One might consider the more general problem of the asymptotic behaviour of deficiencies $\delta(\mathscr{E}^n, \mathscr{F}^n)$ as $n \to \infty$. It is known that we may have $\mathscr{E}^n \geqslant \mathscr{F}^n$ when n is sufficiently large, although $\delta(\mathscr{E}, \mathscr{F}) > 0$. Then $\sqrt[n]{\delta(\mathscr{E}^n, \mathscr{F}^n)} \to 0$. Are there other situations where $\sqrt[n]{\delta(\mathscr{E}, \mathscr{F}^n)} \to 0$?

Clearly,

$$\limsup_n \sqrt[n]{\delta(\mathscr{E}^n, \mathscr{F}^n)} \leqslant \limsup_n \sqrt[n]{\delta_a(\mathscr{E}^n)} \leqslant \sigma(\mathscr{E})$$

while

$$\liminf_n \sqrt[n]{\delta(\mathscr{E}^n, \mathscr{F}^n)} \geqslant \liminf_n \sqrt[n]{[\delta_a(\mathscr{E}^n) - \delta_a(\mathscr{F}^n)]^+}.$$

It follows that $\sqrt[n]{\delta(\mathscr{E}^n, \mathscr{F}^n)} \to \sigma(\mathscr{E})$ whenever $\sigma(\mathscr{E}) > \sigma(\mathscr{F})$. We do not, however, know the limiting behaviour of sequences $\sqrt[n]{\delta(\mathscr{E}^n, \mathscr{F}^n)}$, $n = 1, 2, \ldots$, when $\sigma(\mathscr{E}) < \sigma(\mathscr{F})$.

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Presented to the semester
MATHEMATICAL STATISTICS
September 15–December 18, 1976

MATHEMATICAL STATISTICS
BANACH CENTER PUBLICATIONS, VOLUME 6
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1980

ON THE CRAMÉR–RAO INEQUALITY AND ON A NEW VERSION OF THE CHI-SQUARE STATISTIC

I. VINCZE

Mathematical Institute of the Hungarian Academy of Sciences,
Budapest V, Hungary

In connection with the Cramér–Rao inequality many important investigations were made for the non-regular case, i.e. when the supports of the underlying densities in the sample space do not coincide. In their pioneering papers, D. G. Chapman and H. Robbins [2] (1951), J. Kiefer [4] (1952), D. A. S. Fraser and I. Guttmann [3] (1952) consider mainly the case of a real parameter (using also further restrictions) given various bounds for the variance of an unbiased estimator. The aim of the present lecture is to give a brief account of the results of the above-mentioned papers pointing out that almost no assumption is needed concerning the structure of the parameter space (see also Barankin [1] (1949)).

In the second part of the lecture the following modified form of Pearson's chi-square statistic is investigated:

$$\bar{\chi}^2 = \sum_{i=1}^{r} \frac{(\bar{X}_{(i)} - E_i)^2}{\sigma_i^2} \nu_i,$$

where E_i and σ_i ($i = 1, 2, \ldots, r$) are the conditional expected value and the variance of the variable restricted to the ith interval of the partition, while ν_i is the number of sample elements falling into the ith interval and having arithmetic mean $\bar{X}_{(i)}$. This statistic utilizes besides the number of sample elements lying on the respective intervals of the partition also their *positions* within the intervals. In a joint paper with E. Csáki [7] the authors show that this statistic is asymptotically distributed — when the sample size n tends to infinity — according to the chi-square distribution with parameter r, i.e. the number of intervals chosen — contrary to the $r-1$ belonging to Pearson's statistic. When $n \to \infty$ and $r = O(n^{\alpha})$, $0 < \alpha < 1$, the distribution of

$$\frac{\bar{\chi}^2 - r}{\sqrt{2r}}$$

tends to the normal law $N(0, 1)$. Whenever the relation

$$\sum_{i=1}^{r} \frac{(E_i - E_i^*)^2}{\sigma_i^2} p_i^* > 0$$

holds, where the quantities with asterisks belong to the alternative distribution, the test based on $\overline{\chi}^2$ is asymptotically consistent ($n \to \infty$). Concerning efficiency, the test is convenient for moderate r's, the situation usually met in practice.

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Presented to the semester
MATHEMATICAL STATISTICS
September 15–December 18, 1976

MATHEMATICAL STATISTICS
BANACH CENTER PUBLICATIONS, VOLUME 6
PWN—POLISH SCIENTIFIC PUBLISHERS
WARSAW 1980

SEQUENTIAL ESTIMATION IN PROCESSES WITH INDEPENDENT INCREMENTS

WOLFGANG WINKLER
Sektion Mathematik, Technische Universität Dresden, Dresden, D.D.R.

1. Introduction

Sequential statistical estimation problems, which are characterized by a random sample size, have been considered about thirty years ago in connection with simple random processes with independent increments, especially for the Bernoulli process, Poisson process, and others. (See, for instance, [3], [4], [11] and [20].) The case of the Poisson process has been considered by Trybuła [19] in 1968. In his paper Trybuła solved the problem of describing sequential sample plans for the Poisson process, which are optimal in a certain sense and he studied the properties of such plans concerning efficiently estimable functions.

It is natural to ask for extensions of the statistical results to a wide class of processes with independent increments using arbitrary Markov times to stop the observation of the considered random process. In fact, this is possible for the so-called exponential class of random processes with independent increments.

In the following we give a short summary about some results in this direction, which are due to J. Franz, M. Magiera, S. Trybuła, and W. Winkler. For details see [5]–[10], [13]–[16], [19].

2. The exponential class and stopping times

Let us denote by $X(t) = (X_1(t), \ldots, X_m(t))^T$ an m-dimensional random process defined on the probability space $[\Omega, \mathfrak{F}, P]$, where $X(t)$ takes values in $E = \boldsymbol{R}^m$ and $t \in T$ runs over all non-negative integers or over all non-negative real numbers as usually. Moreover, let $\mathfrak{F}_t$, $t \in T$, denote the σ-algebra generated by the random vectors $X(s)$, $s \leqslant t$.

We assume that the probability measure P depends on an unknown parameter $\boldsymbol{\theta} = (\theta_1, \ldots, \theta_k)^T \in \Theta \subseteq \boldsymbol{R}^k$ and we write $P = P_{\boldsymbol{\theta}}$.

Definition. We shall say that $X(t)$, $t \in T$, *belongs to the exponential class* if the following conditions are satisfied: