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ON THE ASYMPTOTIC DISTRIBUTIONS OF CERTAIN FUNCTIONS OF EIGENVALUES OF CORRELATION MATRICES*

JACK C. LEE
Wright State University, U.S.A.
and
P. R. KRISHNAIAH
University of Pittsburgh, Pittsburgh, Pa., U.S.A.

1. Introduction

In this paper, the authors obtained asymptotic expressions for the joint densities of the linear combinations of the roots as well as the joint densities of the ratios of the linear combinations of the roots of the correlation matrices when the distributions underlying the data are multivariate normal or complex multivariate normal. These expressions are in terms of linear combinations of multivariate normal densities and multivariate Hermite polynomials. The authors have earlier obtained analogous results for the joint densities of the linear combinations and ratios of the linear combinations of the roots of the sample covariance matrices. The results obtained in this paper are useful in the application of simultaneous test procedures for the inference on eigenvalues of the correlation matrices of real and complex multivariate normal populations.

2. Joint distribution of linear combinations of the roots of correlation matrix in the real case

Let the columns of X: $p\times m$ be distributed independently and identically as multivariate normal with zero mean vector and covariance matrix $\Sigma = (\sigma_{gh})$ and $S = (s_{gh}) = XX'$. Then S has the central real Wishart distribution $W_p(\Sigma, m)$.

Let $\Omega = (\varrho_{jh})$, where $\varrho_{jh} = \sigma_{jh}/(\sigma_{jj}\sigma_{hh})^{1/2}$. Since Ω is symmetric, there exists an orthogonal matrix $U = (u_{jh})$ such that

(2.1) $$U'\Omega U = \Lambda,$$

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where Λ is a diagonal matrix with elements $\lambda_1 \geqslant \ldots \geqslant \lambda_p$. The sample correlation matrix $R = (r_{jh})$, where

$$r_{jh} = s_{jh}/(s_{jj}s_{hh})^{1/2},$$

can be expressed as

$$R = S_0^{-1/2} S S_0^{-1/2} \tag{2.2}$$

and

$$S_0 = \mathrm{diag}(s_{11}, \ldots, s_{pp}).$$

In the sequel, we use the notation $A_0 = \mathrm{diag}(a_{11}, \ldots, a_{pp})$ if $A = (a_{jh})$. Let

$$\frac{1}{m}\Sigma_0^{-1/2} S \Sigma_0^{-1/2} = \Omega + \frac{1}{\sqrt{m}} Y, \quad \text{where } Y = (Y_{jh}). \tag{2.3}$$

From (2.1), (2.2) and (2.3) it can be shown that

$$U'RU = \Lambda + \frac{1}{\sqrt{m}} Y^{(1)} + \frac{1}{m} Y^{(2)} + \text{higher order terms}, \tag{2.4}$$

where

$$\begin{aligned} Y^{(1)} &= U'(Y - \tfrac{1}{2}Y_0\Omega - \tfrac{1}{2}\Omega Y_0)U, \\ Y^{(2)} &= \tfrac{1}{8}U'(2Y_0\Omega Y_0 - 4YY_0 - 4Y_0Y + 3Y_0^2\Omega + 3\Omega Y_0^2)U. \end{aligned} \tag{2.5}$$

If the characteristic roots λ_j of Ω are distinct, then the perturbation expansion (see e.g. Wilkinson [6], pp. 69–70) of the hth largest eigenvalue l_h of R is

$$l_h = \lambda_h + \frac{1}{\sqrt{m}} Y_{hh}^{(1)} + \frac{1}{m}\Big(Y_{hh}^{(2)} + \sum_{a \neq h}^{p} \lambda_{ha}^{-1} Y_{ha}^{(1)2}\Big) + \text{higher order terms}, \tag{2.6}$$

where

$$Y^{(j)} = (Y_{ab}^{(j)}), \qquad \lambda_{ab} = \lambda_a - \lambda_b,$$

$$Y_{ab}^{(1)} = \sum_{j=1}^{p}\sum_{h=1}^{p} u_{ja}u_{hb}Y_{hj} - \tfrac{1}{2}(\lambda_a + \lambda_b)\sum_{h=1}^{p} u_{ha}u_{hb}Y_{hh}, \tag{2.7}$$

$$Y_{aa}^{(2)} = \tfrac{1}{4}\sum_{j=1}^{p}\sum_{h=1}^{p} \varrho_{hj}u_{ja}u_{ha}Y_{jj}Y_{hh} - \sum_{j=1}^{p}\sum_{h=1}^{p} u_{ja}u_{ha}Y_{hj}Y_{hh} + \tfrac{3}{4}\lambda_a\sum_{h=1}^{p} u_{ha}^2 Y_{hh}^2.$$

