[2] —, *О полноте статистических алгоритмов распознавания*, ЖВМ и МФ 18, 1 (1978).
[3] —, *О модели распознающих алгоритмов типа потенциальных функций*, ЖВМ и МФ 18, 2 (1978).
[4] Ю. И. Журавлев и др., *Алгоритмы вычисления оценок и их применение*, Фан, Ташкент 1974.
[5] —, *Экстремальные алгоритмы в математических моделях для задач распознавания и классификаций*, Доклады АН СССР 236, 3 (1976).
[6] —, *Корректные алгебры над множествами некорректных (эвристических) алгоритмов*, I, Кибернетика 4 (1977).

Presented to the Semester
Discrete Mathematics
(February 15–June 16, 1977)

DISCRETE MATHEMATICS
BANACH CENTER PUBLICATIONS, VOLUME 7
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1982

A FEW RESULTS ON THE COMPLEXITY OF CLASSES OF IDENTIFIABLE RECURSIVE FUNCTION SETS

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Recently there have been several studies of special classes of identifiable recursive function sets. The complexity of such classes is generally characterized in the literature in the theory of complexity classes of recursive functions. With respect to this question the present paper uses two additional view-points:

(A) The sorting of index sets of identifiable recursive function sets in the arithmetical hierarchy, and

(B) The sorting of the required functionals, to identify recursive function sets of a special type, in the arithmetical hierarchy of function sets.

In this way the paper contributes to the subject of the limiting decision procedures (cf. Gold [2], Barzdin' [1]).

1. Introduction

Let $\mathcal{R}$ be the class of all unary (total) recursive functions on the set N of all non-negative integers. $\mathcal{F}$ is the class of all unary total functions. Let $\{M_z\}_{z\in N}$ be a computable enumeration of Turing machines which defines a standard enumeration φ of all partial recursive functions of one variable. Let $\langle x_0, \ldots, x_t\rangle$, $t \in N$, be an effective enumeration of all finite tuples of non-negative integers.

Strategies (or *inductive inference machines*) are arbitrary recursive functions $F, G, \ldots$; if F is any strategy and if g is a total function,

$$F(g^t) =_{\mathrm{df}} F(\langle g(0), \ldots, g(t)\rangle), \quad t \in N,$$

is the *hypothesis* given by F about g under the assumption that there is considered g^t, where the fixed Gödel numbering φ, being a semantics, always underlies these hypotheses. A hypothesis $a = F(g^t)$ is "true" iff a is a Gödel number of g; therefore $g = \varphi_a$. Every strategy F generates a *limiting recursive functional* $\boldsymbol{F}$ in the following way:

$$\boldsymbol{F}(g) = a \in N \quad \text{iff} \quad \lim_{t\to\infty} F(g^t) = a;$$
$$\text{iff} \quad \exists t_0 \forall t[t \geqslant t_0 \to F(g^t) = a].$$

In the case $F(g) = a$, we denote by a the *final hypothesis* given by F about g. In this paper a *functional* is a unique mapping from $\mathcal{F}$ into N everywhere. Let A be a functional;

$$L(A) =_{\mathrm{df}} \{g \mid g \in \operatorname{domain} A \;\&\; \varphi_{A(g)} = g\}$$

denotes the set of all functions *identifiable* with the functional A, where $L(A) \subseteq \mathcal{R}$. In the sense of the definition of limiting recursive functionals, for a strategy F define

$$L(F) =_{\mathrm{df}} L(F) = \{g \mid g \in \mathcal{R} \;\&\; \lim_{t\to\infty} F(g^t) = a \in N \;\&\; \varphi_a = g\};$$

therefore $L(F)$ is the set of all *limiting identifiable* functions with the strategy F.

Definition 1.

$$GN =_{\mathrm{df}} \{U \mid U \subseteq \mathcal{R} \;\&\; \exists F[F \in \mathcal{R} \;\&\; U \subseteq L(F)]\}$$

is the class of all limiting identifiable recursive function sets.

Example 1. Let

$$V_0 =_{\mathrm{df}} \{g \mid g \in \mathcal{R} \;\&\; \varphi_{g(0)} = g\}.$$

If $F(g^t) =_{\mathrm{df}} g(0)$, $V_0 = L(F)$ is immediately clear. Via the recursion theorem, for every $g_0 \in \mathcal{R}$ there exists a function $g \in V_0$ such that $g(t+1) = g_0(t)$ for every $t \in N$. Consequently V_0 is not a subset of some effective enumerable recursive function set, because otherwise $\mathcal{R}$ would be such a subset, i.e. $\mathcal{R}$ would be effective enumerable.

Definition 2. Let

$$\mathrm{NUM} =_{\mathrm{df}} \{U \mid U \subseteq \mathcal{R} \;\&\; \exists s[s \in \mathcal{R} \;\&\; U = \varphi_{s(N)}]\}$$

be the class of all effective, enumerable recursive function sets;

$$\mathrm{NUM}^{\subseteq} =_{\mathrm{df}} \{U \mid U \subseteq \mathcal{R} \;\&\; \exists V[V \in \mathrm{NUM} \;\&\; U \subseteq V]\}$$

denotes the class of all subsets of sets in NUM.

According to Gold [2] we have $\mathrm{NUM}^{\subseteq} \subseteq \mathrm{GN}$ or, more exactly,

(1) $$\mathrm{NUM} \subseteq L(\mathcal{R}) = \{L(F) \mid F \in \mathcal{R}\};$$

Barzdin' [1] proved the proper inclusion

(2) $$\mathrm{NUM}^{\subseteq} \subset \mathrm{GN}$$

for a set of type V_0. The set V_0 is "directly identifiable"; such sets are an example of finite identifiable function sets. Every strategy F generates an *E-functional* F_e in the following way:(¹)

$$F_e(g) = a \in N \quad \text{iff} \quad \lim_{t\to\infty} F(g^t) = a \;\& \;\; \& [t_0 = \mu t[F(g^t) = F(g^{t+1})] \to F(g^{t_0}) = a];$$

therefore stationarity of the hypotheses is the only *possible* final hypothesis. If F is a strategy, $E(F) =_{\mathrm{df}} L(F_e)$ denotes the set of all *finite identifiable* recursive functions with F.

(¹) "E" from "endlich".

Definition 3.

$$\mathrm{GN}_e =_{\mathrm{df}} \{U \mid U \subseteq \mathcal{R} \;\&\; \exists F[F \in \mathcal{R} \;\&\; U \subseteq E(F)]\}$$

is the class of all finite identifiable recursive function sets.

Example 2. Let

$$U_0 =_{\mathrm{df}} \{1^\omega\} \cup \{1^t 0^\omega \mid t \geqslant 1\},$$

where $g = 1^t 0^\omega$ iff $g(0) = \ldots = g(t-1) = 1$ and $g(n) = 0$, for every $n \geqslant t$. Obviously $U_0 \in \mathrm{NUM}$, because 1^ω is an "accumulation point" in U_0, $U_0 \notin \mathrm{GN}_e$. This topological view-point is worked out in [3] in detail. $U_0 - \{1^\omega\}$ is finite identifiable.

The example proves the proper inclusion

(3) $$\mathrm{GN}_e \subset \mathrm{GN}.$$

Furthermore we regard the complexity of the classes GN and GN_e relative to the view-points (A) and (B) mentioned at the beginning. The definition of the hierarchies Σ_n, Π_n and $\Sigma_n^{(\mathrm{fn})}$, $\Pi_n^{(\mathrm{fn})}$ is that of [7]. References to the subject (A) are in [4], to the subject (B) in [3]. This paper is an extract of [5].

Let $A \subseteq N$ be a recursive enumerable set, let U be a set of partial recursive functions, $\Theta A =_{\mathrm{df}} \{z \mid D_z = A\}$, where $D_z = \operatorname{domain} \varphi_z$, and let $\Omega U =_{\mathrm{df}} \{z \mid \varphi_z \in U\}$ denote the *index sets* of A and U. Use two well-known examples of index sets:

(4) $\Theta N = \{z \mid D_z = N\} = \Omega\mathcal{R}$ is Π_2-complete,

(5) $\Omega\,\mathrm{Bound} = \{z \mid \exists n \forall x[\varphi_z(x) \leqslant n]\} \in (\Sigma_3 \cap \Pi_3) - (\Sigma_2 \cup \Pi_2)$.

