SOME RESULTS ON DUALITY FOR SPECTRAL DECOMPOSITIONS

ȘTEFAN FRUNZĂ

University of Iași, Faculty of Mathematics, Iași, Romania

I should like to begin by saying that it is a honour for me to give lectures at the Banach Center and I should like to address my warmest thanks to Organizing Committee for invitation.

The first part of my lecture will be essentially a survey of some results published between 1971 and 1977 ([18], [20], [21]). In the second part I will discuss some new unpublished results.

Before giving formal definitions, which are probably less known, let me begin with some introductive ideas.

Introduction

1. Consider an operator $T$ on a complex Banach space $X$, whose spectrum consists of two separate parts:

$$\text{sp}(T) = F_1 \cup F_2, \quad F_1 \cap F_2 = \emptyset.$$  

Then, by the Riesz decomposition theorem, $X$ decomposes into the direct sum

$$X = X(F_1) \oplus X(F_2),$$

where $X(F_j)$ are closed subspaces invariant for $T$ and

$$\text{sp}(T, X(F_j)) = F_j, \quad j = 1, 2.$$  

Moreover, $X(F_j)$ is the range of the projection $P_j$ commuting with $T$ which is defined by:

$$P_j = \frac{1}{2\pi i} \int (z - T)^{-1} dz,$$

$T_j$ being an admissible contour of integration which “surrounds” $F_j$ and leaves outside $F_k, k \neq j$.

Consider now the dual operator $T'$, defined on the dual space $X'$ by:

$$(T'u)(x) = u(Tx), \quad u \in X', x \in X.$$  

[263]
Taking into account that \( \text{sp}(T') = \text{sp}(T) = F_1 \cup F_2 \), we have also a decomposition for \( X' \):

\[
X' = X'(F_1) \oplus X'(F_2), \quad \text{sp}(T'X'(F_1)) = F_1, \quad X'(F) = \mathcal{A}(Q),
\]

\[
Q_j = (2\pi j)^{-1} \int \frac{z}{z-T} dz, \quad j = 1, 2.
\]

It is easy to see that \( Q_j = P_j, j = 1, 2 \). By applying the range-kernel duality we get:

\[
X'(F_1) = \mathcal{A}(Q_2) = \mathcal{A}(P_1) = \mathcal{M}(P_1)^\perp = \mathcal{A}(P_2)^\perp = X(F_2)^\perp,
\]

and similarly \( X'(F_2) = X(F_2)^\perp \).

Of course, this example is a very simple one and everybody knows that a situation like that when the spectrum consists of separate parts is a rare avis in the infinite dimensional case. What can we say when the spectrum of \( T \) has no separate parts? If we assume that \( T \) has a good spectral decomposition, then we get a similar duality formula.

2. Let \( T \) be a Dunford spectral operator ([10], [11], [12]); that means, there exists a projection-valued measure on the Borel sets of the complex plane:

\[
\sigma \rightarrow E(\sigma), \quad \sigma \in \mathcal{B}(C),
\]

such that \( \text{sp}(T, E(\sigma)) = \sigma, \forall \sigma \in \mathcal{B}(C). \)

It is easy to see that the map

\[
\sigma \rightarrow E(\sigma), \quad \sigma \in \mathcal{B}(C),
\]

is a projection-valued measure and \( \text{sp}(T', E(\sigma'))X' = \sigma, \forall \sigma \in \mathcal{B}(C). \) Indeed, we have \( E(\sigma)X' = \mathcal{A}(E(\sigma)) = \mathcal{M}(E(\sigma))^\perp = \mathcal{A}(E(\sigma)^\perp) \) (we note that, by the additivity of \( E(\sigma) \), \( E(\sigma) + E(\sigma') = I \)). By applying a well known isomorphism theorem, we get

\[
\mathcal{A}(E(\sigma)^\perp) \cong \langle (E(\sigma)^\perp)X' \rangle \cong \langle (E(\sigma)^\perp)X' \rangle,
\]

so that \( \text{sp}(T', E(\sigma')X') = \text{sp}(T', E(\sigma')X' = \text{sp}(T', E(\sigma)X') = \sigma, \forall \sigma \in \mathcal{B}(C). \) Therefore, if we denote \( X(\sigma) = E(\sigma)X' \) and \( X'(\sigma) = E(\sigma')X' \), we get the following duality formula:

\[
X(\sigma) = X(\sigma)^\perp, \quad \forall \sigma \in \mathcal{B}(C).
\]

3. In order to make another step towards generality let us make the following remark. Consider again a Dunford spectral operator \( T \) and \( \sigma \rightarrow E(\sigma) \), its spectral measure. Then we can integrate, with respect to \( E(\sigma) \), each bounded Borel measurable complex function \( f \) and we obtain an operator:

\[
\Psi(f) = \int f(\sigma)E(d\sigma).
\]

The mapping \( \Psi \) from complex functions to linear bounded operators is linear, multiplicative and \( \Psi(I) = I, \Psi(id) = \int E(d\sigma) = S \) (the scalar part of \( T \)). Moreover, \( T \) has a unique Jordan decomposition \( T = S + Q \) where \( Q \) is a quasinilpotent operator commuting with \( T \). Thus, the scalar part of a Dunford spectral operator admits a functional calculus defined on a very large class of functions; furthermore, the spectral measure may be reaptured from the functional calculus. Now, what can we say when we only know that an operator admits a functional calculus? If the functional calculus is given on the largest algebra (in some sense) of complex functions, then the operator has a spectral decomposition of Dunford type. If the functional calculus is given on the smallest algebra — the algebra of all analytic functions in some neighbourhood of the spectrum — the best thing that we can get is a decomposition of Riesz type, i.e., a decomposition of the spectrum into separate parts. There are many interesting intermediate algebras between the smallest one and the largest one: the algebra \( C^0(C) \) of all complex infinitely differentiable functions on the complex plane is an example. The operators having a \( C^0 \)-functional calculus are called scalar generalized operators and have been introduced and studied by C. Foiaş in 1960 ([13]).