Let $L' = (L_1, \ldots, L_q)$, where

$$L_g = \sqrt{m}\Big\{\sum_{j=1}^{p} c_{gj} l_j - \sum_{j=1}^{p} c_{gj}\lambda_j\Big\} \tag{2.8}$$

for $g = 1, \ldots, q$. Using (2.6) in (2.8) we obtain, in fashion similar to (3.10) of Krishnaiah and Lee [3], the following asymptotic expression for the characteristic function of $L_1, \ldots, L_q$:

$$\varphi_1(t_1, \ldots, t_q) = \exp[-\tfrac{1}{2}t'\Omega_1 t] \times \tag{2.9}$$
$$\times\Big\{1 + \frac{1}{\sqrt{m}}\Big[\sum_{g=1}^{q}(J_{11}+J_{13})(it_g) + \sum_{g_1,g_2,g_3}^{q}(J_{10}+J_{12}+J_{14})(i^3 t_{g_1}t_{g_2}t_{g_3})\Big] + O(m^{-1})\Big\},$$

where

$$\Omega_1 = (\Omega_{gl}^{(1)}),$$

$$\Omega_{gl}^{(1)} = 2\sum_{a,b,h,j}^{p} W_{g,ha}W_{l,jb}\varrho_{aj}\varrho_{gh},$$

$$J_{10} = \tfrac{4}{3}\sum_{a,b,d,h,j,l}^{p} W_{g_1,ha}W_{g_2,jb}W_{g_3,ld}\varrho_{aj}\varrho_{bl}\varrho_{dh},$$

$$J_{11} = \tfrac{1}{2}\sum_{h,j,l}^{p} c_{gj}u_{hj}[\varrho_{hl}u_{lj}(\varrho_{hl}^2 - 4) + 3\lambda_j\delta_{hl}u_{hj}],$$

$$J_{12} = \sum_{a,b,c,d,h,j,l}^{p} c_{g_1 j}W_{g_2,ba}W_{g_3,dc}\varrho_{hc}\varrho_{hd}u_{hj} \times$$
$$\times[\varrho_{hl}\varrho_{la}\varrho_{lb}u_{lj} - 4\varrho_{hb}\varrho_{la}u_{lj} + 3\delta_{hl}\lambda_j\varrho_{ha}\varrho_{hb}u_{hj}],$$

$$J_{13} = \tfrac{1}{2}\sum_{h,j,l}^{p}\sum_{a\neq j}^{p} u_{ha}u_{la}\lambda_{ja}^{-1}c_{gj} \times$$
$$\times\Big[2\sum_{p,k}^{p} u_{sj}u_{kj}(\varrho_{hl}\varrho_{sk} + \varrho_{hk}\varrho_{sl} + (\lambda_j+\lambda_a)^2 u_{hj}u_{lj}\varrho_{hl}^2 - 4(\lambda_j+\lambda_a)\sum_{s=1}^{p} u_{sj}u_{lj}\varrho_{hl}\varrho_{sl}\Big],$$

$$J_{14} = \sum_{b,c,d,h,j,k,l}^{p}\sum_{a\neq j}^{p} u_{ha}u_{la}\lambda_{ja}^{-1}c_{g_1 j}W_{g_2,cb}W_{g_3,kd}\varrho_{hc}\varrho_{lk} \times$$
$$\times\Big[4\sum_{f}^{p} u_{sj}u_{fj}\varrho_{fd}\varrho_{sb} + (\lambda_j+\lambda_a)^2 u_{hj}u_{lj}\varrho_{ld}\varrho_{hb} - 4(\lambda_j+\lambda_l)\sum_{s=1}^{p} u_{sj}u_{lj}\varrho_{ld}\varrho_{sb}\Big],$$

$$W_{g,ha} = \sum_{j=1}^{p} c_{gj}u_{aj}u_{hj}(1 - \delta_{ha}\lambda_j), \qquad \delta_{ha} = \begin{cases} 1 & \text{if } h = a, \\ 0 & \text{if } h \neq a. \end{cases}$$