Lewis [6] proved that:

(6) If $U \subseteq \mathcal{R}$ and $U \neq \emptyset$, then $\Theta N \leqslant_1 \Omega U$;

(7) For every effective enumerable set $U \in \mathrm{NUM}$ we have $\Theta N \leqslant_1 \Omega U$ and $\Omega U \leqslant_1$ $\leqslant_1 \Omega$ Bound.

Functionals, as subset of $\mathcal{F} \times N$, are to be classified in the arithmetical hierarchy of function sets. According to Gold [2] we have:

(8) A functional lies in $\Sigma_2^{(\mathrm{fn})}$ iff it is a limiting recursive functional.

2. Index sets

The set $U_1 =_{\mathrm{df}} \{g \mid g \in \mathcal{R} \;\&\; g =_{\mathrm{a.e.}} 0^\omega\}$ (²) is effective enumerable; consequently $U_1 \in L(\mathcal{R})$. According to [6] we have $\Omega U_1 \equiv \Omega\mathrm{Bound}$. With $\Omega\mathrm{Bound}$ we are already given the upper bound for all index sets of sets in $L(\mathcal{R})$.

Theorem 1. $\forall U[U \in L(\mathcal{R}) \;\&\; U \neq \emptyset \to \Theta N \leqslant_1 \Omega U \leqslant_1 \Omega\mathrm{Bound}]$.

In this way, by assertion (1) assertion (7) is an immediate conclusion from Theorem 1.

Proof. The lower bound ΘN follows directly by (6). For the proof of the upper bound use the set

$$U_\tau =_{\mathrm{df}} \{g \mid g \in \mathcal{R} \;\&\; \forall t \exists t_0[t_0 > t \;\&\; \pi_1(g(t_0)) > \pi_1(g(t))] \;\& \;\; \& \exists t_1 \forall t[t \geqslant t_1 \to \pi_2(g(t)) = \pi_2(g(t_1))]\},$$

(²) $=_{\mathrm{a.e.}}$ means "equal almost everywhere".

where τ is an effective pairing function whose inverses are denoted by π_1, π_2; thus $z = \tau(\pi_1(z), \pi_2(z))$. Use a Blum measure Φ. Let F be a strategy, the following relations are recursive enumerable:

$$\begin{array}{lll} A(z,t) & \text{iff} & \forall i[i \leqslant t+1 \to \exists y \Phi_z(i) = y] \\ & \text{iff} & \bigwedge_{i=0}^{t+1} \varphi_z(i)\downarrow; \\ B(z,t,t_0) & \text{iff} & t_0 = \mu t_1[t_1 \leqslant t \,\&\, \forall x[t_1 \leqslant x \leqslant t \to \varphi_z^x\downarrow \& \\ & & \& F(\varphi_z^x) = F(\varphi_z^t)]]; \\ C(z,t,a) & \text{iff} & a = \max\{0\} \cup \{x \mid x \leqslant t+1 \,\&\, \varphi_z^t\downarrow \,\&\, \forall j[j \leqslant x \to \\ & & \to \varphi_z(j)\downarrow \,\&\, \exists y[y \leqslant t \,\&\, \Phi_{F(\varphi_z^t)}(j) = y] \,\& \\ & & \& \varphi_{F(\varphi_z^t)}(j) = \varphi_z(j)]\}. \end{array}$$

By means of

$$\varphi_{f(z)}(t) =_{\mathrm{df}} \begin{cases} \tau(a, t_0) & \text{if} \quad A(z,t) \,\&\, B(z,t,t_0) \,\&\, C(z,t,a), \\ \text{divergent} & \text{otherwise}, \end{cases}$$

an injective recursive function f is defined. For $z \in \Omega L(F)$ we first have $\varphi_z \in \mathscr{R}$; consequently $\varphi_{f(z)}$ is fully defined. Furthermore, in conformity with B, for some t_0 $\pi_2(\varphi_{f(z)}(t)) = \pi_2(\varphi_{f(z)}(t_0))$ is always true; then $F(\varphi_z)$ exists. Finally, according to C $\pi_1(\varphi_{f(z)}(t))$ is infinitely increasing, because $\varphi_{F(\tau_z)} = \varphi_z$. Consequently $f(z)$ lies in ΩU_τ. For $f(z) \in \Omega U_\tau$ in respect of relation A we must have $\varphi_z \in \mathscr{R}$, and by relation B $F(\varphi_z)$ is defined. Finally, according to C, $F(\varphi_z)$ is a Gödel number of φ_z. Altogether $z \in \Omega L(F)$ comes true. We have proved $\Omega L(F) \leqslant_1 \Omega U_\tau$.

Now, as step 2, the relation $\Omega U_\tau \leqslant_1 \Omega\text{Bound}$ is to be shown. Therefore the following recursive enumerable relations are to be defined:

$$\begin{array}{lll} A(z,t) & \text{iff} & \bigwedge_{i=0}^{t+1} \varphi_z(i)\downarrow; \\ B(z,t) & \text{iff} & \exists t_0[t_0 > t \,\&\, \varphi_z(t_0)\downarrow \,\&\, \pi_1(\varphi_z(t_0)) > \pi_1(\varphi_z(t))]; \\ C(z,t) & \text{iff} & \varphi_z(t) \,\&\, \pi_2(\varphi_z(t)) = \pi_2(\varphi_z(t+1)). \end{array}$$

The injective reducing function f is obtained by

$$\varphi_{f(z)}(t) =_{\mathrm{df}} \begin{cases} \pi_2(\varphi_z(t)) & \text{if} \quad A(z,t) \,\&\, B(z,t) \,\&\, C(z,t), \\ t & \text{if} \quad A(z,t) \,\&\, C(z,t), \\ \text{divergent} & \text{otherwise}. \end{cases}$$

For $z \in \Omega U_\tau$ A and B are always true. For a certain t_0, if $t < t_0$, then $\varphi_{f(z)}(t)$ equals t (with possible interruptions $\pi_2(\varphi_z(t))$), and if $t \geqslant t_0$, then at all times $\varphi_{f(z)}(t) = \pi_2(\varphi_z(t)) = \pi_2(\varphi_z(t_0))$. Therefore, $f(z) \in \Omega\text{Bound}$ comes true. On the other hand, let $f(z) \in \Omega\text{Bound}$; then $\varphi_{f(z)}$ cannot be divergent anywhere. Furthermore, the case $\varphi_{f(z)}(t) = t$ is limited to a finite frequency. Consequently, if $t \geqslant t_0$, the case $\varphi_{f(z)}(t) = \pi_2(\varphi_z(t))$ is the only possible one, i.e. according to A, $\varphi_z \in \mathscr{R}$ must be true, by relation B for every $t \in N$ there exists such a $t_0 > t$ that $\pi_1(\varphi_z(t_0)) > \pi_1(\varphi_z(t))$ holds, and in conformity with C there exists such a t_1 that the identity

$\pi_2(\varphi_z(t)) = \pi_2(\varphi_z(t_1))$ holds for every $t \geqslant t_1$. Altogether $z \in \Omega U_\tau$ is shown. Thus the theorem is proved. ([3])

In order to prove the acceptance of the upper bound ΩBound a set $U_1 \in \text{NUM}$ was given. Function sets of the type

$$\text{MOD}(m) =_{\mathrm{df}} \{g \mid g \in \mathscr{R} \,\&\, \exists t_0 \forall t[t \geqslant t_0 \to g(m \cdot t_0) = g(m \cdot t) \,\& \\ \& \varphi_{g(m \cdot t_0)} = g]\} \qquad (m \geqslant 1)$$

are not contained in an effective enumerable recursive function set, i.e. not in $\text{NUM}^{\subseteq}$, but of course in $L(\mathscr{R})$. For these sets $\text{MOD}(m)$, $m \geqslant 1$, $\Omega\,\text{MOD}(m) \equiv \Omega\text{Bound}$ also holds [5].

