Remark 2. If $Z (= S)$ is a group, and $\gamma(t) = f_t$, then the above statement is just the assertion of Naimark dilation theorem — see [4], [5], [6].

Remark 3. Using standard arguments of dilation theory — see [3], [5], one can prove suitable theorems, which explain how some continuity properties of $B$ and $F(\cdot, \cdot)$ imply such properties of $\sigma(\cdot)$.

References


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In 1949 Beurling gave the now famous factorization of analytic functions on the unit disk as the product of an inner function and an outer function. Years earlier Szegő had solved the somewhat similar problem of describing the positive functions on the unit circle that can be written as the modulus of an analytic function. (The best known contemporary reference for these theorems is Hoffman’s book [3].)

Within the past few decades many theorems in operator theory have exhibited some resemblance to the above results. Not always has the relationship between the operator theory and the classical function theory been clear. The purpose here is to describe one direction in which such research has moved, to tie together some theorems that have appeared in the past few years, and to answer a few previously unanswered questions concerning these theorems. Most of the ideas contained herein arose in conversations with Ralph Gellar. In particular, the multiplicity theorem (Theorem 4) was his conjecture.

First we briefly review the necessary language of Hardy spaces. For each of the $L^p$ spaces of functions on the unit circle, $H^p$ is the subspace consisting of those functions whose Fourier coefficients of negative index are all zero. In $H^p$, an outer function is a function $f$ having the property that $fH^p$ is dense in $H^p$, and an inner function is a function that is of modulus one almost everywhere on the unit circle.

The bilateral shift is the operator $U: L^2 \to L^2$ defined by $U: f(z) \to zf(z)$. The unilateral shift $S$ on $H^p$ is the operator $S = U|H^p$.

Recall also that there is an easy description of the operators that commute with $U$ or $S$. (We shall assume throughout that “operator” means “bounded operator.”) Also, for simplicity in the multiplicity discussion surrounding Theorem 4 we shall assume that the underlying Hilbert space is separable.) The commutant of $U$ is $L^2$, i.e. each operator $B$ that commutes with $U$ is of the form $A = A_0$: $f \to qf$ for all $f \in L^2$, where $q \in L^2$. In the same sense, $H^\infty$ is the commutant of $S$.

The original factorization theorems of Beurling and of Szegő are as follows:

**Theorem 1.** If $f \in H^2$, then $f = gh$ where $g$ is inner and $h$ is outer.