Inverting (2.9) we obtain the following asymptotic expression for the joint density of $L_1, \ldots, L_q$:

$$f_1(L_1, \ldots, L_q) = N(L; \Omega_1)\Big\{1 + \frac{1}{\sqrt{m}}\Big[\sum_{p=1}^{q}(J_{11}+J_{13})H_g(L) + \tag{2.10}$$
$$+ \sum_{g_1,g_2,g_3}^{q}(J_{10}+J_{12}+J_{14})H_{g_1,g_2,g_3}(L)\Big] + O(m^{-1})\Big\},$$

where $L' = (L_1, \ldots, L_q)$, $N(L; \Sigma)$ stands for the multivariate normal with zero mean vector and covariance matrix Σ, $H_{g_1,\ldots,g_s}(L)$ is the multivariate Hermite polynomials defined by

$$H_{g_1,\ldots,g_s}(x) = \frac{(-1)^s\partial^s}{N(x;\Sigma)\partial x_{j_1}\ldots\partial x_{j_s}} N(x;\Sigma), \tag{2.11}$$

for s integers $g_1, \ldots, g_s$ such that $1 \leqslant g_j \leqslant p$ and $x' = (x_1, \ldots, x_p)$.

3. Joint distribution of the ratios of linear combinations of the roots of the correlation matrix in the real case

In this section we give an asymptotic expression for the joint density of the ratios of linear combinations of the roots of the real correlation matrix when the population roots are distinct.

Let

$$T_g = \sqrt{m}\Big[\sum_{j=1}^{p} c_{gj} l_j \Big(\sum_{j=1}^{p} d_{gj} l_j\Big)^{-1} - \sum_{j=1}^{p} c_{gj}\lambda_j \Big(\sum_{j=1}^{p} d_{gj}\lambda_j\Big)^{-1}\Big] \quad \text{for} \quad g = 1, \ldots, q. \tag{3.1}$$

Using (2.6) in (3.1) we obtain the following asymptotic expression for the joint characteristic function of $T' = (T_1, \ldots, T_q)$:

$$\varphi_2(t) = \exp[-\tfrac{1}{2}t'Q_2 t]\times \Big\{1+\frac{1}{\sqrt{m}}\Big[\sum_{g=1}^{q}(J_{21}+J_{23}+J_{25}+J_{27}+J_{29})(it_g)+ + \sum_{g_1,g_2,g_3}^{q}(J_{20}+J_{22}+J_{24}+J_{26}+J_{28}+J_{210})(i^3 t_{g_1}t_{g_2}t_{g_3})\Big]+O(m^{-1})\Big\}, \tag{3.2}$$

where

$$Q_2 = (Q_{gl}^{(2)}), \tag{3.3}$$

$$Q_{gl}^{(2)} = 2\sum_{a,b,h,j}^{p} Z_{g,ha} Z_{l,jb}\varrho_{aj}\varrho_{bh},$$

$$Z_{g,ha} = \sum_{j=1}^{p} \Lambda_g(c_{gj}-\tilde{\lambda}_g\Lambda_g d_{gj})u_{aj}u_{hj}(1-\delta_{ah}\lambda_j),$$

$$\Lambda_g^{-1} = \sum_{j=1}^{p} d_{gj}\lambda_j,$$

$$\tilde{\lambda}_g = \sum_{j=1}^{p} c_{gj}\lambda_j,$$

$$J_{25} = \sum_{a,b,c,d,j,l}^{p} \Lambda_g^2(\tilde{\lambda}_g\Lambda_g d_{gj}-c_{gj})d_{gl}u_{aj}u_{bj}u_{cl}u_{dl}(\varrho_{bd}\varrho_{ac}+\varrho_{bc}\varrho_{ad}),$$