DEFINITION 4. Let A, B be subsets of N, define the *segment* from A to B as

$$\text{Sgm}(A \Rightarrow B) =_{\mathrm{df}} \{d_1(X) \mid A \leqslant_1 X \leqslant_1 B\}.$$

If $\mathscr{U}$ is a class of partial recursive function sets, then

$$\text{Sp}(\mathscr{U}) =_{\mathrm{df}} \{d_1(\Omega U) \mid U \in \mathscr{U} \,\&\, U \neq \emptyset \,\& \\ \& \forall V[V \in \mathscr{U} \,\&\, U \subseteq V \to \Omega U \leqslant_1 \Omega V]\}$$

denotes the *spectrum* of $\mathscr{U}$.

EXAMPLE 3. If A non$\leqslant_1 B$, then $\text{Sgm}(A \Rightarrow B) = \emptyset$. By the completeness theorem of the arithmetical hierarchy we obtain $\text{Sgm}(\emptyset^{(n)} \Rightarrow \emptyset^{(m)}) = \{d_1(X) \mid X \in \Sigma_m - \Pi_n\}$ as an example. We have $\text{Sp}(2^{\mathscr{R}}) = \{d_1(\Theta N)\} = \text{Sgm}(\Theta N \Rightarrow \Theta N)$.

CONCLUSION 1. $\text{Sp}(\text{GN}) \subseteq \text{Sgm}(\Theta N \Rightarrow \Omega\text{Bound})$

This assertion is an immediate conclusion from Theorem 1. It is not known whether the proper inclusion holds or does not hold here. For sets $E(F)$ of the finite identification [5] holds:

(9) $\forall U[U \in (\mathscr{R}) \,\&\, U \neq \emptyset \to \Theta N \leqslant_1 \Omega U \leqslant_1 \Omega\text{Bound}]$;

(10) $\exists U[U \in E(\mathscr{R}) \,\&\, \Omega U \equiv \Omega\text{Bound}]$.

But the upper bound ΩBound will be accepted here only by "unnatural" sets, for which in $E(\mathscr{R})$ there already exist upper sets with Π_2-complete index sets.

THEOREM 2. $\text{Sp}(\text{GN}_e) = \{d_1(\Theta N)\}$.

Proof. To outline the general idea, first define a special kind of functionals and show that these functionals are sufficient to generate the class GN_e. Then for a functional A of this special type we shall prove $\Omega L(A) \equiv \Theta N$ if $L(A) \neq \emptyset$.

Let F be a strategy, and let g be a total function. Define

$$F_{ee}(g) =_{\mathrm{df}} \begin{cases} a & \text{if} \quad \exists t_0[F(g_0^t) = F(g_0^{t+1}) = a \,\&\, \forall t \\ & \qquad [t < t_0 \to F(g^t) \neq F(g^{t+1})]], \\ \text{divergent} & \text{otherwise}. \end{cases}$$

Such functionals F_{ee} are called *functionals of the effective finite identification* or short *E_e-functionals*. Therefore, in the case of effective finite identification, already

([3]) Of course the reduction $\Omega L(F) \leqslant_1 \Omega$ Bound is possible in one step. But in this case the construction is very complicated.

the first consecution consisting of one hypothesis is sufficient to fix that hypothesis as final. Let F be a strategy, and let $E_e(F) =_{df} L(F_{ee})$ denote the set of all *effective, finite identifiable* recursive functions by this strategy F.

LEMMA 1. *For every strategy F you can construct a strategy G for which G_{ee} is an extension of the functional F_e.*

Proof. Let F be a recursive function, and define

$G(g^t) =_{df}$ "If for some $t_0 < t$ and for some $b \in N$ the identity $F(g^{t_0}) = F(g^{t_0+1}) = b$ holds, and if t_0 is minimal relative to this property, then the output is b. Otherwise the output is t."

