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SOME REMARKS ON LARGE DEVIATIONS FOR WEIGHTED SUMS IF CRAMÉR'S CONDITION IS NOT SATISFIED

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1. Introduction

1.1. We consider a sequence of independent identically distributed random variables $X_1, X_2, \dots$ with $EX_1 = 0$ and $D^2X_1 = 1$ and a double array $\{a\} = \{a_{nk}, 1 \leqslant k \leqslant n, 1 \leqslant n < \infty\}$ of nonnegative numbers. We want to study the asymptotic behaviour of the probabilities

$$(1.1)\qquad P\{a_{n1}X_1 + \dots + a_{nn}X_n > x\} \quad \text{or} \quad P\{a_{n1}X_1 + \dots + a_{nn}X_n < -x\}$$

in the case where if $n \to \infty$ also $x = x(n) \to \infty$. Large deviation theorems for weighted sums under Cramér's condition were studied by S. A. Book [1], [2], L. Saulis and V. Statulevičius [6]. Our aim is to derive asymptotic representations for the probabilities (1.1) if Cramér's condition is not satisfied.

1.2. In the following, g always denotes a function with the following properties: $g(x)$ is nondecreasing and continuous if $x > C(g)$ and satisfies the conditions

$$(1.2)\qquad \varrho(x)\ln x \leqslant g(x) \leqslant C^*(g)x^{\alpha}, \quad 0 < \alpha < 1$$

and

$$(1.3)\qquad g(x)x^{-1} \text{ is strictly decreasing.}$$

(Here $\varrho(x)$ is an arbitrary monotone increasing function with $\lim_{x\to\infty} \varrho(x) = \infty$, $C(g)$ and $C^*(g)$ are positive constants depending on g.)

Furthermore, let the array $\{a\}$ satisfy the following condition (see [6]):

There exist numbers δ and β, $0 < \delta \leqslant 1$, $0 < \beta \leqslant 1$, such that, for every sufficiently large n, for at least δn of the a_{nk}'s the inequalities

$$(1.4)\qquad a_{nk} \geqslant \beta\gamma_n$$

hold; here

$$(1.5)\qquad \gamma_n = \max_k\{a_{nk}, 1 \leqslant k \leqslant n\}.$$

1.3. We introduce the following notations:

$$V(x) = P\{X_1 < x\}, \quad v(t) = Ee^{itX_1}, \quad S_n = \sum_{j=1}^{n} a_{nj}X_j, \quad B_n^2 = \sum_{j=1}^{n} a_{nj}^2,$$

$$(1.6) \qquad a_n = \sum_{j=1}^{n} a_{nj}, \quad H_n = B_n(\sqrt{\delta}\,\beta\gamma_n)^{-1} \quad W_n^2 = B_n^2 H_n^{-2},$$

$$\varphi(x) = (\sqrt{2\pi})^{-1}\int_{-\infty}^{x} \exp(-t^2/2)\,dt, \qquad \omega_k(z) = \frac{\int_z^\infty \exp(-t^2/2)(t-z)^k\,dt}{\int_z^\infty \exp(-t^2/2)\,dt}.$$

The cumulant of order k of the random variable X_1 is denoted by γ_k. Let $\Lambda_a(n)$ be the root of the equation

$$(1.7) \qquad lx^2 = H_n^2 g(x) \quad (l > 1).$$

Then from (1.2)

$$(1.8) \qquad \Lambda_a(n) \leqslant (C^*(g)B_n^2(\delta\beta^2\gamma_n^2)^{-1})^{1/(2-\alpha)}.$$

If p is a nonnegative integer, then

$$\lambda_n^p(t) = \sum_{j=0}^{p-1} \lambda_{jn} t^j,$$

where $\lambda_n(t)$ is the Cramér–Petrov power series [3] and

$$L_n(z,p) = \sum_{\nu=1}^{p-1} N_{\nu n}(z)\,(z/\sqrt{n})^{\nu} + \sum_{\nu=1}^{p-1}\sum_{l=1}^{\nu}\sum_{i=0}^{[3l/2]} e_{il\nu-ln} n^{-\nu/2} z^{\nu-l}\omega_{3l-2i}(z) +$$
$$+ \sum_{\nu=1}^{p-3}\sum_{l=1}^{\nu}\sum_{i=0}^{[3l/2]} e_{il\nu-ln} n^{-\nu/2} z^{\nu-l} \sum_{\tilde{l}=1}^{p-\nu-1} M_{\tilde{l}n}(z)\,(z/\sqrt{n})^{\tilde{l}}.$$