$$J_{26} = 4\sum_{a,b,c,d,f,h,j,l,r,s}^{p} \Lambda_{g_1}^2(\tilde{\lambda}_{g_1}\Lambda_{g_1}d_{g_1 j}-c_{g_1 j})d_{g_1 l}u_{aj}u_{bj}u_{cl}u_{dl}Z_{g_2,sr}Z_{g_3,hf}\varrho_{bs}\varrho_{ar}\varrho_{dh}\varrho_{cf},$$

$$J_{27} = 2\sum_{b,d,j,l}^{p} \Lambda_g^2(\tilde{\lambda}_g\Lambda_g d_{gj}-c_{gj})d_{gl}\lambda_j\lambda_l u_{bj}^2 u_{dl}^2\varrho_{bd},$$

$$J_{28} = 4\sum_{b,d,f,h,j,l,r,s}^{p} \Lambda_{g_1}^2(\tilde{\lambda}_{g_1}\Lambda_{g_1}d_{g_1 j}-c_{g_1 j})d_{g_1 l}\lambda_j\lambda_l u_b^2 u_{bj}^2 Z_{g_2,sr}Z_{g_3,hf}\varrho_{bs}\varrho_{br}\varrho_{dh}\varrho_{df},$$

$$J_{29} = -\sum_{b,c,d,j,l}^{p} \Lambda_g^2(\tilde{\lambda}_g\Lambda_g d_{gj}-c_{gj})d_{gl}(\lambda_j u_{bj}^2 u_{cl}u_{dl}+\lambda_l u_{bl}^2 u_{cj}u_{dj}),$$

$$J_{210} = -4\sum_{b,c,d,f,h,j,l,r,s}^{p} \Lambda_{g_1}^2(\tilde{\lambda}_{g_1}\Lambda_{g_1}d_{g_1 j}-c_{g_1 j})d_{g_1 l}\times (\lambda_j u_{bj}^2 u_{cl}u_{dl}+\lambda_l u_{bl}^2 u_{cj}u_{dj})Z_{g_2,sr}Z_{g_3,hf}\varrho_{bs}\varrho_{br}\varrho_{dh}\varrho_{df},$$

and for $j = 0, 1, 2, 3, 4$, J_{2j} are exactly in the same forms as those of J_{1j} except $W_{g,ha}$ replaced by $Z_{g,ha}$, c_{gj} replaced by $\Lambda_g(c_{gj}-\tilde{\lambda}_g\Lambda_g d_{gj})$.

Inverting (3.2) we obtain the following expression for the joint density of $T_1, \ldots, T_q$:

$$f_2(T_1, \ldots, T_q) = N(T; Q_2)\times \Big\{1+\frac{1}{\sqrt{m}}\Big[\sum_{g=1}^{q}(J_{21}+J_{23}+J_{25}+J_{27}+J_{29})H_g(T)+ + \sum_{g_1,g_2,g_3}^{q}(J_{20}+J_{22}+J_{24}+J_{26}+J_{28}+J_{210})H_{g_1,g_2,g_3}(T)\Big]+O(m^{-1})\Big\}. \tag{3.4}$$

4. Asymptotic distributions in the complex case

Let $Z = Z_1+iZ_2$ be a $p\times m$ matrix and let the rows of $(Z_1':Z_2')$ be distributed independently as a multivariate normal with zero mean vector and covariance matrix

$$\begin{bmatrix} \Sigma_1 & \Sigma_2 \\ -\Sigma_2 & \Sigma_1 \end{bmatrix},$$

where Σ_1 and Σ_2 are of order $p\times p$. Then, the columns of Z are distributed independently as complex multivariate normal (in the sense of Wooding [7]) with covariance matrix $\tilde{\Sigma} = 2(\Sigma_1+i\Sigma_2) = (\tilde{\sigma}_{ab})$. Also, the distribution of $\tilde{S} = Z\bar{Z}' = (\tilde{W}_{ab})$, $\tilde{W}_{ab} = W_{ab}+iV_{ab}$, is known to be a central complex Wishart distribution. In this paper we denote by $\bar{Z}$ and Z^* the complex conjugate and the transpose of the complex conjugate of Z, respectively.

