_1, S_2 \in \mathcal{U}^{\mathbb{N}}(E, E)\) such that \(S_1 + S_2 \notin \mathcal{U}^{\mathbb{N}}(E, E)\).

**Proof.** Choose a natural number \(n\) and a real number \(q\) such that \(2^n > 2q > (2n-1)p \geq 4\). Take any sequence \((a_i) \in \ell_2^n\), and let \((b_i) \in \ell_{2n-1}\), and define the diagonal operator \(S \in \mathcal{U}(E, E)\) by \(S(\xi) := (\xi, \xi, \ldots)\). Furthermore, let \(J \in \mathcal{U}(E, E)\) be the canonical embedding.

In the following \(\mathcal{P}\) stands for the ideal of absolutely \(r\)-summing operators. Obviously we have \(S \in \mathcal{P}_p(E, E)\). It has been proved in [2] that \(\mathcal{P}_p \subseteq \mathcal{U}^{\mathbb{N}}\). So from Proposition 1 we get \(\mathcal{U}^{\mathbb{N}} \subseteq \mathcal{P}_p^{\mathbb{N}}\). Therefore \((JS)^{-1} \in \mathcal{P}_p^{\mathbb{N}}\).

On the other hand, by [6], 22.4.2, we have \(U(\ell_p, \ell_q) \subseteq \mathcal{P}_p(\ell_p, \ell_q)\). This implies that \((JS)^{-1} \in \mathcal{P}_p^{\mathbb{N}}(E, E)\) for all \(E \in \mathcal{P}_p(\ell_p, \ell_q)\). Therefore \((JS)^{-1} \in \mathcal{U}^{\mathbb{N}}(E, E)\).

Form the Cartesian product \(E := \ell_p \times \ell_q\) equipped with any suitable norm. Then the operators \(S_1, S_2 \in \mathcal{U}(E, E)\) defined by

\[
S_1: (x, y) \to (S(JS)^{-1} y, 0)
\]

and

\[
S_2: (x, y) \to (0, (JS)^{-1} x)
\]

are of Riesz type \(\ell_p\). It follows from

\[
S_1 + S_2(\xi, \xi, \ldots) \to (0, (JS)^{-1} 0)
\]

that \(\lambda S_1 + S_2 \in \mathcal{P}_p^{\mathbb{N}}(E, E)\). This implies that \(S_1 + S_2 \notin \mathcal{U}^{\mathbb{N}}(E, E)\).

**Remark.** If \(p = 2\) or \(p = 1\), then the above proof can be essentially simplified.

In contrast to the preceding result it turns out that \(\mathcal{U}^{\mathbb{N}}(H, H)\) is an ideal in the operator algebra \(\mathcal{U}(H, H)\) of the separable infinite-dimensional Hilbert space \(H\). More precisely, if \(\mathcal{S}_p(H, H)\) denotes the Schatten class of type \(\ell_p\), then we have

**Proposition 4.** Let \(0 < p < \infty\). Then \(\mathcal{S}_p^{\mathbb{N}}(H, H) \subseteq \mathcal{U}^{\mathbb{N}}(H, H)\).

**Proof.** Obviously, \(\mathcal{S}_p^{\mathbb{N}}(H, H) \subseteq \mathcal{U}^{\mathbb{N}}(H, H)\) is immediate consequence of Weyl’s Theorem; cf. [6], 274.3.

The converse inclusion can be checked in two steps. First we observe that every operator \(S \in \mathcal{U}^{\mathbb{N}}(H, H)\) is approximable. Otherwise, by [6], 5.1.1 (Lemma 3), there would exist operators \(B, X \in \mathcal{U}(H, H)\) such that \(BSX = I_2\). This is a contradiction by Proposition 2. Now it is clear that every operator \(S \in \mathcal{U}^{\mathbb{N}}(H, H)\) admits a Schmidt factorization; cf. [6], D.3.3. In other terms, there are operators \(U \in \mathcal{U}(H, I_2)\) and \(V \in \mathcal{U}(I_2, H)\) as well as a diagonal operator \(S_0 \in \mathcal{U}(I_2, I_2)\) generated by a sequence \((a_i) \in \ell_2\) such that \(S = V S_0 U^*\) and \(S_0 = V^* S U\). Therefore \(S_0 \in \mathcal{U}^{\mathbb{N}}(I_2, I_2)\), and it follows from

\[
\sum_{i \in \mathbb{N}} |a_i|^p = \sum_{i \in \mathbb{N}} |a(\lambda, S_0)| \lambda |\ell|^p < \infty
\]

that \(S_0 \in \mathcal{S}_p(I_2, I_2)\). So we also have \(S \in \mathcal{S}_p(H, H)\). This completes the proof.

**Proposition 5.** If \(\mathcal{U}\) is an operator ideal such that \(\mathcal{U} \subseteq \mathcal{U}^{\mathbb{N}}\), then \(\mathcal{U} \subseteq \mathcal{P}_p\).

