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ON HYPERGRAPHS OF MAXIMAL SIMPLE PATHS OF A CLASS OF HAMILTONIAN GRAPHS

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Let $G = \langle V, X, \varphi\rangle$ be an arbitrary simple graph. A *hypergraph of maximal simple paths of* G is a hypergraph $H = \langle X, \mathscr{E}\rangle$, where $\mathscr{E} = \{E_i\}_{i\in I}$, $E_i = E_j \Leftrightarrow i = j$ is a family of subsets of X corresponding to subsets of edges of an arbitrary maximal path of G. In [1] some fundamental problems in structural hypergraph theory have been formulated. Some solutions to these problems related to hypergraphs of maximal simple paths of a graph are given in [2]. In this paper necessary and sufficient conditions for a hypergraph H to be a hypergraph of maximal simple paths of G in certain subclass $\mathscr{G}$ of Hamiltonian graphs are established. Also, the unicity of reconstruction of $G \in \mathscr{G}$ based on corresponding hypergraph H will be proved. The class $\mathscr{G}$ contains, as a proper subclass, all Hamiltonian graphs for which $r(v) \geqslant 3$.

A graph $G \in \mathscr{G}$ iff there exists a Hamiltonian cycle C of G such that if a vertex $v \in C$ is not incident with any chord of cycle C, then there is a chord d linking both neighbours of v in C. The class of hypergraphs of maximal paths of elements in $\mathscr{G}$ is denoted by $\mathscr{H}$. The following properties of a hypergraph $H = \langle X, \mathscr{E}\rangle \in \mathscr{H}$ are evident:

(1) $E_i \in \mathscr{E},\ A \subset E_i,\ A \neq E_i \Rightarrow A \notin \mathscr{E}$.

(2) There exists a set $C \subset X$ such that $A \subset C$, $|A| = |C|-1 \Rightarrow A \in \mathscr{E}$.

The set C called a *Hamiltonian cycle of hypergraph* H. The set $X\setminus C$ will be called a *set of chords of cycle* C and its elements—*chords of cycle* C.

Let d be an arbitrary chord of a Hamiltonian cycle of G. There exist in $D = C\cup\{d\}$ exactly two maximal paths of G with the length $|D|-2$ containing edge d. Hence, hypergraph H has to satisfy:

(3) $d \in X\setminus C \Rightarrow$ there exist exactly two sets $E_i, E_j \in \mathscr{E}$; $E_i, E_j \subset C\cup\{d\}, d \in E_i$, $d \in E_j$, $|E_i| = |E_j| = |C\cup\{d\}|-2$.

Let E_i and E_j be maximal paths in G determined by chord d in condition (3). Then, it is easy to notice that

(4) For every $x \in C\setminus E_i$ there exists exactly one $y \in C\setminus E_j$ such that $\{x, y, d\} \notin E_k$ for every $E_k \in \mathscr{E}$.

Let us denote $\{x_1, x_2\} = C \setminus E_i$ and $\{y_1, y_2\} = C \setminus E_j$. By condition (4) it follows that for a fixed chord d three cases are possible:

(a) $\{x_1, y_1, d\}$ and $\{x_2, y_2, d\}$ are not contained in any set $E_k \in \mathscr{E}$; for the remaining two sets: $\{x_1, y_2, d\} \in E_p$, $\{x_2, y_1, d\} \in E_q$; $p, q \in I$,

(b) only $\{x_2, y_1, d\}$ is contained in E_k, $k \in I$,

(c) none of the sets $\{x_i, y_j, d\}$ is contained in $E_k \in \mathscr{E}$.

Let us form a family of subsets of X denoted by $\mathscr{F}$. Let d be an arbitrary chord of cycle C. In case (a) we include sets $\{x_1, y_1, d\}$ and $\{x_2, y_2, d\}$ into family $\mathscr{F}$, in case (b) the sets $\{x_1, y_1, d\}$, $\{x_2, y_2, d\}$ and $\{x_1, y_2\}$, and in case (c): $\{x_1, y_1, d\}$, $\{x_2, y_2, d\}$, $\{x_1, y_2\}$, $\{x_2, y_1\}$ or $\{x_1, y_2, d\}$, $\{x_2, y_1, d\}$, $\{x_1, y_1\}$, $\{x_2, y_2\}$.

Now, let $\mathscr{F}$ be an arbitrary family of subsets of X and C an arbitrary non-empty subset of the set. Let us denote by $\mathscr{F}^{(2)}$ the least family of a subset in X such that

(i) $\mathscr{F} \subset \mathscr{F}^{(2)}$,

(ii) $F_i, F_j \in \mathscr{F}^{(2)}$, $|F_i \cap F_j| \geqslant 2 \Rightarrow F_i \cup F_j \in \mathscr{F}^{(2)}$.