Let $\tilde{\Omega} = (\tilde{\varrho}_{ab})$, where $\tilde{\varrho}_{ab} = \dfrac{\tilde{\sigma}_{ab}}{(\tilde{\sigma}_{aa}\tilde{\sigma}_{bb})^{1/2}}$. Since $\tilde{\Omega}$ is Hermitian positive definite, there exists a unitary matrix $\tilde{U} = (\tilde{u}_{ab})$ such that

$$\tilde{U}^*\Omega\tilde{U} = \tilde{\Lambda}, \tag{4.1}$$

where $\tilde{\Lambda}$ is a diagonal matrix with elements $\delta_1 \geqslant \delta_2 \geqslant \ldots \geqslant \delta_p$. The sample correlation matrix $\tilde{R} = (\tilde{r}_{ab})$, where

$$\tilde{Y}_{ab} = \frac{\tilde{W}_{ab}}{(W_{aa}W_{bb})^{1/2}},$$

can be expressed as

$$\tilde{R} = \tilde{S}_0^{-1/2}\tilde{S}\tilde{S}_0^{-1/2}. \tag{4.2}$$

Similarly to (2.3) we let

$$\frac{1}{m}\tilde{\Sigma}_0^{-1/2}\tilde{S}\tilde{\Sigma}_0^{-1/2} = \tilde{\Omega}+\frac{1}{\sqrt{m}}\tilde{Y} \tag{4.3}$$

where $\tilde{Y} = (\tilde{Y}_{ab})$.

From (4.1)–(4.3) we have the following expression for $\tilde{U}^*\tilde{R}\tilde{U}$:

$$\tilde{U}^*\tilde{R}\tilde{U} = \tilde{\Lambda}+\frac{1}{\sqrt{m}}\tilde{Y}^{(1)}+\frac{1}{m}\tilde{Y}^{(2)}+\text{higher order terms}, \tag{4.4}$$

where

$$\begin{aligned} \tilde{Y}^{(1)} &= \tilde{U}^*\tilde{Y}\tilde{U}-\tfrac{1}{2}\tilde{U}^*\tilde{Y}_0\tilde{U}\tilde{\Lambda}-\tfrac{1}{2}\tilde{\Lambda}U^*\tilde{Y}_0\tilde{U}, \\ \tilde{Y}^{(2)} &= \tfrac{1}{8}(2\tilde{U}^*\tilde{Y}_0\tilde{\Omega}\tilde{Y}_0\tilde{U}-4\tilde{U}^*\tilde{Y}\tilde{Y}_0\tilde{U}-4\tilde{U}^*\tilde{Y}_0\tilde{Y}\tilde{U}+ \\ &\qquad +3\tilde{U}^*\tilde{Y}_0^2\tilde{U}\tilde{\Lambda}+3\tilde{\Lambda}\tilde{U}^*\tilde{Y}_0^2\tilde{U}). \end{aligned} \tag{4.5}$$

When the eigenvalues δ_j of $\tilde{\Omega}$ are distinct, the perturbation expansion of the hth largest eigenvalue θ_h of $\tilde{R}$ is

$$\theta_h = \delta_h+\frac{1}{\sqrt{m}}\tilde{Y}_{hh}^{(1)}+\frac{1}{m}\Big(\tilde{Y}_{hh}^{(2)}+\sum_{a\neq h}^{p}\tilde{\lambda}_{ha}^{-1}\tilde{Y}_{ha}^{(1)}\tilde{Y}_{ha}^{(1)*}\Big), \tag{4.6}$$

where $\tilde{\lambda}_{ha} = \delta_h-\delta_a$, $\tilde{Y}^{(j)} = (Y_{ab}^{(j)})$.