The inclusion $F_e \subseteq G_{ee}$ follows immediately. By means of the same construction you can find

LEMMA 2. *For every strategy F a strategy G is constructible such that $G_e = F_{ee}$ holds.*

Lemma 1 and Lemma 2 imply

$$\mathrm{GN}_e = \{U \mid U \subseteq \mathscr{R} \;\&\; \exists F[F \in \mathscr{R} \;\&\; U \subseteq E_e(F)]\}. \tag{11}$$

LEMMA 3. *For every strategy F, $E_e(F) \neq \emptyset$ implies $\Omega E_e(F) \equiv \Theta N$.*

Proof. The connection $\Theta N \leqslant_1 \Omega E_e(F)$ holds by (6). For the proof of the relation $\Omega E_e(F) \leqslant_1 \Theta N$ we define the following recursive enumerable relations:

$$\begin{aligned}
A(z,t) \quad &\text{iff} \quad \textstyle\bigwedge_{i=0}^{t+1} \varphi_z(i)\downarrow;\\
B(z) \quad &\text{iff} \quad \exists t_0[F(\varphi_z^{t_0})\downarrow \;\&\; F(\varphi_z^{t_0}) = F(\varphi_z^{t_0+1})];\\
C(z,t) \quad &\text{iff} \quad \exists t_0 \exists a[t_0 \leqslant t \;\&\; F(\varphi_z^{t_0}) = F(\varphi_z^{t_0+1}) = a \;\&\; \forall t_1\\
&\qquad [t_1 < t_0 \to F(\varphi_z^{t_1}) \neq F(\varphi_z^{t_1+1})] \Rightarrow\\
&\qquad \textstyle\bigwedge_{i=0}^{t} \varphi_a(i) = \varphi_z(i)];
\end{aligned}$$

we obtain the injective reducing function f using

$$\varphi_{f(z)}(t) =_{df} \begin{cases} 0 & \text{if } A(z,t) \;\&\; B(z) \;\&\; C(z,t),\\ \text{divergent} & \text{otherwise.}\end{cases}$$

(i) $z \in \Omega E_e(F)$. Because of $\varphi_z \in \mathscr{R}$, $A(z,t)$ is always true. $F_{ee}(\varphi_z)\downarrow$ implies $B(z)$. The relation C describes the behaviour of F on φ_z; because of $F_{ee}(\varphi_z) = a \in N$ and $\varphi_a = \varphi_z$, C always holds; for $t < t_0$ the premise (separation at $\Rightarrow$ in C) is not satisfiable, and therefore $C(z,t)$ is true for every $t < t_0$.

(ii) $f(z) \in \Theta N$. By A we have $\varphi_z \in \mathscr{R}$. According to B, for sufficiently large t there exist everywhere t_0 and a such that the premise in C is satisfiable. Consequently for sufficiently large t the conclusion in C is true. If $t_0 = \mu t_1[F(\varphi_z^{t_1}) = F(\varphi_z^{t_1+1})]$, then because of C with $a = F(\varphi_z^{t_0}) = F_{ee}(\varphi_z)$ will be given an index of φ_z. By assertion (11) and Lemma 3 the theorem is proved.

3. Functionals

According to Gold's theorem (8), we immediately have

THEOREM 3. *For every set $U \subseteq \mathscr{R}$ the following statements are equivalent:*

(a) $U \in \mathrm{GN}$;

(b) *There exists a functional $A \in \Sigma_2^{(fn)}$ such that $U \subseteq L(A)$.*

In this paper we shall prove

THEOREM 4. *For every set $U \subseteq \mathscr{R}$ the following statements are equivalent:*

(a) $U \in \mathrm{GN}_e$;

(b) *There exists a functional $A \in \Sigma_1^{(fn)}$ such that $U \subseteq L(A)$.*

Observe that for sets in $\mathrm{NUM}^{\subseteq}$ the following propositions are true [5]:

THEOREM 5. *For every set $U \in \mathrm{GN} \cap \mathrm{NUM}^{\subseteq}$ there exists a functional $A \in \Sigma_2^{(fn)} \cap \cap \Pi_2^{(fn)}$ such that $U \subseteq L(A)$.*

THEOREM 6. *For every set $U \in \mathrm{GN}_e \cap \mathrm{NUM}^{\subseteq}$ there exists a functional $A \in \Sigma_0^{(fn)}$ such that $U \subseteq L(A)$.*

Theorems of this type characterize the complexity of sets in GN, GN_e, $\mathrm{GN} \cap \cap \mathrm{NUM}^{\subseteq}$ etc.(4) relative to the functionals required for the identification of recursive function sets. For example, we can say that in GN_e all "$\Sigma_1^{(fn)}$-hard" identifiable recursive function sets are exactly.