Let us denote by $\mathscr{F}^{(2)}_{\max}$ the subfamily of all maximal sets of family $\mathscr{F}^{(2)}$. We say that family $\mathscr{F}$ *determines a Hamiltonian structure in X with respect to a set C* if the following conditions are satisfied:

(iii) $F_i \in \mathscr{F}^{(2)}_{\max} \Rightarrow |F_i \cap C| = 2$,

(iv) $x \in X \Rightarrow$ there exist exactly two sets $F_i, F_j \in \mathscr{F}^{(2)}_{\max}$ such that $x \in F_i$, $x \in F_j$,

(v) $x \in X$; $F_i, F_j \in \mathscr{F}^{(2)}_{\max}$, $F_i \neq F_j \Rightarrow (y \in X, y \neq x, y \in F_i \Rightarrow y \notin F_j)$.

Let $\mathscr{F}$ determine in X a Hamiltonian structure with respect to C. A subset S of X is called *elementary* if for every $F_i \in \mathscr{F}^{(2)}_{\max}$ we have $|F_i \cap S| \leqslant 2$. In particular, C is an elementary set. $n(s)$ denotes the number of those $F_i \in \mathscr{F}^{(2)}_{\max}$ for which $|F_i \cap S| = 1$. The number $n(s)$ is an index of an elementary set S. $\mathscr{S}$ denotes a family of subsets of X, all elementary maximal sets of a given Hamiltonian structure with index 2.

It is easy to see that if $\mathscr{F}$ is a family of subsets of X determined by conditions (a), (b), (c), then for a hypergraph H of maximal paths of a Hamiltonian graph G the following condition should be satisfied:

(5) the family $\mathscr{F}$ determines a Hamiltonian structure in X with respect to cycle C and $E_i \in \mathscr{E}$ iff $E_i \in \mathscr{S}$.

Now, we shall prove the following

THEOREM. *Hypergraph $H \in \langle X, \mathscr{E} \rangle$ is a hypergraph of maximal simple paths for a graph $G = \langle V, X, \varphi \rangle \in \mathscr{G}$ iff for H conditions (1)–(5) are satisfied. There is a one-to-one correspondence (to isomorphism) between $H \in \mathscr{H}$ and $G \in \mathscr{G}$.*

Proof. The necessity of conditions (1)–(5) is evident.

Let $H = \langle X, \mathscr{E} \rangle$ be a hypergraph satisfying (1)–(5) and C its Hamiltonian cycle. We form graph G in the following way: let $G = \langle V, X, \varphi \rangle$ where $V = \mathscr{F}^{(2)}_{\max}$ function $\varphi\colon X \to V^2$ is defined according to condition (iv): $x \in F_i$, $x \in F_j \Leftrightarrow \varphi(x) = (F_i, F_j)$.

By condition (5), and taking into account (iii), (iv), and (v), we obtain the fact that G is a simple graph and for every vertex $F_j \in \mathscr{F}^{(2)}_{\max}$ (iii) it follows that there exist exactly two edges $x \in C$ of the graph which are incident with F_j and hence C is a Hamiltonian cycle of G. By conditions (2)–(4) the unicity of construction for family $\mathscr{F}$ follows, and so it follows consequently for $\mathscr{F}^{(2)}_{\max}$. Further, by condition (5) and (iv) the unicity of construction for G is obtained. Only in case (c) a fictious ambiguity occurs for family $\mathscr{F}$ when for a chord d none of the sets $\{x_i, x_j, d\}$ is contained in $E_k \in \mathscr{E}$. By conditions (i)–(iv) it follows that if G has a chord with this property, then its Hamiltonian cycle contains exactly four edges and one or two diagonals. Here, the unicity (to isomorphism) of G is evident. The vertices of G, as it follows by construction of $\mathscr{F}$, are determined either by a chord incident with a vertex or by a pair of edges belonging to a Hamiltonian cycle and incident with the same chord d. Hence, it follows that $G \in \mathscr{G}$.

By the construction of sets in family $\mathscr{S}$ it follows that each $S \in \mathscr{S}$ is a set of edges of G belonging to an arbitrary maximal path of G. A set of edges of G belonging to an arbitrary maximal simple path of the graph is an elementary set with index 2 and it belongs to $\mathscr{S}$. Hence, according to condition (5) hypergraph $H = \langle X, \mathscr{E} \rangle$ is a hypergraph of maximal simple paths of G.

The set of conditions (1)–(5) allows to formulate a simple algorithm for verification whether $H \in \mathscr{H}$. It seems that analysis of independence and reduction of the set of conditions should be interesting.

References

[1] L. Szamkołowicz, *On problems of the elementary theory of graphical matroids*, in: *Recent advances in graph theory*, Praha 1975, 501–505.

[2] —, *On reproducity of a hypergraph of maximal paths in a graph*, in: *Graphs, hypergraphs and block systems*, Zielona Góra 1976, 297–303.

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