**Proof.** Suppose that \(S \in \mathcal{U}(F, F)\). Let \((a_i)\) be any weakly 2-summable sequence in \(E\). Choose functions \(b_i \in F\) such that \(\langle S(x_i), b_i \rangle = \|Sx_i\|\) and \(\|b_i\| = 1\). Take \((\beta_i) \in I_2\). Then by

\[
X: (x_i) \to \sum_{i \in \mathbb{N}} \beta_i x_i
\]
and define operators $X \in \mathfrak{U}(l_1, F)$ and $Y \in \mathfrak{U}(F, l_2)$. By Proposition 4 it follows that $\mathfrak{U} \in \mathfrak{U}(l_1, F) \subseteq \mathfrak{U}(l_1, l_2) \subseteq \mathfrak{U}(l_1, l_2)$. Using [6], 15.4.3 we see that

$$\sum_{n=1}^{\infty} |b_n| |\langle Sx_n, x_n \rangle| = \sum_{n=1}^{\infty} |b_n| |\langle Sx_n, x_n \rangle| < \infty$$

for all $(b_n) \in l_2$. Hence the sequence $(Sx_n)$ is absolutely 2-summable. This proves that $S \in \mathfrak{U}(E, F)$.

The following $\mathfrak{N}$ denotes the ideal of nuclear operators.

THEOREM. Let $0 < p < \infty$. If $\mathfrak{N}$ is an operator ideal such that $\mathfrak{N} \subseteq \mathfrak{U}^{2\mathfrak{N}}$, then $\mathfrak{N}^{\mathfrak{N}} \subseteq \mathfrak{N}$ whenever $n > p$.

Proof. By Proposition 1, we have $\mathfrak{N} \subseteq \mathfrak{U}^{2\mathfrak{N}}$. Now Proposition 5 implies that $\mathfrak{N} \subseteq \mathfrak{P}_1$. Therefore $\mathfrak{N}^{\mathfrak{N}} \subseteq \mathfrak{P}_1 \subseteq \mathfrak{R}$; cf. [6], 26.4.6.5.

Let us recall that $\mathfrak{N}$, the ideal of Goldberg operators, is the largest operator ideal possessing the property that every $S \in \mathfrak{N}(E, F)$ is a Riesz operator; cf. [6], 26.4.7.2. It is well known that $\mathfrak{N}$ contains all operators $S \in \mathfrak{U}(E, F)$ which have some compact power $S^k$.

COROLLARY. Let $0 < p < \infty$. If $\mathfrak{N}$ is an operator ideal such that $\mathfrak{N} \subseteq \mathfrak{U}^{2\mathfrak{N}}$, then $\mathfrak{N} \subseteq \mathfrak{N}$.

2. Examples

Let $\mathfrak{N}_{0, 2, 2, 1}$ with $1 < r < \infty$ denote the ideal of absolutely $(r, 2, 2, 1)$-summing operators; cf. [6], 17.1.1.

Clearly $\mathfrak{N}_{0, 2, 2, 1} \subseteq \mathfrak{U}^{2\mathfrak{N}}$. On the other hand, since $\mathfrak{N}_{0, 2, 2, 1}$ contains the identity map of $l_1$, we have $\mathfrak{N}_{0, 2, 2, 1}$ non $\subseteq \mathfrak{U}^{2\mathfrak{N}}$ for $0 < p < \infty$. These borderline cases support König's

CONJECTURE 1. If $1 < r < 2$ and $1/p = 1/r - 1/2$, then $\mathfrak{N}_{0, 2, 2, 1} \subseteq \mathfrak{U}^{2\mathfrak{N}}$.

As shown in [3] we have a somewhat weaker inclusion, namely $\mathfrak{N}_{0, 2, 2, 1} \subseteq \mathfrak{U}^{2\mathfrak{N}}$, for all $p > 0$. This, however, is enough to establish

PROPOSITION 6. If $1 < r < 2$ and $n > 2r/(2-r)$, then $\mathfrak{N}_{0, 2, 2, 1} \subseteq \mathfrak{N}$.

Let $\mathfrak{N}_{0, 2, 2, 1}$ with $2 < p < \infty$ denote the ideal of absolutely $(p, 2, 2, 1)$-summing operators; cf. [6], 17.2.1.

For these operator ideals we now formulate König's

CONJECTURE 2. If $2 < p < \infty$, then $\mathfrak{N}_{0, 2, 2, 1} \subseteq \mathfrak{U}^{2\mathfrak{N}}$.

Remark. At present it seems to be unknown whether every $\mathfrak{U}_{0, 2, 2}$ is contained in some $\mathfrak{U}^{2\mathfrak{N}}$. The only result along this line is the inclusion $\mathfrak{U}_{0, 2, 2} \subseteq \mathfrak{U}^{2\mathfrak{N}}$ for $q > 2p/(4-p)$ and $2 < p < 4$ which has been recently checked by König. Moreover, we have $\mathfrak{U}_{0, 2, 2} \subseteq \mathfrak{N}$ for $n > p/2$, where $\mathfrak{N}$ denotes the ideal of compact operators; cf. [4].

References