In this section we give asymptotic expressions for the joint density of $\tilde{L} = (\tilde{L}_1, \ldots, \tilde{L}_q)$ as well as the joint density of $\tilde{T} = (\tilde{T}_1, \ldots, \tilde{T}_q)$, where

$$\tilde{L}_g = \sqrt{m}\sum_{j=1}^{p}c_{gj}(m^{-1}\theta_j-\delta_j), \tag{4.7}$$

$$\tilde{T}_g = \sqrt{m}\Big[\Big(\sum_{j=1}^{p}c_{gj}\theta_j\Big)\Big(\sum_{h=1}^{p}d_{gh}\theta_h\Big)^{-1}-\Big(\sum_{j=1}^{p}c_{gj}\delta_j\Big)\Big(\sum_{h=1}^{p}d_{gh}\delta_h\Big)^{-1}\Big], \tag{4.8}$$

where $\theta_1 \geqslant \ldots \geqslant \theta_p$ are the roots of $\tilde{R}$.

We obtain the following expression for the joint densities of $\tilde{L}_1, \ldots, \tilde{L}_q$ and $\tilde{T}_1, \ldots, \tilde{T}_q$ as in the real case:

$$\begin{aligned} f_3(\tilde{L}_1, \ldots, \tilde{L}_q) &= N(\tilde{L}; Q_3)\times \\ &\times\Big\{1+\frac{1}{\sqrt{m}}\Big[\sum_{g=1}^{q}(J_{31}+J_{33})H_g(\tilde{L})+ \\ &+\sum_{g_1,g_2,g_3}^{q}(J_{30}+J_{32}+J_{34})H_{g_1,g_2,g_3}(\tilde{L})\Big]+O(m^{-1})\Big\} \end{aligned} \tag{4.9}$$

and

$$\begin{aligned} f_4(\tilde{T}_1, \ldots, \tilde{T}_q) &= N(\tilde{T}; Q_4)\times \\ &\times\Big\{1+\frac{1}{\sqrt{m}}\Big[\sum_{g=1}^{q}(J_{41}+J_{43}+J_{45}+J_{47}+J_{49})H_g(\tilde{T})+ \\ &+\sum_{g_1,g_2,g_3}^{q}(J_{40}+J_{42}+J_{44}+J_{46}+J_{48}+J_{410})+H_{g_1,g_2,g_3}(\tilde{T})\Big]+O(m^{-1})\Big\}, \end{aligned} \tag{4.10}$$

