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THE STRUCTURE OF A CLASS OF BANACH ALGEBRAS GENERATED BY A BANACH SPACE

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1

Let E be a complex Banach space. We shall investigate the structure of a commutative Banach algebra with identity A which has the following three properties:

(i) A contains (an isometric copy of) E.

(ii) Every linear functional (l.f.) on E of norm at most one can be extended to a multiplicative linear functional (m.l.f.) on A.

(iii) The algebra generated by E is dense in A.

Property (ii) will be called the *multiplicative extension property* (m.e.p.).

2

It was historically first proved that some subspaces of certain given Banach algebras had the multiplicative extension property. The problem of giving a general characterization of subspaces with the m.e.p. was then raised. There are examples of algebras which have no subspaces with the m.e.p.: for instance, every finite-dimensional algebra and the algebra of continuous functions on a compact scattered space.

On the other hand, the following are two positive examples:

(a) Let X be a compact convex and balanced subset of a Hausdorff locally convex topological vector space. Then the subspace of continuous linear functionals on X has the m.e.p. in $C(X)$.

(b) If A is any function algebra on an uncountable compact metrizable space there exists an isometry $T\colon A \to A$ such that $T(A)$ has the m.e.p. in A.

We now give some necessary and sufficient conditions for a subspace E of a Banach algebra A to have the m.e.p.

(1) Let $\{x_i\colon i \in I\}$ be a set of linearly independent elements of E whose span is dense in E. Then E has the m.e.p. if and only if the joint spectrum of the x_i's is balanced and convex.

(2) E has the m.e.p. if and only if every finite-dimensional subspace of E has the m.e.p.

Let us now look at a single linear functional on E and find conditions under which it can be extended to a m.l.f. on $A(E)$, the closed subalgebra generated by E.

The result is as follows:

THEOREM 1. *A linear functional L on E can be extended to a m.l.f. on A(E) if and only if for every finite subset* $\{x_1, \ldots, x_n\}$ *of E and every complex polynomial in n variables the following inequality holds:*

(3) $$|p[L(x_1), \ldots, L(x_n)]| \leqslant \|p(x_1, \ldots, x_n)\|_\infty$$

where $\|p(x_1, \ldots, x_n)\|_\infty$ *is the spectral radius of* $p(x_1, \ldots, x_n)$.

As a consequence we have the

COROLLARY 1. *Suppose that the subspace E of A has the m.e.p. Then every linearly independent subset of E is also algebraically independent.*

3

Let us go back to the problem posed in Section 1.

Let E be a Banach space, A a Banach algebra with properties (i), (ii) and (iii) and $\{x_i: i \in I\}$ a subset of E as described in (1) of Section 2.

Finally, let A_I be the algebra generated in A by the x_i's and the identity and let $\boldsymbol{C}[(X_i)]$ be the algebra of complex polynomials in the indeterminates $\{X_i: i \in I\}$. By Corollary 1, A_I and $\boldsymbol{C}[(X_i)]$ are isomorphic and it then follows from condition (iii) that A is the completion of $\boldsymbol{C}[(X_i)]$ for a norm which is compatible with conditions (i) and (ii). It is a consequence of Theorem 1 that a norm $\|\cdot\|_1$ on A is compatible with condition (ii) if and only if for every $p \in \boldsymbol{C}[(X_i)]$, $p = p(X_1, \ldots, X_n)$, we have

(4) $$\|p(x_1, \ldots, x_n)\|_1 \geqslant \sup_L |p[L(x_1), \ldots, L(x_n)]|$$

where L ranges over the closed unit ball of the dual of E.

Hence we have the following theorem.

THEOREM 2. *It is always possible to construct an algebra A which has properties* (i), (ii) *and* (iii). *Any such A is the completion of* $\boldsymbol{C}[(X_i)]$ *for a norm which coincides with the initial norm on E and verifies inequality* (4) *for every polynomial.*

4. Examples

[α] Let E_1' be the unit ball of the dual of E. Then E can be viewed as the space of continuous linear functionals on E_1'. As in example (a) E has the m.e.p. in $C(E_1')$ and the algebra generated by E in $C(E_1')$ verifies conditions (i), (ii) and (iii).

Here $\|p\|_1 = \sup_{L \in E_1'} |p[L(x_1), \ldots, L(x_n)]| = \|p\|_\infty$.

[β] Let E^n be the nth tensor power of E and $E^{\hat{n}}$ the projective nth power ($E^{\hat{n}} = E^{\widehat{n-1}} \hat{\otimes} E$). Put

$$T(E) = \sum_{n=0}^{\infty}{}^{*} E^{\hat{n}} = \Big\{x = (x_n) \in \prod_{n=0}^{\infty} E^{\hat{n}}:\ \|x\| = \sum_{n=0}^{\infty} \|x_n\| < \infty\Big\}.$$

$T(E)$ is a Banach algebra under the product

$$(x \cdot y)_n = \sum_{k+r=n} x_k \otimes y_r.$$

Let K be the closed ideal generated by the set $\{x \otimes y - y \otimes x:\ x, y \in E\}$. Then the algebra $T(E)/K$ contains an isometric copy of E and has properties (ii) and (iii). For this construction see Leptin [1].

(γ) In the case $E = \boldsymbol{C}$ construction (α) leads to the disc algebra and construction (β) to the algebra of analytic functions on the closed unit disc with absolutely convergent Fourier series.

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