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COMPLETENESS FOR A FAMILY OF NORMAL DISTRIBUTIONS

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1. Introduction

In this paper the problem of completeness of minimal sufficient statistics for a family of multivariate normal distributions $\mathscr{P} = \{N(\mu, \Sigma)\colon \mu \in \mathscr{X}, \Sigma \in \mathscr{V}\}$ is considered. Here $\mathscr{X}$ is a subspace of n-dimensional Euclidean space R^n and $\mathscr{V}$ is a set of $n\times n$ positive definite matrices containing non-empty open set relative to the sp $\mathscr{V}$ (the smallest linear space containing $\mathscr{V}$).

For special cases this problem has been considered by Graybill and Hultquist [1] and Seely [3], [4].

2. Minimal sufficient statistics

Without loss of generality we assume that the identity operator I belongs to $\mathscr{V}$. Let $\mathscr{W} = \{\Sigma^{-1}\colon \Sigma \in \mathscr{V}\}$ and let $W_1, \ldots, W_k$ form a basis for sp $\mathscr{W}$. Then for $W \in \mathscr{W}$ we have

$$W = \sum_{i=1}^{k} w_i(\Sigma) W_i, \tag{1}$$

where $w(\Sigma) = (w_1(\Sigma), \ldots, w_k(\Sigma))'$ is the vector of linearly independent functions from $\mathscr{V}$ to R^1. Let $\mathscr{R}$ be the smallest subspace of R^n such that $\Sigma^{-1}\mathscr{X} \subset \mathscr{R}$ for all $\Sigma \in \mathscr{V}$. Note that $\mathscr{X} \subset \mathscr{R}$, because $I \in \mathscr{V}$. Let $x_1, \ldots, x_p$ and $x_1, \ldots, x_p, x_{p+1}, \ldots, x_r$ be a basis for $\mathscr{X}$ and $\mathscr{R}$, respectively. Moreover, let for $\mu \in \mathscr{X}$ and $\Sigma \in \mathscr{V}$, $f(y|\mu, \Sigma)$ denote the $N(\mu, \Sigma)$ density functions with respect to Lebesgue's measure.

LEMMA 1. *The functions* $T(y) = (x_1'y, \ldots, x_r'y)'$ *and* $W(y) = (y'W_1y, \ldots, y'W_ky)$ *are minimal sufficient statistics for* $\mathscr{P}$.

Proof. Let R denote the $n\times r$ matrix $[x_1 \ldots x_r]$. Since $R(R'R)^{-1}R'$ is the orthogonal projection on $\mathscr{R}$, we have

$$\Sigma^{-1}\mu = R(R'R)^{-1}R'\Sigma^{-1}\mu = R\alpha(\Sigma, \mu),$$

where $\alpha(\Sigma, \mu) = (\alpha_1(\Sigma, \mu), \ldots, \alpha_r(\Sigma, \mu))'$ is the vector of functions from $\mathscr{X} \times \mathscr{V}$ to R^r. From this and (1) it is easy to find that f can be expressed in the following form:

$$f(y|\mu, \Sigma) = c \exp\{(\alpha(\Sigma, \mu), T(y)) - \tfrac{1}{2}(w(\Sigma), W(y)) + \Psi(\mu, \Sigma)\}, \tag{2}$$

where $(\cdot\,, \,\cdot)$ denotes the usual inner product, $W(y) = (y'W_1 y, \ldots, y'W_k y)'$, and $\Psi(\mu, \Sigma) = 2^{-1}\{\ln|\Sigma| + \mu'\Sigma^{-1}\mu\}$. From the fact that $w_1(\Sigma), \ldots, w_k(\Sigma), \alpha_1(\Sigma, \mu), \ldots, \alpha_r(\Sigma, \mu)$ are linearly independent and that $x_1'y, \ldots, x_r'y,\ y'W_1 y, \ldots, y'W_k y$ are linearly independent the lemma follows.

Now we obtain another representation of minimal sufficient statistics under assumption that $\mathscr{R} = \mathscr{X}$, i.e. $P\Sigma = \Sigma P$ for $\Sigma \in \mathscr{V}$, where P is the orthogonal projection on $\mathscr{X}$. Let $\mathscr{V}_1 = \{P\Sigma P \colon \Sigma \in \mathscr{V}\}$ and let $\mathscr{V}_2 = \{M\Sigma M \colon \Sigma \in \mathscr{V}\}$, where $M = I - P$. If $P\Sigma = \Sigma P$, then it is easy to find that $\Sigma^{-1} = (P\Sigma P)^+ + (M\Sigma M)^+$, where A^+ denotes the Moore–Penrose general inverse of A.