where

$$\begin{aligned} Q_3 &= (Q_{gl}^{(3)}) \\ Q_{gl}^{(3)} &= \sum_{a,b,j,h}\tilde{W}_{g,ha}\tilde{W}_{l,jb}\tilde{\varrho}_{aj}\tilde{\varrho}_{bh}, \\ Q_4 &= (Q_{gl}^{(4)}), \\ Q_{gl}^{(4)} &= \sum_{a,b,j,h}\tilde{Z}_{g,ha}\tilde{Z}_{l,jb}\tilde{\varrho}_{aj}\tilde{\varrho}_{bh}, \\ \tilde{W}_{g,ha} &= \sum_{j=1}^{p}c_{gj}\tilde{u}_{aj}^*\tilde{u}_{jh}(1-\delta_{ah}\delta_j), \\ \tilde{Z}_{g,ha} &= \sum_{j=1}^{p}\tilde{\Lambda}_g(c_{gj}-\lambda_g^*\tilde{\Lambda}d_{gj})\tilde{u}_{aj}^*\tilde{u}_{hj}(1-\delta_{ah}\delta_j), \\ \lambda_g^* &= \sum_{j=1}^{p}c_{gj}\delta_j, \\ \tilde{\Lambda}_g^{-1} &= \sum_{j=1}^{p}d_{gj}\delta_j, \\ J_{30} &= \tfrac{1}{3}\sum_{a,b,d,h,j,l}^{p}\tilde{W}_{g_1,ha}\tilde{W}_{g_2,jb}\tilde{W}_{g_3,ld}\tilde{\varrho}_{aj}\tilde{\varrho}_{bl}\varrho_{dh}, \\ J_{31} &= \tfrac{1}{2}\sum_{j,h,l}^{p}c_{gj}[\tilde{\varrho}_{hl}\tilde{\varrho}_{lh}^2\tilde{\mu}_{lj}\tilde{\mu}_{hj}^*-\tilde{\varrho}_{hl}\tilde{\mu}_{hj}^*\tilde{\mu}_{lj}-\tilde{\varrho}_{lh}\tilde{\varrho}_{hj}\tilde{\mu}_{lj}^*+\delta_j\delta_{lh}\tilde{\mu}_{hj}^*\tilde{\mu}_{hj}], \\ J_{32} &= \tfrac{1}{4}\sum_{a,b,c,d,h,j,l}^{p}c_{g_1j}\tilde{W}_{g_2,ba}\tilde{W}_{g_3,dc}\tilde{\varrho}_{hc}\tilde{\varrho}_{hd}\tilde{U}_{hj}\times \\ &\qquad\times[\tilde{\varrho}_{hl}\tilde{\varrho}_{la}\tilde{u}_{lb}\tilde{\varrho}_{lj}-2\tilde{\varrho}_{hb}\tilde{\varrho}_{la}\tilde{u}_{lj}+3\delta_{lh}\delta_j\tilde{\varrho}_{ha}\tilde{\varrho}_{hb}\tilde{u}_{hj}], \\ J_{33} &= \tfrac{1}{4}\sum_{h,j,l}^{p}\sum_{a\neq j}c_{gj}\tilde{\lambda}_{ja}^{-1}\Big[4\sum_{s,f}^{p}\tilde{\varrho}_{ls}\tilde{\varrho}_{hf}\tilde{u}_{lj}^*\tilde{u}_{ha}\tilde{u}_{sj}\tilde{u}_{fa}^*- \\ &\qquad-2(\delta_j+\delta_a)\sum_{s=1}^{p}(\tilde{\varrho}_{sl}\varrho_{hl}\tilde{u}_{lj}\tilde{u}_{ha}^*\tilde{u}_{sj}^*\tilde{u}_{sa}+\tilde{\varrho}_{sh}\tilde{\varrho}_{ls}\tilde{u}_{lj}^*\tilde{u}_{ha}\tilde{u}_{sj}\tilde{u}_{sa}^*)+ \\ &\qquad+(\delta_j+\delta_a)^2\tilde{\varrho}_{hl}\tilde{\varrho}_{lh}\tilde{u}_{lj}^*\tilde{u}_{la}\tilde{u}_{hj}\tilde{u}_{ha}\Big], \end{aligned} \tag{4.11}$$

$$J_{34} = \frac{1}{2} \sum_{b,c,d,h,j,k,l}^{p} \sum_{a \neq j} c_{g_1 j} \tilde{\lambda}_{ja} \tilde{W}_{g_2, cb} \tilde{W}_{g_3, kd} \Big[2 \sum_{s,f}^{p} \tilde{\varrho}_{lc} \tilde{\varrho}_{hb} \tilde{\varrho}_{fk} \tilde{\varrho}_{sd} \tilde{u}_{lj}^{*} \tilde{u}_{ha} \tilde{u}_{sj} \tilde{u}_{fa}^{*} -$$

$$- (\delta_j + \delta_a) \sum_{s=1}^{p} (\tilde{u}_{lj}^{*} \tilde{u}_{ha} \tilde{u}_{sj} \tilde{u}_{sa}^{*} + \tilde{u}_{lj} \tilde{u}_{ha}^{*} \tilde{u}_{sj}^{*} \tilde{u}_{sa}) \tilde{\varrho}_{hk} \tilde{\varrho}_{sb} \tilde{\varrho}_{sc} \tilde{\varrho}_{ld} +$$

$$+ 2(\delta_j + \delta_a)^2 \tilde{u}_{lj}^{*} \tilde{u}_{la} \tilde{u}_{hj} \tilde{u}_{ha}^{*} \tilde{\varrho}_{hb} \tilde{\varrho}_{hc} \tilde{\varrho}_{ld} \tilde{\varrho}_{lk} \Big],$$

$$J_{45} = \sum_{a,b,c,d,j,l}^{p} \tilde{\Lambda}_g^2 (\lambda_g^{*} \tilde{\Lambda}_g d_{gj} - c_{gj}) d_{gj} \tilde{\varrho}_{ad} \tilde{\varrho}_{cb} \tilde{u}_{aj}^{*} \tilde{u}_{bj} \tilde{u}_{cl} \tilde{u}_{dl},$$