Here

$$N_{\nu n}(z) = \sum_{l=1}^{\nu} \frac{(-1)^l}{l!}\omega_l(z) z^l b_{l\nu n},$$

$$M_{\tilde{l}n}(z) = \sum_{r=1}^{\tilde{l}} \frac{(-1)^r}{r!} z^r \omega_{r+3l-2i}(z) b_{r\tilde{l}n},$$

$$b_{lkn} = \sum_{\substack{k_j \geqslant 1 \\ k_1+\ldots+k_l=k}} \prod_{j=1}^{l} b_{k_j n}.$$

$L_n(z,-1) = 0$, $L_n(z,1) = 0$. In our case the coefficients λ_{jn}, b_{lkn}, and e_{lkn} are expressed in terms of the cumulants of the random variable X_1 and of the sums $\frac{1}{n}\sum_{k=1}^{n} a_{kn}^l$, $l = 2, \ldots$ For example,

$$(1.10) \qquad \lambda_{jn} = f\left(\frac{1}{n}\Big(\sum_{k=1}^{n} a_{nk}^l\Big)\gamma_l,\ l = 2, \ldots, j+3\right)$$

and the coefficients of the series $L_n(z,p)$ are defined by the first cumulants $\gamma_2, \ldots, \gamma_{p+2}$ and the sums $n^{-1}\sum_{k=1}^{n} a_{nk}^l$, $l = 2, \ldots, p+2$. The series $L(z,p)$ was first introduced by L. Saulis [5].

2. Large deviation limit theorems

2.1. In this paper the following condition plays an important role:

$$(\mathrm{A}) \qquad E\exp\{g(|X_1|)\} < \infty.$$

THEOREM 1. *If $x \geqslant 0$ and if condition* (A) *is satisfied, then*

$$(2.1) \qquad \begin{aligned} \frac{P\{S_n > xW_n\}}{1-\varphi\left(\frac{x}{H_n}\right)} &= \exp\left\{\frac{x^3}{H_n^4}\lambda_n^{[s+1]}\left(\frac{x}{H_n^2}\right)\right\}\left[1+O\left(\frac{x+H_n}{H_n^2}\right)\right], \\ \frac{P\{S_n < -xW_n\}}{\varphi\left(-\frac{x}{H_n}\right)} &= \exp\left\{-\frac{x^3}{H_n^4}\lambda_n^{[s+1]}\left(-\frac{x}{H_n^2}\right)\right\}\left[1+O\left(\frac{x+H_n}{H_n^2}\right)\right] \end{aligned}$$

as $n \to \infty$ in the domain $0 \leqslant x \leqslant \Lambda_a(n)$. Here $s = \left[\frac{\alpha}{1-\alpha}\right]$. If $\left[\frac{\alpha}{1-\alpha}\right] = \frac{\alpha}{1-\alpha}$, then it is allowed in (2.1) *to consider only the first s terms in the Cramér–Petrov series.*

We have some consequences of this result.

THEOREM 2. *If condition* (A) *is satisfied with $\alpha = \frac{r}{r+1}$ for a fixed positive integer r, then*

$$(2.2) \qquad \begin{aligned} \frac{P\{S_n > xW_n\}}{1-\varphi\left(\frac{x}{H_n}\right)} &= \exp\left\{\frac{x^3}{H_n^4}\lambda_n^{[r]}\left(\frac{x}{H_n^2}\right)\right\}\left[1+O\left(\frac{x+H_n}{H_n^2}\right)\right], \\ \frac{P\{S_n < -xW_n\}}{\varphi\left(-\frac{x}{H_n}\right)} &= \exp\left\{-\frac{x^3}{H_n^4}\lambda_n^{[r]}\left(-\frac{x}{H_n^2}\right)\right\}\left[1+O\left(\frac{x+H_n}{H_n^2}\right)\right] \end{aligned}$$

as $n \to \infty$ in the domain $0 \leqslant x \leqslant \Lambda_a(n)$ with $\Lambda_a(n) = (C^*(g)H_n^2)^{(1+r)/(2+r)}$.

Theorem 2 implies the following result:

THEOREM 3. *Suppose that the conditions of Theorem 2 are satisfied. If*

$|x| \leqslant (C^*(g))^{(1+r)/(2+r)} H_n^{r/(2+r)}$ *and* $\gamma_k = 0$ *for* $k = 3, \ldots, r+2$, *then*

$$P\left\{\frac{S_n}{B_n} < x\right\} - \varphi(x) = O\left(\frac{1}{H_n} e^{-x^2/2}\right). \tag{2.3}$$

This assertion is a consequence of relations (2.2), (1.10) and the inequality

$$1-\varphi(x) = \varphi(-x) < \frac{1}{x\sqrt{2\pi}} e^{-x^2/2} \quad \text{for} \quad x > 0.$$

EXAMPLE. Let $a_{nk} = 1$ for every k, $(1 \leqslant k \leqslant n)$; then $\gamma_n = 1$, $\delta = 1$, $\beta = 1$, $B_n^2 = n$, $H_n = \sqrt{n}$, $\Lambda_a(n) = \Lambda(n) \leqslant (C^*(g)n)^{1/(2-\alpha)}$. Under the condition of Theorem 1 we obtain from (2.1) Theorem 1 in [8]:

$$\frac{P\{X_1+\ldots+X_n > x\}}{1-\varphi\left(\frac{x}{\sqrt{n}}\right)} = \exp\left\{\frac{x^3}{n^2} \lambda_n^{[s+1]}\left(\frac{x}{n}\right)\right\}\left[1+O\frac{x+\sqrt{n}}{n}\right)\Big],$$

(2.4)

$$\frac{P\{X_1+\ldots+X_n < -x\}}{\varphi\left(-\frac{x}{\sqrt{n}}\right)} = \exp\left\{-\frac{x^3}{n^2}\lambda_n^{[s+1]}\left(-\frac{x}{n}\right)\right\}\left[1+O\left(\frac{x+\sqrt{n}}{n}\right)\right]$$

as $n \to \infty$ in the domain $0 \leqslant x \leqslant \Lambda(n)$. The same result holds if $a_{nk} = n^{-1/2}$ for every k $(1 \leqslant k \leqslant n)$.

2.2. In the following we give a large deviation theorem under the condition

$$E|X_1|^k < \infty \tag{B}$$

for a certain fixed $k \geqslant 3$.

THEOREM 4. *If condition* (B) *is satisfied, then*

$$\frac{P\{S_n > xW_n\}}{1-\varphi\left(\frac{x}{H_n}\right)} \to 1, \quad \frac{P\{S_n < -xW_n\}}{1-\varphi\left(-\frac{x}{H_n}\right)} \to 1 \tag{2.5}$$

as $n \to \infty$ *in the domain* $0 \leqslant x \leqslant \sqrt{(k/2-1)H_n^2 \ln H_n^2}$.

Relations (2.5) express the large deviation problem in the central limit theorem (see e.g. [7]).

2.3. If condition (A) is satisfied, then there exists the kth moment of the random variable X_1 and we are able to get asymptotic expansions for large deviations for weighted sums. Asymptotic expansions for large deviations were first obtained by L. Saulis [4].

In the following we suppose that

$$\limsup_{|t|\to\infty} |v(t)| < 1. \tag{C}$$

THEOREM 5. *If conditions* (A) *and* (C) *are satisfied, then*

$$\frac{P\{S_n > xW_n\}}{1-\varphi\left(\frac{x}{H_n}\right)} = \exp\left\{\frac{x^3}{H_n^4}\lambda_n^{[s+q]}\left(\frac{x}{H_n^2}\right)\right\}\left[1+L_n\left(\frac{x}{H_n}, q\right)+O\left(\frac{x}{H_n^2}\right)^q\right],$$

(2.6)

$$\frac{P\{S_n < -xW_n\}}{\varphi\left(-\frac{x}{H_n}\right)} = \exp\left\{-\frac{x^3}{H_n^4}\lambda_n^{[s+q]}\left(-\frac{x}{H_n^2}\right)\right\}\left[1+L_n\left(-\frac{x}{H_n}, q\right)+O\left(\frac{x}{H_n^2}\right)\right]$$

as $n \to \infty$ *in the domain* $H_n < x \leqslant \Lambda_a(n)$. *Here* $q \geqslant 1$ *is an arbitrary integer and* s *was given in Theorem* 1.

Theorem 5 implies some results similar to Theorems 2 and 3. We shall give one of them.

THEOREM 6. *If condition* (C) *and the conditions of Theorem* 2 *are satisfied and* $\gamma_k = 0$ *for* $k = 3, \ldots, r+q+2$, *then*

$$P\left\{\frac{S_n}{B_n} < x\right\} - \varphi(x) = O\left(\frac{x^{q-1}}{H_n^q} e^{-x^2/2}\right) \tag{2.7}$$

in the domain $1 < |x| \leqslant (C^*(g))^{(1+r)/(2+r)} H_n^{r/(2+r)}$.

3. Some remarks about the proof of the large deviations limit theorems

Let $c_1, c_2, \ldots$ be positive and $\varepsilon_1, \varepsilon_2, \ldots$ small positive constants. $\overset{n}{\underset{j=1}{\boxed{*}}} V_j$ denotes the composition of the distribution functions $V_1, V_2, \ldots, V_n$. Furthermore, $V_{nj}(x) = V\left(\frac{x}{a_{nj}}\right)$ is the distribution function of $Y_{nj} = a_{nj}X_j$. $F_n(x)$ denotes the d.f. of S_n. Then we have $F_n(x) = \overset{n}{\underset{j=1}{\boxed{*}}} V_{nj}(x)$. For $y > 0$ we define new distribution functions $V_{nj}^y(x)$:

$$V_{nj}^y(x) = \begin{cases} V_{nj}(x), & x \leqslant 0, \\ 1-V_{nj}(y)+V_{nj}(x), & 0 < x \leqslant y, \\ 1, & x > y. \end{cases}$$

We denote

$$F_n^y(x) = \overset{n}{\underset{j=1}{\boxed{*}}} V_{nj}(x).$$

Then we can write

$$1-F_n(xW_n) = 1-F_n^y(xW_n)+F_n^y(xW_n)-F_n(xW_n).$$

The following inequality holds:

$$F_n^y(xW_n)-F_n(xW_n) \leqslant n\left(1-V\left(\frac{y}{\gamma_n}\right)\right).$$

Put $y = \Lambda_a(n)\gamma_n$. We introduce a parameter h with

$$\frac{c_1}{H_n} \leqslant h \leqslant c_2 \frac{\Lambda_a(n)}{H_n^2}.$$

Further, using the method proposed in [9], we obtain relations (2.6) or (2.1) of Theorem 5 or Theorem 1.

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ROBUSTNESS: A QUANTITATIVE APPROACH

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According to Box and Anderson [1] who introduced the notion, a test is "robust" if it is "sensitive to change, of a magnitude likely to occur in practice, in extraneous factors". Furthermore, a test is said to be "powerful" if it is "sensitive to change in the specific factor tested". In the note a real valued function on the parameter space of a statistical problem is constructed which measures robustness of a test similarly as the power function measures its "sensitivity to change in the factor tested".

More precisely, given a statistical structure $M_0 = (\mathscr{X}, \mathscr{A}, \mathscr{P}_0)$, $\mathscr{P}_0 \subset \mathscr{P}$, $\mathscr{P}$ being the set of all probability measures on $\mathscr{A}$, we will use a larger structure $M_1 \supset M_0$ to express "changes, of a magnitude likely to occur in practice, in extraneous factors". Let $\pi\colon \mathscr{P}_0 \to 2^{\mathscr{P}}$ be a function such that $\pi(P) \ni P$ and define $M_1 = (\mathscr{X}, \mathscr{A}, \mathscr{P}_1)$ with $\mathscr{P}_1 = \bigcup_{P \in \mathscr{P}_0} \pi(P)$. Let t be a fixed statistic and ϱ a real valued function on $\mathscr{P}_1^t$, $\mathscr{P}_1^t = \{P^t(\cdot) = P\big((t^{-1}(\cdot)\big),\ P \in \mathscr{P}_1\}$. A function $r_t\colon \mathscr{P}_0 \to \mathbf{R}^1$ defined as

$$r_t(P) = \sup\{\varrho(Q^t)\colon Q \in \pi(P)\} - \inf\{\varrho(Q^t)\colon Q \in \pi(P)\}$$

is called *ϱ-robustness* of the statistic t in the extension M_1 of M_0.

EXAMPLE. Let d be a metric in the space $\mathscr{P}$ and for a given statistic t let d_t be a metric in $\mathscr{P}^t$. For a given statistical structure $M_0 = (\mathscr{X}, \mathscr{A}, \mathscr{P}_0)$ consider M_1 defined as ε-extension of M_0 constructed by the mapping $\pi(P) = \{Q \in \mathscr{P}\colon d(P, Q) < \varepsilon\}$. The distribution-robustness of the statistic t in ε-extension of M_0 is given by

$$r_{t,\varepsilon}(P) = \sup\{d_t(P^t, Q^t)\colon Q \in \pi(P)\},$$

A qualitative Hampel's [2] definition of robustness is: t is robust in a neighbourhood of P if for any $\delta > 0$ there exists $\varepsilon > 0$ such that $r_{t,\varepsilon}(P) < \delta$; t is robust in the structure M_0 if for any positive δ there exists $\varepsilon > 0$ such that $\sup_{\mathscr{P}_0} r_{t,\varepsilon}(P) < \delta$.

The full text containing some further discussion and examples (power-robustness of the two-sided Student test with respect to change of variance; a risk-robustness of sample mean and sample median in estimating expected value of a normal dis-