And thus, \( S \) is called a scalar generalized operator if there exists a mapping \( \Psi: C^0(C) \rightarrow L(X) \), which is linear, multiplicative and \( \Psi(I) = I, \Psi(id) = S \). Such a mapping is called a spectral distribution of \( S \).

Any scalar generalized operator has a spectral decomposition of a generalized type. In order to describe such a decomposition, denote:

\[
X(F) = \{ x \in X; \supp \Psi(\cdot) x \subset F \},
\]

where \( F \) is a closed subset of \( C \). Then \( X(F) \) is a closed linear subspace, invariant for \( S \). A very useful and important fact is that \( X(F) \) has the following intrinsic characterization:

\[
X(F) = \{ x \in X; \text{the resolvent equation } (z - T)f(x) = x \text{ has an analytic solution } f \text{ outside } F \}.
\]

It might be worth mentioning that just by using such a characterization, N. Dunford was able to prove the uniqueness of the spectral measure. The function \( f \) appearing in (2) is unique for any given \( x \) and \( f(x) = R(z; T)x \) if \( z \in F \cap \rho(T) \); this means that \( f \) is, in some sense, a local resolvent of \( T \) in \( x \). By using the characterization (2) of the space \( X(F) \) one can see that \( \text{sp}(S, X(F)) \subset F \) and \( X(F) \) contains any subspace \( Y \) invariant for \( S \), whose spectrum is contained in \( F \); \( X(F) \) is called the maximal spectral space corresponding to \( F \).

Now take an arbitrary finite open covering \( \{ G_j \} \) of \( \text{sp}(S) \). Then there exists a corresponding partition of unity \( \{ \eta_j \} \) subordinate to this covering; that means:

\[
\sum_{j=1}^{n} \eta_j = 1, \quad \supp \eta_j \subset G_j, \quad j = 1, \ldots, n.
\]

Consequently, we get the following decomposition of the space \( X \):

\[
X = \sum_{j=1}^{n} X(F_j), \quad F_j \subset G_j, \quad j = 1, \ldots, n.
\]
It is known that, unlike a Dunford scalar operator whose spectral measure is unique, a scalar generalized operator may have several spectral distributions and it is even possible that no one of them has values in the bioboominant of the operator (I1). Nevertheless, on account of (2), the spaces $X(F)$ are the same for all possible functional calculi associated to the operator. If $S$ is a scalar Dunford operator then $X(F)$ is the range of the projection $E(F)$. In the more general case we cannot say that $X(F)$ is associated to a projection but the family $\{X(F), F \in C\}$ acts as an effective partial substitute of a spectral measure. The generalized decomposition discussed before was formalized by C. Foiaş ([H1; see also [25]) in the following manner.

**Definition 1.** A bounded linear operator $T$ on a complex Banach space $X$ is called decomposable (or spectral in the sense of Foiaş) if the following conditions are fulfilled:

(i) for each closed set $F \subset C$, there exists a maximal spectral space $X(F)$ of $T$, whose spectrum is contained in $F$, 

(ii) for each finite open covering $\{G_j\}_{j=1}^n$ of $\text{sp}(T)$, we have:

$$X = \sum_{j=1}^n X(F_j) \quad \text{for some } F_j \subset G_j, \quad j = 1,\ldots,n.$$  

The maximality property of $X(F)$ stated in (i) means that $Y \subset X(F)$ whenever $Y$ is invariant for $T$ and $\text{sp}(T, Y) \subset F$. A natural candidate for $X(F)$ would be the following:

$$M(F) = \{x \in X: \text{the resolvent equation } (\lambda - T)x = x \text{ has an analytic solution } \lambda \text{ outside } F\}.$$  

It is easy to see, looking at the definition, that $M(F)$ is a linear subspace, invariant for $T$ and that $Y \subset M(F)$ whenever $Y$ is invariant for $T$ and $\text{sp}(T, Y) \subset F$ (indeed, for each $x \in Y$, the resolvent equation has the obvious solution $(\lambda - T)^{-1}x$, $\lambda \in \text{sp}(T, Y) \subset F$). Therefore $M(F)$ has the maximality property. The problems that arise in connection with $M(F)$ are the following:

1. Is $M(F)$ closed?
2. If $M(F)$ is closed, is it true that $\text{sp}(T, M(F)) \subset F$?

If $M(F)$ is closed and $\text{sp}(T, M(F)) \subset F$, then obviously $M(F)$ is the maximal spectral space whose spectrum is contained in $F$. Generally speaking, the answer to problem 1 is negative ([9]). As regards problem 2, the answer is always affirmative as is shown by the following basic result of Foiaş: If the resolvent equation has a unique analytic solution on $F^c$ for each $x \in M(F)$ and $M(F)$ is closed (for a certain set $F$), then $\text{sp}(T, M(F)) \subset F$ (application of the closed graph theorem). Another result of Foiaş is that if $T$ is decomposable then, for each closed set $F$, the resolvent equation has a unique analytic solution on $F^c$ for any $x \in M(F)$. In other words, a decomposable operator has the single-valued extension property (condition A of Dunford). Moreover, $M(F)$ is closed for each closed subset $F$ (condition C of Dunford). Therefore the conditions A and C, which are necessary for spectrality in the sense of Dunford, are necessary for spectrality in the sense of Foiaş, too.

