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RECURSIVE INTERPOLATION OF PARTIALLY OBSERVABLE RANDOM FIELDS

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The problem considered in this paper can, in a general way, be formulated as follows. Let (θ_z, ξ_z) be a partially observable field with the n-dimensional parameter $z \in D$, D being a set either in R^n or in an integer-valued lattice Z^n, $n \geqslant 1$, where θ_z and ξ_z are unobservable and observable fields, respectively. Our aim is to find a (stochastic) equation for the estimate

$$\bar{\theta}_z = E(\theta_z | F_D^\xi),$$

where F_D^ξ is the σ-algebra generated by ξ_z, $z \in D$.

In accordance with terminology for stochastic processes, introduced in [1], the problem given above will be referred to as the interpolation problem of partially observable field (θ_z, ξ_z).

This paper is confined only to the cases of dimensions $n = 1$ and $n = 2$, and — concerning a set D — to rectangles of the following forms: $\Pi_T = [0, T]$ or $\{0, 1, \ldots, T\}$ for $n = 1$ and $\Pi_T = [0, T_1] \times [0, T_2]$ or $\{0, 1, \ldots, T_1\} \times \{0, 1, \ldots, T_2\}$, $T = T_1 \times T_2$, for $n = 2$. As to the structure of the fields θ_z and ξ_z one assumes conditions analogous to those made for the Kalman–Bucy filtering scheme (cf. [1]). One may note that the result for $n = 1$ presented below can be generalized to the case of the so-called *conditionally Gaussian processes* for which a solution (to the interpolation problem) was obtained in [1] in a different way.

Assume that the partially observable field (θ_z, ξ_z) is described by the following equations:

(1) $$L_z \theta_z = b(z)\varepsilon_z + a(z),$$

(2) $$K_z \xi_z = A(z)\theta_z + B(z)\tilde{\varepsilon}_z + c(z),$$

where the operators L_z and K_z are defined as follows:

$$L_z = \frac{d}{dz} + a_0(z), \quad K_z = \frac{d}{dz} + c_0(z) \quad \text{for} \quad z \in (0, T);$$

$$L_z = \Delta_1 + a_0(z), \quad K_z = \Delta_1 + c_0(z) \quad \text{for} \quad z \in \{0, 1, \ldots, T-1\};$$

$$L_z = \frac{\partial^2}{\partial z_1 \partial z_2} + a_1(z)\frac{\partial}{\partial z_1} + a_2(z)\frac{\partial}{\partial z_2} + a_0(z),$$

$$K_z = \frac{\partial^2}{\partial z_1 \partial z_2} + c_1(z)\frac{\partial}{\partial z_1} + c_2(z)\frac{\partial}{\partial z_2} + c_0(z), \quad \text{for} \quad z \in (0, T_1) \times (0, T_2);$$

$$L_z = \Delta_{(1,1)} + a_1(z)\Delta_{(1,0)} + a_2(z)\Delta_{(0,1)} + a_0(z),$$

$$K_z = \Delta_{(1,1)} + c_1(z)\Delta_{(1,0)} + c_2(z)\Delta_{(0,1)} + c_0(z),$$

$$\text{for} \quad z \in \{0, 1, \ldots, T_1 - 1\} \times \{0, 1, \ldots, T_2 - 1\};$$

and Δ_u denotes the shift operator, i.e. $\Delta_u f(z) = f(z+u)$. Functions a, c, a_i, c_i $(i = 0, 1, 2)$, b $(b \neq 0)$ and B $(B \neq 0)$ are assumed to be known. In the case of *continuous* parameter z it is also assumed that these functions as well as $\frac{\partial}{\partial z_i} a_i$ and $\frac{\partial}{\partial z_i} c_i$ $(i = 1, 2)$ are continuous; ε_z and $\tilde{\varepsilon}_z$ are independent Gaussian white noises (i.e. corresponding derivatives of the Wiener field w_z defined on rectangles) and solutions of the equations under consideration are meant in a generalized sense, i.e. for $n = 2$, solution of equation (1), for instance, is meant in the sense of satisfying the identity

$$\int_{\Pi_T} L_z^*(\varphi(z))\theta_z dz = \int_{\Pi_T} \varphi(z) b(z) dw_z + \int_{\Pi_T} \varphi(z) a(z) dz,$$

for any function $\varphi(z)$ from the class of all finite functions on Π_T with continuous second order derivatives, and with L_z^* denoting the adjoint operator. In the case of *discrete* parameter z the noises ε_z and $\tilde{\varepsilon}_z$ are assumed to be independent and Gaussian with $E\varepsilon_z = E\tilde{\varepsilon}_z = 0$ and $E\varepsilon_z\varepsilon_u = E\tilde{\varepsilon}_z\tilde{\varepsilon}_u = \delta(z, u)$, where $\delta(z, u)$ is the Kronecker symbol. For simplicity, initial conditions for θ_z and ξ_z are assumed to be zeroes. In case $n = 2$ the last condition means that

$$\theta_z = \xi_z = 0 \quad \text{for} \quad z \in \partial\Pi_T \equiv \{0 \leqslant z_1 \leqslant T_1, z_2 = 0\} \cup \{0 \leqslant z_2 \leqslant T_2, z_1 = 0\}.$$

If $n = 1$, then $\partial\Pi_T = \{z = 0\}$. We shall also use the following notations $\partial\check{\Pi}_T^{\,*} = \{z = T-1\}$ or, depending on the dimension n, $\partial\check{\Pi}_T = \{z_1 = 0, 1, \ldots, T_1 - 1;\ z_2 = T_2 - 1\} \cup \{z_2 = 0, 1, \ldots, T_2 - 1;\ z_1 = T_1 - 1\}$ for the *discrete* parameter z and $\partial\check{\Pi}_T = \{z = T\}$ or, respectively,

$$\partial\check{\Pi}_T = \{0 < z_1 \leqslant T_1, z_2 = T_2\} \cup \{0 < z_2 \leqslant T_2, z_1 = T_1\}$$

for the *continuous* parameter z.

THEOREM. *Suppose that the forementioned assumptions are fulfilled. Then the estimate $\bar{\theta}_z$ satisfies the following equation*

(3) $$L_z^*[b^{-2}(z)(L_z\bar{\theta}_z - a(z))] = \frac{A(z)}{B^2(z)}(K_z\xi_z - c(z) - A(z)\bar{\theta}_z)$$

with the boundary conditions

(4) $$\bar{\theta}_z = 0, \quad z \in \partial\Pi_T,$$

(5) $$L_z\bar{\theta}_z = a(z), \quad z \in \partial\check{\Pi}_T.$$

An analogous theorem is valid for the vector fields θ_z and ξ_z.

In the case of the one dimensional discrete parameter one can solve the boundary problem (3)–(5) by using a method of pursuit.

Reference

[1] R. S. Liptzer, A. N. Shiryaev, *Statistics of random processes II. Applications*, Springer-Verlag, 1978.

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