$$J_{46} = \sum_{a,b,c,d,f,h,j,l,r,s,}^{p} \tilde{\Lambda}_{g_1}^2 (\lambda_{g_1}^{*} \tilde{\Lambda}_{g_1} d_{g_1 j} - c_{g_1 j}) d_{g_1 j} \tilde{\varrho}_{af} \tilde{\varrho}_{bs} \tilde{\varrho}_{cr} \tilde{\varrho}_{dh} \tilde{u}_{aj}^{*} \tilde{u}_{bj} \tilde{u}_{cl} \tilde{u}_{dl} \tilde{Z}_{g_2, fs} \tilde{Z}_{g_3, rh},$$

$$J_{47} = \sum_{b,d,j,l}^{p} \tilde{\Lambda}_g^2 (\lambda_g^{*} \tilde{\Lambda}_g d_{gj} - c_{gj}) d_{gj} \delta_j \delta_l \tilde{\varrho}_{bd} \tilde{\varrho}_{ba} \tilde{u}_{bj}^{*} \tilde{u}_{bj} \tilde{u}_{dl}^{*} \tilde{u}_{dl},$$

$$J_{48} = \sum_{b,d,f,h,j,l,r,s}^{p} \tilde{\Lambda}_{g_1}^2 (\lambda_{g_1}^{*} \tilde{\Lambda}_{g_1} d_{g_1 j} - c_{g_1 j}) d_{g_1 l} \tilde{\varrho}_{bd} \tilde{\varrho}_{db} \tilde{\varrho}_{bf} \tilde{\varrho}_{bs} \tilde{\varrho}_{dr} \tilde{\varrho}_{dh} \times$$

$$\times \tilde{u}_{bj}^{*} \tilde{u}_{bj} \tilde{u}_{dl}^{*} \tilde{u}_{dl} \tilde{Z}_{g_2, fs} \tilde{Z}_{g_3, rh},$$

$$J_{49} = - \sum_{b,c,d,j,l}^{p} \tilde{\Lambda}_g^2 (\lambda_g^{*} \tilde{\Lambda}_g d_{gj} - c_{gj}) d_{gl} (\delta_j \tilde{u}_{bj}^{*} u_{bj} \tilde{u}_{cl}^{*} u_{dl} + \delta_l \tilde{u}_{bl}^{*} \tilde{u}_{bl} u_{cj}^{*} u_{dj}) \tilde{\varrho}_{bc} \tilde{\varrho}_{db},$$

$$J_{410} = - \sum_{b,c,d,f,h,j,l,r,s}^{p} \tilde{\Lambda}_{g_1}^2 (\lambda_{g_1}^{*} \tilde{\Lambda}_{g_1} d_{g_1 j} - c_{g_1 j}) d_{g_1 l} \times$$

$$\times (\delta_j \tilde{u}_{bj}^{*} u_{bj} \tilde{u}_{cl}^{*} \tilde{u}_{dl} + \delta_l \tilde{u}_{bl} \tilde{u}_{cj}^{*} \tilde{u}_{dj}) \tilde{\varrho}_{bf} \tilde{\varrho}_{bs} \tilde{\varrho}_{dr} \tilde{\varrho}_{ch} \tilde{Z}_{g_2, fs} \tilde{Z}_{g_3, rh},$$

and J_{4j} are exactly in the same forms as those of J_{3j}, for $j = 0, 1, 2, 3, 4$, except $\tilde{W}_{g,ha}$ replaced by $\tilde{Z}_{g,ha}$ and c_{gj} replaced by $\tilde{\Lambda}_g (c_{gj} - \lambda_g^{*} \tilde{\Lambda}_g d_{gj})$.

Note added in proof: Very recently, Konishi derived asymptotic distributions of the individua roots l_α as well as $\sum_{\alpha=1}^{q} l_\alpha$ where $q < p$ and $l_1, \ldots, l_p$ are the roots of the real sample correlation matrix, and the roots of the population correlation matrix are simple. Konishi also used the well-known perturbation method in deriving his results and his expressions involving linear combinations of the density of the normal distribution and the derivatives of this density.

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