If an operator $T$ has the single-valued extension property, then, for each $x \in X$, there exists a smallest closed set $\sigma(x)$ outside which the resolvent equation has an analytic solution. This set $\sigma(x)$ is called the spectrum of $x$ with respect to $T$ and its complement $\sigma(x)^c = \sigma(x)$ is called the resolvent set of $x$ with respect to $T$. Then we may formulate Definition 1.1 in the following equivalent way:

**Definition 1.** $T$ is decomposable if the following conditions are fulfilled:

(i) $T$ has the single-valued extension property,

(ii) for each closed set $F$ the space $M(F)$ is closed,

(iii) for each finite open covering $\{G_j\}_{j=1}^n$ of $\text{sp}(T)$ and any element $x \in X$, we have a decomposition:

$$x = \sum_{j=1}^n x_j, \quad \sigma(x) \subset G_j, \quad j = 1,\ldots,n.$$  

The decomposition (4) may be considered as an abstract non-classical partition of unity. It should be observed that there exist spectral decompositions of this type which are not obtained via a functional calculus (69). If the property (iii) from Definition 1 holds for coverings consisting of a fixed number $k$ of open sets, then $T$ is called $k$-decomposable ([30]). If the decomposition (4) holds for any finite open covering $\{G_j\}_{j=1}^n$ of $\sigma(x)$, then $T$ is called decomposable with almost localized spectrum ([69]). Finally, if any element $x \in X$ has a decomposition:

$$x = \sum_{j=1}^n x_j, \quad \sigma(x) \subset \sigma(x) \cap G_j, \quad j = 1,\ldots,n$$  

then $T$ is called strongly decomposable ([5]). This last decomposition property is the closest to the classical partition of unity.

The duality for generalized spectral decompositions

Let $T$ be a scalar generalized operator and $\Psi: C^0(C) \to L(X)$ be a spectral distribution of $T$. Then the mapping $\Psi: C^0(C) \to L(X)$ defined by:

$$\Psi(f) = (\Psi(f)), \quad f \in C^0(C),$$  

is a spectral distribution for $T^*$ (the dual of $T$). By using simple arguments of partition of unity and the range-kernel duality, it is easy to see that the duality of spectral spaces that we have got in more particular cases, holds in this case, as well. Namely, we have:

$$X^*(F) = X(F)^*.$$
for each closed set $F \subset C$. Indeed, we can write successively:

$$X'(F) = \left\{ u \in X', \supp \Psi(\cdot) u \subset F \right\} = \bigcap \left\{ \Psi'(F), \supp f \subset F \right\}$$

$$= \bigcap \left\{ \Psi'(f), \supp f \subset F \right\}$$

$$= \left\{ \bigcup_{\xi \in F} \left\{ \Psi'(\xi), \supp g \subset \xi \right\} \right\} = X'(F).$$

Therefore, if $T$ is a scalar generalized operator then $T'$ is a scalar generalized operator, too, and the duality formula for spectral spaces holds.

We are now asking, up to what extent can this statement be generalized to decomposable operators. The basic question to be answered is the following.

**Question:** Is the dual of a decomposable operator decomposable and does it satisfy the duality formula for spectral spaces?

As we will see in what follows, the answer is positive. Even more, it will be shown that the dual of a 2-decomposable operator is decomposable with almost localized spectrum. The first main result in this direction is the following.

**Theorem 1** ([19]). If $T$ is a 2-decomposable operator, then

$$\text{sp}(T', X'(F)) \subset F$$

for any closed subset $F$ of $C$. Moreover, $X'(F)$ is the maximal invariant subspace for $T'$ whose spectrum is contained in $F$.

We send to [18] for the proof. By using Theorem 1 it is relatively easy to get the following duality result.

**Theorem 2** ([18]). If $T$ is a 2-decomposable operator, then $T'$ is also a 2-decomposable operator.

**Proof.** First of all, condition (i) in Definition 1 is satisfied, by Theorem 1. In order to verify condition (ii), take an open covering $\{G_j, G_j'\}$ of $\text{sp}(T)$ and let $\{U_1, U_2\}$ be another covering of $\text{sp}(T)$ such that $U_j \subset G_j$, $j = 1, 2$. Denote $H_j = U_j \cap \text{sp}(T) = U_j \cap \text{sp}(T)$, $j = 1, 2$. Then $H_1 \cap H_2 = \emptyset$, so that by applying the Riesz decomposition theorem ([18]), we get the following direct decomposition:

$$X(H_1 \cup H_2) = X(H_1) \oplus X(H_2).$$

Let now $u \in X'$ be an arbitrary element of the dual space. Define $u_1, u_2 \in X(H_1), x \in X(H_2)$ by $u_1(x) = u(x)$ if $x = x_1 + x_2$, $x_1 \in X(H_1), x_2 \in X(H_2)$. We obtain a continuous linear functional $u_i$ on $X(H_1 \cup H_2)$. By the Banach–Bolzano theorem, $u_i$ may be extended to the whole space $X$ by preserving the linearity and continuity. Denote by $u_1$ such an extension and by $u_2 = u - u_1$. From the definitions of $u_1$ and $u_2$, it follows that $u_1 \in X(H_1)'$ and $u_2 \in X(H_2)'$; moreover, $u = u_1 + u_2$. Now we have:

$$X(H_1)' = X(H_2)' \oplus M(U) = X(U)' \text{ and } X(H_2)' \oplus X(U)' = X(U), \quad j = 1, 2.$$ 

Therefore we have obtained the decomposition:

$$X' = X'(U_1) + X'(U_2), \quad U_j \subset G_j, \quad j = 1, 2.$$

The proof is finished.
Proof (19)]. A simple inductive argument shows that an operator T is decomposable with almost localized spectrum if T is 2-decomposable and
\[ X(F) \subset X(G_1) + X(G_2) \]
for any closed set F and each open covering \( \{ G_1, G_2 \} \) of F.

