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ON SOME ASYMPTOTIC RESULTS IN STATISTICS BY WAY OF CONTIGUITY

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The material presented during the four hour lectures at the Stefan Banach International Mathematical Center in the Fall Semester, 1976, was a condensed summary of results most of which appeared in the literature, in some form or another, and others which have been submitted for publication. It is therefore advisable to confine ourselves here to a brief outline of that material.

To this end, for $n \geqslant 1$, integer, consider the measurable spaces $(\mathscr{X}, \mathscr{A}_n)$, where the σ-fields $\mathscr{A}_n$ form a non-decreasing sequence. For each n as above and each $\theta \in \Theta$, a k-dimensional open subset of $\boldsymbol{R}^k$, $k \geqslant 1$, let $P_{n,\theta}$ be a probability measure on $\mathscr{A}_n$. Suppose that $P_{n,\theta^*} \approx P_{n,\theta}$ for all θ, $\theta^* \in \Theta$ and $n \geqslant 1$, and consider the simplest possible case, where there are available n i.i.d. real-valued r.v.'s $X_1, \ldots, X_n$ defined on $(\mathscr{X}, \mathscr{A}_n)$. Actually, $\mathscr{A}_n$ is the σ-field induced by these r.v.'s, so that the relevant log-likelihood function is

$$\log[dP_{n,\theta^*}/dP_{n,\theta}] = \Lambda(X_1, \ldots, X_n; \theta, \theta^*).$$

For n sufficiently large so that $\theta_n = \theta + \frac{h_n}{\sqrt{n}}$ belongs in Θ ($h_n \to h \in \boldsymbol{R}^k$ as $n \to \infty$), set

$$\Lambda_n(\theta) = \log[dP_{n,\theta_n}/dP_{n,\theta}].$$

Let $\dot{\varphi}_0(\theta)$ be the derivative in quadratic mean of the random function $[dP_{1,\theta^*}/dP_{1,\theta}]^{1/2}$ with respect to θ^* at θ when the probability measure $P_{1,\theta}$ is used, and set

$$\Delta_n(\theta) = \frac{2}{\sqrt{n}} \sum_{j=1}^{n} \dot{\varphi}_j(\theta).$$

Then, under some further suitable conditions, one obtains the asymptotic distribution (as $n \to \infty$) of $\Lambda_n(\theta)$ as well as that of $\Delta_n(\theta)$, under both $P_{n,\theta}$ and P_{n,θ_n}. Also, one obtains a local approximation of the probability measure P_{n,θ_n} by an exponential probability measure. This approximation is then utilized for establishing the asymptotic sufficiency of a certain statistic (a suitable truncated version of $\Delta_n(\theta_0)$) for the family of probability measures $\{P_{n,\theta}; \theta \in \Theta\}$ in the neighborhood of a given point $\theta_0 \in \Theta$.

The results obtained above are then used for discussing some hypotheses testing problems, and also establishing the asymptotic efficiency (in the Weiss–Wolfowitz sense) of the maximum probability estimates.

Finally, it is indicated that, under suitable regularity conditions, the general results mentioned above can be extended to the following cases: The r.v.'s involved are independent but not necessarily identically distributed; they are coming from a stationary and ergodic Markov process; they are coming from a fairly general stochastic process.

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TESTING FOR NORMALITY

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Let $X_1, \ldots, X_n$ denote identically and independently distributed random variables with the distribution function $F(x)$.

Let H_0 denote the hypothesis

$$H_0\colon F(x) = \Phi\left(\frac{x-\mu}{\sigma}\right),$$

where $\Phi(x)$ is the standardized normal distribution function. The constants μ and σ are unspecified (nuisance parameters).

Tests for H_0 are called (one-sample) *normality tests*. We deal with such tests in Section 1.

Sometimes we have more than one sample, with different nuisance parameters, for assessing normality. The corresponding tests are called multisample normality tests; they are discussed in Section 2.

1. One sample case

Any goodness of fit test can be used as a test for normality if the empirical moments are substituted in the theoretical distribution function. This modification, however, changes the distribution of the test statistics. For the χ^2-test, the same tables may be used, only the number of the degrees of freedom is to be diminished by the number of estimated constants (Fisher [11] (1924)). Although this solution is of approximative character (see Chernoff and Lehmann [3] (1954)) the accuracy is sufficient in most of the practical cases provided the sample size is large. For other goodness of fit tests, separate tables have been prepared for the modified case. For the Kolmogorov test, the critical values have been tabulated by Lillefors [13] (1967), for the Cramér–Mises, Anderson–Darling and some other tests by Stephens [30] (1974), see Biometrika Tables, Volume 2, Table 54.

A further group of normality tests are the tests of Shapiro and Wilk [27] (1965), Shapiro and Francia [26] (1972), DeWet and Venter [7] (1972) and d'Agostino