Let $\mathscr{W}_1 = \{(P\Sigma P)^+ \colon \Sigma \in \mathscr{V}\}$ and $\mathscr{W}_2 = \{(M\Sigma M)^+ \colon \Sigma \in \mathscr{V}\}$. Select linear independent matrices $U_1, \ldots, U_u$ such that $\operatorname{sp}\mathscr{W}_2 = \operatorname{sp}\{U_1, \ldots, U_u\}$; the expression $-\frac{1}{2}(w(\Sigma), W(y))$ in (2) can be rewritten

$$-\tfrac{1}{2}\Big\{y'(P\Sigma P)^+ y + \sum_{i=1}^{u} u_i(\Sigma) y' U_i y\Big\}. \tag{3}$$

Since $y'(P\Sigma P)^+ y$ is a function of $T(y)$ and $\mathscr{R} = \mathscr{X}$, we have the following.

LEMMA 2. *Let $P\Sigma = \Sigma P$ for $\Sigma \in \mathscr{V}$. Then the functions $x_1'y, \ldots, x_p'y, y'U_1 y, \ldots$ $\ldots, y'U_u y$ are minimal sufficient statistics for $\mathscr{P}$.*

3. Completeness

In the following theorem a necessary and sufficient conditions for completeness of $T(y)$ and $W(y)$ are given.

THEOREM 1. *The minimal sufficient statistics are complete iff*

$$P\Sigma = \Sigma P \quad \textit{for} \quad \Sigma \in \mathscr{V} \tag{4}$$

and

(5) $\operatorname{sp}\mathscr{V}_2$ *is a quadratic subspace of all symmetric matrices.*

Proof. (a) Let $P\Sigma \neq \Sigma P$ for some $\Sigma \in \mathscr{V}$. Then $\mathscr{R} \neq \mathscr{X}$ and this implies that there exists a vector $a \neq 0$ such that $a = \sum_{i=1}^{r} c_i x_i$ for some c_i and a is orthogonal to $\mathscr{X}$. The minimal sufficient statistics are not complete because the expectation of $a'y$ is equal to zero.

Now, we assume that $P\Sigma = \Sigma P$ for all $\Sigma \in \mathscr{V}$ and that $\mathscr{V}_2$ is not a quadratic subspace. Similarly as in [2] (see Lemma 1) it is easy to find that $\mathscr{W}_2 \neq \mathscr{V}_2$. It implies that there exists a matrix $A \neq 0$ such that $A = \sum_{i=1}^{u} d_i U_i$ for some d_i and

that the trace of AV is equal to zero for $V \in \mathscr{V}_2$. It is clear that the expectation of $y'Ay$ is equal to zero. From Lemma 2 it follows that $y'Ay$ is a function of minimal sufficient statistics. From above it follows that the minimal sufficient statistics are not complete.

(b) Now, let conditions (4) and (5) are satisfied. Then it follows from Lemma 2 that $x_1'y, \ldots, x_p'y$ and $y'U_1 y, \ldots, y'U_u y$ are minimal sufficient statistics. Define the matrix $X = [x_1, \ldots, x_p]$ and $z = X'y$; it is clear that z is normally distributed with expectation ξ, where ξ is running over R^p. This implies that the family of distributions of z is strong complete. Moreover, z and $y'U_i y$ are independent for $i = 1, \ldots, u$ because $X'\Sigma U_i = 0$. Now it is sufficient to prove that the family of distributions of $y'U_1 y, \ldots, y'U_u y$ is complete. Since $\mathscr{V}$ contains a non-empty open set relative to the sp $\mathscr{V}$ and since $M \otimes M$ is a linear operator, $\mathscr{V}_2$ contains a non-empty open set in sp $\mathscr{V}_2$. Since $\mathscr{V}_2$ is a quadratic subspace, we have $\mathscr{W}_2 = \mathscr{V}_2$. Similarly as in Lemma 8 of Seely [3] it is easy to find that $(u_1(\Sigma), \ldots, u_u(\Sigma))$ contains an open set in R^u. Now, from the known theorem of Lehmann and Scheffé, the completeness of $y'U_1 y, \ldots, y'U_u y$ follows.

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