We will prove that the dual operator \( T' \) has that property. First of all, by applying Theorem 2, it follows that \( T' \) is 2-decomposable and \( X'(F') = X'(F')_{\lambda} \) for any closed set F. In view of the duality between the spectral spaces of T and \( T' \), it will be sufficient to show that
\[ X(G') \subset X(F')_{\lambda} + X(F')_{\lambda}, \]
for any open set G and any closed sets \( F_1, F_2 \) such that \( F_1 \cap F_2 \subset G \). Take two open sets \( D_1, D_2 \) such that \( F_1 \subset D_1 \) and \( D_1 \cap D_2 \subset G \). Since \( T' \) is decomposable with almost localized spectrum, we have:
\[ X(D_1) = X(D_2), \]
Consider an element \( u \in X(G') \). Define \( \bar{u}_i : X(F_i \cup F_2) \to C \) by:
\[ \bar{u}_i(x) = u(x_2) \quad \text{if} \quad x = x_1 + x_2, \ x_1 \in X(D_i), \ j = 1, 2. \]

It is easy to see that the definition of \( \bar{u}_i \) is consistent and that \( \bar{u}_i \) is a linear functional. In order to show that \( \bar{u}_i \) is continuous we shall apply the open mapping theorem (36). Consider the mapping \( S : X(D_i) \otimes X(D_i) \to X \), defined by \( S(x_1 \otimes x_2) = x_1 + x_2 \). It is obvious that \( S \) is linear and continuous and that the range of \( S \) contains \( X(F_i \cup F_2) \). Consequently, if we restrict \( S \) to \( S^{-1}(X(F_i \cup F_2)) \), we obtain a continuous linear surjection between two Banach spaces; by the open mapping theorem, this surjection must be open, so that there exists a constant \( K > 0 \) such that for any element \( x \in X(F_i \cup F_2) \) we have a representation \( x = x_1 + x_2, x_1 \in X(D_i), j = 1, 2, \| x_1 \| + \| x_2 \| \leq K \| x \| \). By using such a representation, it is easy to show that \( \bar{u}_i \) is bounded. By applying the Hahn–Banach theorem, \( \bar{u}_i \) may be extended to a continuous linear functional \( u_i \) on X. If we define \( u_1 = u_1 - u_2 \), then we get \( u = u_1 + u_2, u_1 \in X(F_1)_{\lambda}, j = 1, 2 \), and the proof is complete.

The arguments used before suggest that it would not be possible to obtain more than 2-decomposability of the dual operator. However, as we shall see in what follows, the result of Theorem 2 can be considerably improved.

Theorem 4 (20). If T is 2-decomposable then \( T' \) is decomposable.

Proof. By applying the duality of spectral spaces, it is easy to see that \( T' \) is decomposable if and only if the mapping
\[ \mathcal{J} : X(F_1) \otimes X(F_2) \otimes \cdots \otimes X(F_n) \to X', \]
deﬁned by \( \mathcal{J}(u \otimes u_2 \otimes \cdots \otimes u_n) = u_1 + u_2 + \cdots + u_n \), is surjective for any family \( \{ F_i \}_{i=1}^n \) of closed sets such that \( \bigcap_{i=1}^n F_i = \emptyset \). It is quite natural to look for an equivalent property regarding the space X. Since \( X(F_1)_{\lambda} \) is isometrically isomorphic to the dual of the quotient space \( X(X(F_j)), j = 1, \ldots, k \), the mapping \( \mathcal{J} \) is implemented (by the isomorphisms above) by the dual of the mapping:
\[ \mathcal{J} : X \to X(X(F_1)_{\lambda} \otimes X(F_2)_{\lambda} \otimes \cdots \otimes X(F_k)_{\lambda}, \]
deﬁned by \( \mathcal{J}(x) = (x + X(F_1)) \otimes (x + X(F_2)) \otimes \cdots \otimes (x + X(F_k)) \). Consequently, it will be enough to prove that the mapping \( \mathcal{J} \) has null kernel and closed range. The condition \( \bigcap_{i=1}^n F_i = \emptyset \) implies that \( \mathcal{J} \) has null kernel. It is more difﬁcult to prove that \( \mathcal{J} \) has closed range. In order to do this we shall proceed by a reducible ad absurdum. Suppose there exists a sequence \( x_n \in X \) such that \( \mathcal{J}(x_n) = 0 \) for \( n \to \infty \), \( 1 \leq j \leq k \), and \( \| x_n \| = 1 \) for any natural number \( n \); that implies that there exist sequences \( x_n^{(1)} \in X(F_1)_{\lambda} \) such that \( \| x_n - x_n^{(1)} \| \to 0 \) for \( n \to \infty \), \( 1 \leq n \in N \), it follows that \( x_n^{(1)} \) are bounded sequences. Taking into account that \( x_n^{(1)} \in X(F_1)_{\lambda} \) and \( \mathcal{J}(x_n^{(1)}) = 0 \), we obtain:
\[ x_n^{(1)} = (x - T) f_n(x), \quad x \in F_1, \]
where \( f_n(x) = R(z; T X(F_1)) x_n^{(1)}, \quad x \in F_1 \). It is obvious that the sequences \( f_n(x) \) are uniformly bounded on compact sets. Let us now put this situation in a more adequate framework. Consider the space \( L_0(X_0) / c_0(X) \) and denote by \( x_0 \) the class deﬁned by the sequence \( x_n^{(1)} \) and by \( f_0 \) the function deﬁned by:
\[ f_0(z) = (f_n(z)) + c_0(X), \quad x \in F_1. \]

Then we have \( f_0(x) = f_0(x), \quad x \in F_1 \cap F_2 \), and consequently we can deﬁne an analytic function on \( \mathbb{C} \) by \( f_0(z) = f_0(z) \) if \( z \in F_1 \). If \( T \) is the operator deﬁned by \( T \) on \( L_0(X_0) / c_0(X) \), then we have \( x_0 = (x - T) f_0(x) \), \( x \in X \), and consequently, by taking a suitable function \( \mathcal{J} \) contained in the resolvent set of \( T \) and surrounding the spectrum of \( T \), we deduce:
\[ x_0 = (2\pi i)^{-1} \int_{\gamma} \mathcal{J}(z) x_0 dz = (2\pi i)^{-1} \int_{\gamma} f_0(z) dz = 0. \]

We are thus led to a contradiction, since \( x_0 \) is the class deﬁned by the sequence \( x_n^{(1)} \) and \( \| x_n^{(1)} \| = 1, n \in N \).

Corollary 1. If T is decomposable then \( T' \) is also decomposable.

Corollary 2. On a reﬂexive Banach space, any 2-decomposable operator is decomposable.

By using arguments of topological dimension theory, E. Albrecht and F.-H. Vasilescu have proved in [3] that any 3-decomposable operator is decomposable (on any Banach space). Recently, I have read in Notices of the American Math. Soc. that M. Radjabalipour has proved that any 2-decomposable operator is decomposable (33), thus improving both the result in Corollary 2 and the result of Albrecht and Vasilescu. Such a result justiﬁes the analogy between the spectral decompositions in the sense of Foiaş and the partition of unity.
The result in Theorem 4 is yet not the best possible. By using a Cousin type theorem, it turned out possible to prove that the dual of a 2-decomposable operator is a decomposable operator with almost localized spectrum.

**Theorem 5 (21).** If \( T \) is a 2-decomposable operator, then \( T' \) is a decomposable operator with almost localized spectrum.

**Proof.** By taking into account the duality between spectral spaces of \( T \) and \( T' \), we see that the statement of Theorem 5 is equivalent to the following: for any two closed sets \( F_1, F_2 \) and any open set \( G \) such that \( F_1 \cap F_2 \subset G \), we have:

\[
X(G) := (X(F_1) + X(F_2))^0.
\]

This, in its turn, is equivalent to say that for any sequence \((x_n) \subset X\) such that \(d(x_n, X(F_j)) \to 0\) for \( n \to \infty, j = 1, 2\), we have \(d(x_n, X(G)) \to 0\) for \( n \to \infty\). The first idea was to follow the same way as in the proof of the preceding theorem, by taking \(X(G)\) instead of \(X\). From Corollary 4 of Theorem 2 we know that \(sp(T, X(G)) \subset \mathcal{G}'\) and from hypothesis it follows that \(\mathcal{G}'\) is covered by \(F_1\) and \(F_2\). This argument does not work because we need a decomposition property for the space \(X(G)\) and we do not know such a property. In order to take over this difficulty, the following theorem of Cousin type can be used:

**Theorem.** Let \( G_1 \) and \( G_2 \) be two open sets in the complex plane such that \( G_1 \cap G_2 \neq \emptyset \) and let \((x_n) \subset A(G_1, X), j = 1, 2\) be two sequences of analytic functions such that \(f_{x_n} - f_{x_2} \to 0\) in \(A(G_1 \cap G_2, X)\) for \( n \to \infty\). Then there exists a sequence \((x_n) \subset A(G_1 \cap G_2, X)\) such that \(f_{x_n} - f_{x_2} \to 0\) in \(A(G_1, X)\) for \( n \to \infty\), \( j = 1, 2\).

The idea of the proof is to apply the open mapping theorem (for Fréchet spaces) for the mapping:

\[\mathcal{O}: A(G_1, X) \oplus A(G_2, X) \to A(G_1 \cap G_2, X)\]

defined by \(\mathcal{O}(f_{x_1} \oplus f_{x_2}) = f_{x_1} - f_{x_2}\). This mapping is continuous, linear and, by the vectorial variant of the classical theorem of Cousin, is also surjective (24). To go on with the proof of Theorem 5, let \((x_n) \subset X\) be a sequence such that \(d(x_n, X(F_j)) \to 0\) for \( n \to \infty, j = 1, 2\). We must prove that \(d(x_n, X(G)) \to 0\) for \( n \to \infty\). Let \((x_n) \subset X(F_j)\) be sequences such that \(d(x_n, x_{n_0}) = ||x_n - x_{n_0}|| \to 0\) for \( n \to \infty, j = 1, 2\). Since \(sp(T, X(F_j)) \subset F_j\), we may write:

\[
x_{n_j} = (z - T) R(z; X(F_j)) x_{n_j}, \quad z \in \mathbb{C}, \quad j = 1, 2, n \in N.
\]

Let us denote, for simplicity, \(f_{x_n}(z) = R(z; X(F_j)) x_{n_j}, z \in F_j, j = 1, 2, n \in N\). Then we have:

\[
f_{x_n}(z) - f_{x_2}(z) = R(z; X(F_j)) x_{n_j} - x_{n_2} \to 0
\]

for \( n \to \infty\), uniformly on each compact subset of \(F_1 \cap F_2\). Let us now transpose our situation to the quotient space \(Y = X/G\). Denote by \( S \) the operator induced by \( T\):

\[
S^\perp = (FX), \quad x \in X/Y
\]
set $F$. Let $Y_1, Y_2$ be two invariant subspaces for $T$ such that $\text{sp}(T, Y_1) \cap \text{sp}(T, Y_2) = \emptyset$. Then $Y_j \subseteq H_T(F_j)$, where $F_j = \text{sp}(T, Y_j)$, $j = 1, 2$. Since a normal operator has an orthogonal resolution of the identity, we have $H_T(F_1) \perp H_T(F_2)$ and consequently $Y_1 \perp Y_2$. Therefore it remains to prove that if $T$ is an orthogonally decomposable operator, then $T = N + Q$ where $N$ is normal and $Q$ is quasinilpotent commuting with $N$. It will be enough to show that $T$ is a Dunford spectral operator. Once this is proved, we have the canonical decomposition $T = S + Q$ where $S$ is a scalar operator and $Q$ is a quasinilpotent commuting with $S$ ([10], [11], [12]). Since $T$ is orthogonally separate, $S$ has an orthogonal resolution of the identity and consequently is a normal operator. In order to prove that $T$ is a Dunford spectral operator we shall use the characterization of Dunford spectral operators on weakly (sequentially) complete Banach spaces ([12]): an operator $T$ on a weakly complete Banach space is a Dunford spectral operator if and only if it satisfies the conditions (A), (B), (C) and (D) of Dunford. Therefore we shall prove that an orthogonally decomposable operator satisfies the conditions (A), (B), (C) and (D). Condition (A) means the single-valued extension property and is satisfied since $T$ is decomposable (B). Another direct consequence of the decomposability of $T$ is condition (C), which means that the maximal space $M(T)$ is closed for any closed set $F$. Condition (B) requires that $|x| \leq K|x+y|$ for any elements $x$ and $y$ having disjoint spectra $K$ being a positive constant. Since $T$ is orthogonally separate and decomposable, it is easy to see that any two elements whose spectra are disjoint are orthogonally and consequently condition (B) is also satisfied. As regards condition (D), the verification is more difficult. Let us recall in what consists condition (D). Denote by $\mathcal{S}(T)$ the family of all subsets $\sigma$ of $\mathbb{C}$ such that the elements of the form $x + y \in \sigma$ where $n(x) \in \sigma$ and $n(y) \in \sigma'$ constitute a dense set in $H$. Denote by $\mathcal{S}(T)$ the family of all subsets $\sigma$ of $\mathbb{C}$ such that for any $x \in H$ and any $\varepsilon > 0$ there exist $x_1, x_2 \in H$ with the properties: $n(x_1) \in \sigma$, $n(x_2) \in \sigma'$ and $|x_1 + x_2 - x| < \varepsilon$. If $\sigma$ belongs to $\mathcal{S}(T)$, then there exists a projection $E(\sigma)$ such that $E(\sigma)x = x$ if $x \in \sigma$ and $E(\sigma)x = 0$ if $x \notin \sigma$. Denote by $\mathcal{S}(T)$ the family of all subsets $\sigma$ of $C$ such that there exist two sequences of sets $\mu_{n}, \mu_{n} \subseteq \sigma$, $\mu_{n}, \mu_{n} \subseteq \mathcal{S}(T)$, $n \in \mathbb{N}$, with the property:

$$
x = \lim_{n} E(\mu_{n} + E(\mu_{n}) x,
$$

for any $x \in H$. The condition (D) requires that any complex number have a neighbourhood of arbitrarily small diameter which belongs to $\mathcal{S}(T)$. We shall prove that $\mathcal{S}(T)$ contains any closed subset of $C$ (if $T$ is orthogonally decomposable). The main steps in proving this are the following.

**Lemma 1.** If $T$ is an orthogonally decomposable operator on a Hilbert space $H$, then the adjoint $T^{*}$ of $T$ is orthogonally decomposable and $s_{r}(x) = s_{r}(x)^{*}$ for any element $x \in H$.

**Proof.** By applying Corollary 2 it follows that $T^{*}$ is 2-decomposable and $H_{T}(F) = H_{T}(F^{*})^{\perp}$ for any closed subset $F$ of $C$ $(F^{*})^{\perp}$ denotes the complex conjugate set of $F^{*}$). In order to prove the equality $s_{r}(x) = s_{r}(x)^{*}$, write $F = s_{r}(x)$ and denote by $T_{r}$ the restriction of $T$ to $H_{T}(F)$. It is easy to see that the function $x^{*}(z) = R(z^{*}, T_{r}x, z \in (F^{*})^{\perp}$, is analytic on $(F^{*})^{\perp}$ and $(z - T_{r})^{*}(x) = x$, $z \in (F^{*})^{\perp}$. Consequently, $s_{r}(x) = s_{r}(x)^{*}$, for any element $x \in H$. Since $T$ and $T^{*}$ may be interchanged, we have also the opposite inclusion and so the proof is finished.