Proof (Theorem 4). (a) $\Rightarrow$ (b). If $U \in \mathrm{GN}_e$ ((by (11)), then there exists such a strategy F that $U \subseteq L(F_{ee})$ holds.

LEMMA 4. *For every strategy F the generated E_e-functional F_{ee} lies in $\Sigma_1^{(fn)}$.*

Proof. Let F be a strategy, $g \in \mathscr{F}$ and $a \in N$; then we have

$$[g,a] \in F_{ee} \quad \text{iff} \quad \exists t_0[F(g^{t_0}) = F(g^{t_0+1}) = a \;\&\; \textstyle\bigwedge_{i=0}^{t_0-1} F(g^i) \neq F(g^{i+1})] \text{ everywhere.}$$

Consequently there exists a recursive relation $R \subseteq \mathscr{F} \times N^2$ such that

$$[g,a] \in F_{ee} \quad \text{iff} \quad \exists t_0 R(g,a,t_0),$$

therefore $F_{ee} \in \Sigma_1^{(fn)}$ is proved.

(b) $\Rightarrow$ (a). We use

LEMMA 5. *For every functional $A \in \Sigma_1^{(fn)}$ there exists a strategy F such that $A = F_{ee}$ holds.*

Proof. For A let $R \subseteq \mathscr{F} \times N^2$ be a recursive relation relative to which

$$[g,a] \in A \quad \text{iff} \quad \exists x R(g,a,x)$$

always holds. Define by means of

$F(g^t) =_{df}$ "For some $b \leqslant t$ there exists an $x \leqslant t$ such that $R(g^{t_0\omega}, b, x)$ is true and for this [during the computation to establish that $R(g^{t_0\omega}, b, x)$ holds] the question will be asked ["Is $[i,j] \in g$ true or not?"] only on tuples $[i,j]$ with $i \leqslant t$. [Since A is a unique mapping, there exists at most one b of this kind.] Then the output is b. [We have $A(g) = b$.] If such b does not exist, then the output is t."

a recursive function F. By definition, $F_{ee} = A$ is true.

(4) For some other classes see for instance [5].

Now assume that for some $U \subseteq \mathscr{R}$ there exists a functional $A \in \Sigma_1^{(\mathrm{fn})}$ with $U \subseteq L(A)$. By Lemma 5, for $F \in \mathscr{R}$ let $A = F_{ee}$; therefore $U \subseteq E_e(F)$ and, together with (11), $U \in \mathrm{GN}_e$. Thus the theorem is proved.

References

[1] [I. M. Barzdin'] Я. М. Барздинь, *Сложность и частотное решение некоторых алгоритмических неразрешимых массовых проблем*, Дисс. Новосибирск 1971.

[2] E. M. Gold, *Limiting recursion*, J. Symb. Logic 30 (1965), 28–48.

[3] R. Klette, *Erkennung allgemein rekursiver Funktionen*, Elektron. Informationsverarbeit. Kybernetik 12 (1976), 227–243.

[4] —, *Indexmengen und Erkennung rekursiver Funktionen*, Z. Math. Logik Grundlagen Math. 22 (1976), 231–238.

[5] —, *Hierarchiebetrachtungen zur Erkennung rekursiver Funktionen*, to be published.

[6] F. D. Lewis, *Classes of recursive functions and their index sets*, ibid. 17 (1971), 291–294.

[7] H. Rogers, *Theory of recursive functions and effective computability*, McGraw-Hill, New York 1967.

Presented to the Semester
Discrete Mathematics
(February 15–June 16, 1977)