**Lemma 2.** If $T$ is orthogonally decomposable, then $H_{T}(\sigma) \perp H_{T}(\sigma) = H$ for any closed set $\sigma$.

**Proof.** Taking into account that $H_{T}(\sigma) \perp H_{T}(\sigma) = H$ implies $x = 0$. By using the reflexivity of $H$ and the duality of spectral spaces, we see that the last statement is equivalent to $H_{T}(\sigma) \cap H_{T}(\sigma) = \{0\}$. This statement, in turn, is a consequence of the fact that $T$ is orthogonally decomposable.

**Lemma 3.** If $T$ is orthogonally decomposable, then $H_{T}(\sigma) = H_{T}(\sigma \cap \sigma) \oplus H_{T}(\sigma \cap \sigma)$ for any two closed sets $\sigma$, $\sigma$.

**Proof.** By Lemma 2, we have: $H = H_{T}(\sigma) \oplus H_{T}(\sigma)$, Denote by $P$, the orthogonal projection of $H$ on $H_{T}(\sigma)$. Since both $H_{T}(\sigma)$ and $H_{T}(\sigma) = H_{T}(\sigma)$ are invariant for $T$, $P_{r}$ commutes with $T$. Consequently, we have $s_{r}(x) = s_{r}(x)$ and $s_{r}(x) \in \sigma$, whence $H_{T}(\sigma) = H_{T}(\sigma \cap \sigma) \oplus H_{T}(\sigma \cap \sigma)$. It is easy to see that $H_{T}(\sigma) \cap H_{T}(\sigma) = H_{T}(\sigma \cap \sigma)$ and so Lemma 3 is proved.

In order to close the proof of Theorem 6, it suffices to remark that, by Lemma 3, the family $\mathcal{S}(T)$ contains any closed subset of the complex plane.

**The operators with strongly decomposable dual**

We have already seen (Theorem 5) that the dual of a 2-decomposable operator is decomposable with almost localized spectrum. So it is quite natural to conjecture the dual might even be strongly decomposable. We have not succeeded in solving this difficult problem but, by trying to reduce it to more transparent equivalent questions, we have obtained several characterizations of the operators with strongly decomposable dual, as well as a new characterization of strongly decomposable operators. In this way, we have found a new class of decomposable operators which is, in some sense, dual to the class of strongly decomposable operators. Moreover, the results of this type reveal some more complete aspects of duality. We shall begin with a characterization of strongly decomposable operators by a spectral condition.

**Theorem 7.** Let $T$ be a 2-decomposable operator. Then the following statements are equivalent:

(i) $T$ is strongly decomposable,
(ii) \( \text{sp}(T, X(F)/X(H)) \subseteq F \setminus H \) for any two closed sets \( F, H \) such that \( H = F \).

The implication (i) \( \Rightarrow \) (ii) is essentially a result of C. Apostol ([5], [6]). The implication (ii) \( \Rightarrow \) (i) may be proved by using the Riesz decomposition theorem in a way similar to that used in [33] and [26].

By applying Theorem 7 and the duality of spectral spaces we get the following characterization of the operators having strongly decomposable dual, by a spectral condition.

**Theorem 8.** Let \( T \) be a 2-decomposable operator. Then the following statements are equivalent:

(i) \( T \) is strongly decomposable,

(ii) \( \text{sp}(T, X(V)/X(U)) \subseteq V \setminus U \) for any two open sets \( U, V \) such that \( U = V \).

Before stating the next result let us reformulate a basic result for strongly decomposable operators ([5]).

**Theorem 9.** A 2-decomposable operator \( T \) is strongly decomposable if and only if \( T \) is decomposable on any spectral space \( X(F) \). If this condition holds, then \( T \) is decomposable on any quotient space \( X(F) \).

The following result is a dual variant of Theorem 9.

**Theorem 9’.** The dual of a 2-decomposable operator is strongly decomposable if and only if \( T \) is decomposable on any space \( X(G) \). If this condition holds, then \( T \) is decomposable on any space \( X(G) \).

Another characterization of the operators with strongly decomposable dual may be given in the terms of geometry of spectral spaces.

**Theorem 10’.** Let \( T \) be a 2-decomposable operator. The following statements are equivalent:

(i) \( T \) is strongly decomposable,

(ii) \( \overline{X(G_1 \cup G_2)} = \overline{X(G_1)} + \overline{X(G_2)} \) for any two open sets \( G_1, G_2 \) with the property \( G_1 \cap G_2 \cap G_1 = \emptyset \),

(iii) The subspace \( X(G_1) \) and \( X(G_2) \) are gapping for any two open sets \( G_1, G_2 \) with the property \( G_1 \cap G_2 \cap G_1 = \emptyset \).

**Theorem 10’** is a dual result to the following new characterization of strongly decomposable operators.

**Theorem 10.** Let \( T \) be a 2-decomposable operator. The following statements are equivalent:

(i) \( T \) is strongly decomposable,

(ii) \( \overline{X(F_1 \cup F_2)} = \overline{X(F_1)} + \overline{X(F_2)} \) for any two closed sets \( F_1, F_2 \) with the property \( F_1 \cap F_2 \cap F_1 = \emptyset \),

(iii) \( X(F_1) \) and \( X(F_2) \) are gapping for any two closed sets \( F_1, F_2 \) such that \( F_1 \cap F_2 \cap F_1 = \emptyset \).

The preceding results point out the importance of the spectral spaces \( X(G) \), as well as their dual position with respect to spectral spaces \( X(F) \). A comparison with the Dunford spectral operators would be useful. In that case we have \( X(F) = E(F) \) and \( X(G) = E(G) \) where \( E \) denotes the spectral measure of \( T \).

Therefore both \( X(F) \) and \( X(G) \) are complemented in \( X \) and \( X(F) \cong X(G) \cong X(F) \cong X(G) \) are complemented in \( X \) and \( \cong X(F) \). In the more general case of decomposable operators the spaces \( X(F) \) and \( X(G) \) are not necessarily complemented so that we must consider four different types of spectral spaces \( X(F), X(F'), X(G), X(G') \). It would be interesting to study in great detail the operators which are strongly decomposable and have strongly decomposable dual.

**Duality for general operators**

A very deep result due to E. Bishop ([8]) shows that the essence of the duality relations between spectral spaces remains valid for general operators on reflexive Banach spaces. Furthermore, in a more recent paper ([29]), V. Lomonosov, Ju. Lusin and V. Matzaev extend Bishop’s results to nonreflexive Banach spaces.

In order to formulate his duality theorem, Bishop introduced two types of spectral manifolds associated to a closed set \( F \) of the complex plane.

The strong spectral manifold \( M(F, T) \) is the closure of the set of all vectors \( x \) in \( X \) which have the property that the resolvent equation \( (z - T)x = x \) has an analytic solution \( f \) outside \( F \).

The weak spectral manifold \( N(F, T) \) is the set of all vectors \( x \) in \( X \) with the property that for any \( \epsilon > 0 \) there exists an \( \epsilon \)-valued analytic function \( f \) defined outside \( F \) such that \( |(z - T)x - f(x)| < \epsilon, x \in X \).

**Theorem 8.** For any (continuous linear) operator \( T \) on a reflexive Banach space \( X \), the following duality relations hold:

\[ M(F_1, T)^\perp \subseteq N(F_2, T)^\perp, \quad N(F_1, T) \subseteq M(F_2, T)^\perp \]

for any two disjoint closed sets \( F_1, F_2 \) and

\[ M(G_1, T)^\perp \subset N(G_2, T)^\perp, \quad N(G_1, T)^\perp \subset M(G_2, T)^\perp \]

for any two open sets \( G_1, G_2 \) which cover the complex plane.

A very interesting consequence of Theorem 8 is a general decomposition theorem. By following [8], we say that an operator \( T \) satisfies condition \( \beta \) if, given any open set \( G \) and any sequence \( (f_n) \) of analytic functions on \( G \) such that \( (z - T)f_n(x) \to 0 \) for \( n \to \infty \) uniformly on \( G \), it follows that \( f_n \) is uniformly bounded on compact subsets of \( G \). There exist several equivalent reformulations of the condition \( \beta \). One of them is the following. \( T \) satisfies the condition \( \beta \) if, given any open set \( G \) and any sequence \( (f_n) \) of analytic functions on \( G \) such that \( (z - T)f_n(x) \to 0 \) uniformly on compact sets, it follows that \( f_n \to 0 \) uniformly on compact sets.

**Theorem 9.** Let \( X \) be a reflexive Banach space and \( T \) be an operator on \( X \). If both \( T \) and \( T^* \) satisfy the condition \( \beta \), then to any open covering \( \{G_j\} \) of the complex plane corresponds a family \( \{M_j\} \) of invariant subspaces of \( T \) such that:

\[ \text{sp}(T, M_j) \subseteq G_j, \quad j = 1, \ldots, n \quad \text{and} \quad X = \text{c.l.m.}(M_j, j = 1, \ldots, n). \]
By using some ideas similar to those occurring in the proof of Theorem 2 and Theorem B, it is possible to improve the result of Theorem B.

Theorem 11. Let \( X \) be a reflexive Banach space and \( T \) be an operator on \( X \). If both \( T \) and \( T' \) satisfy condition \( \beta \), then \( T \) is decomposable.

Actually, the statement converse to that of Theorem 11 is also true. Indeed, by a result proved in [17] and [16], any decomposable operator satisfies condition \( \beta \). Consequently, on account of Corollary 1, Theorem 4, it follows that \( T' \) also satisfies condition \( \beta \). In this way we get the following interesting characterization of decomposable operators on reflexive Banach spaces.

Theorem 12. An operator \( T \) on a reflexive Banach space is decomposable if and only if both \( T \) and \( T' \) satisfy the condition \( \beta \).

It is very probable that a similar characterization holds on nonreflexive Banach spaces.

It can be expected that all operators satisfying a reasonable growth condition of the resolvent satisfy the condition \( \beta \), too, and consequently, by Theorem 11, are decomposable.

I should like to close my lecture with a remark concerning Theorem B. The key to the proof of Theorem B, was a duality theorem for spaces of analytic functions. Recently I discovered in a paper of A. Grothendieck ([23]) a definitive result concerning the duality of spaces of analytic functions due to J. Silva, G. Köthe ([35], [27]) and A. Grothendieck. Although Grothendieck's paper was published much before Bishop's paper, the duality result contained there was probably not known neither by Bishop nor by the authors of the more recent paper [29].

It seems to me that the result of Silva-Köthe-Grothendieck will be the key to obtaining a definitive duality result for general operators on Banach spaces.

References


[34] J. Silva, As Funções Analíticas e a Análise Funcional, Portuguese Math. 9 (1950), 1-130.

Presented to the semester Spectral Theory September 23—December 16, 1977