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ON THE ADMISSIBILITY OF TESTS FOR DISCRETE EXPONENTIAL FAMILIES

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1. Introduction

In this paper we state some necessary and sufficient conditions for the admissibility of tests for the following testing problem.

A random vector (X, Y), where $X \in \boldsymbol{R}^m$, $Y \in \boldsymbol{R}^k$, has the density

$$p_{\theta,\omega}(x, y) = c(\theta, \omega)\exp(\langle\theta, x\rangle + \langle\omega, y\rangle) \tag{1}$$

with respect to a finite measure, say l, having a finite support. The parameters θ and ω range over $\boldsymbol{R}^m$ and $\boldsymbol{R}^k$, respectively and $\langle\theta, x\rangle = \sum_{i=1}^{m} \theta_i x_i$, $\langle\omega, y\rangle = \sum_{i=1}^{k} \omega_i y_i$.

The null hypothesis, H, asserts that

$$H\colon \omega = 0,\ \theta \in \boldsymbol{R}^m,$$

while the alternative is of the form

$$K\colon \omega \in \Omega \subset \boldsymbol{R}^k \setminus \{0\},\ \theta \in \boldsymbol{R}^m,$$

where Ω satisfies the following requirements:

(i) there exists a point $\omega_0 \in \boldsymbol{R}^k$, $\omega_0 \neq 0$, such that $\Omega \subset \{\omega \in \boldsymbol{R}^k\colon \langle\omega_0, \omega\rangle \leqslant 0\}$,

(ii) $n\omega \in \Omega$ if $\omega \in \Omega$ and if $n \geqslant 1$,

(iii) Ω is convex,

(iv) Ω contains an open sphere in $\boldsymbol{R}^k$.

Earlier results on the characterization of admissible tests for testing a simple null hypothesis in some exponential families have been done by E. Lehmann [6] and A. Birnbaum [1]. Certain sufficient conditions for admissibility with applications to multivariate normal distributions have been obtained by C. Stein [10] and also by J. Kiefer and R. Schwartz [3], [9]. Moreover, a description of the class of admissible tests for H against $K_1\colon \omega \neq 0$, $\theta \in \boldsymbol{R}^m$ on density (1) has been worked out by T. K. Matthes and D. R. Truax [7].

Numerous applications of T. K. Matthes' and D. R. Truax's, and also the present author's results, are listed in Section 3 and treated in detail in [4]. The proofs of the theorems stated below are contained in [5], which is submitted to *Mathematische Operationsforschung und Statistik*.

2. Characterization of admissible tests

The description of the class of admissible tests for H against K is given in two steps.

First we give some necessary and sufficient conditions for a test to be admissible for testing a simple null hypothesis

$$H^x\colon\ \omega = 0,$$

against

$$K^x\colon\ \omega \in \Omega \setminus \{0\}$$

on the conditional distribution of Y, given $X = x$.

To formulate this result we need the following notation:

N_x stands for the support of the conditional distribution of Y, given $X = x$. V denotes the smallest convex cone (with vertex at $0 \in \boldsymbol{R}^k$) containing Ω while by V^- we denote the normal cone of V, e.g.

$$V^- = \{w \in \boldsymbol{R}^k\colon \langle w, v\rangle \leqslant 0 \text{ for all } v \in V\}.$$

Moreover, Φ stands for the class of all non-empty closed convex sets in $\boldsymbol{R}^k$ and

$$\Phi(V) = \{C\colon\ C \in \Phi \text{ and } V^- \subset C - c \text{ for each } c \in \partial C\}.$$

Finally, E^D denotes the set of extremal points of D.

We describe the class of admissible tests for testing H^x against K^x as follows:

THEOREM 1. *Test $\varphi(y)$ is admissible if and only if there exists a set $D \in m(V)$ such that*

$$(\boldsymbol{R}^k \setminus D) \cap N_x = \{y\colon\ \varphi(y) = 1\} \cap N_x,$$
$$\mathrm{Int}(D) \cap N_x \subset \{y\colon\ \varphi(y) = 0\} \cap N_x,$$
$$E^D \cap N_x \supset \{y\colon\ 0 < \varphi(y) < 1\} \cap N_x.$$

We now give some connections between admissibility of tests for conditional distribution and the initial testing problem.

THEOREM 2. *Test $\psi(x, y)$ is admissible for testing H against K if and only if for every fixed $X = x$ the test $\psi(x, \cdot)$ is admissible for testing H^x against K^x.*

So, by combining these two theorems an explicit characterization of an admissible test of H against K follows.

Finally note that under the assumptions given in Section 1 every admissible test for H against K is also admissible for H against $K_2\colon \omega \in \Omega_1 \supset \Omega$. Besides the class of tests described above is minimal complete and minimal essentially complete.

3. Applications

By appropriate specification of density (1) the admissibility of the following tests can be deduced.

1. The maximum likelihood tests for testing H against K is admissible and remains admissible for testing H against $K_1\colon \omega \neq 0$, $\theta \in \boldsymbol{R}^m$. The converse is not true.

2. We give an admissible and unbiased test for testing

$$P_{1j} = P_{2j}, \quad j = 1, \ldots, k,$$

against

$$P_{1j} \leqslant P_{2j}, \quad j = 1, \ldots, k,$$

where P_{1j} and P_{2j}, $j = 1, \ldots, k$, are parameters of independent binomial distributions.

3. Some of the results given above are proved, in fact, under slightly weaker assumptions and can be applied also for comparing the intensities of k independent Poisson distributions. Both the Fisher test and its generalization for unequal sample sizes (see [8], p. 369) are admissible for testing

$$\lambda_1 = \ldots = \lambda_k$$

against

$$\lambda_i \neq \lambda_j \quad \text{for some } i \neq j,$$

and neither of these tests is admissible for the "one-sided" alternative

$$\lambda_1 \leqslant \lambda_2 \leqslant \ldots \leqslant \lambda_k.$$

4. Some tests of the no-interaction hypothesis in a three-way contingency table which are based on the likelihood ratio and chi-square statistics with parameters fitted by an iterative method (see [2]) are admissible when the sample size and the number of classes are fixed.

5. The chi-square test for the independence in a three-way contingency table is admissible.

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ON THE ASYMPTOTIC DISTRIBUTIONS OF CERTAIN FUNCTIONS OF EIGENVALUES OF CORRELATION MATRICES*

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1. Introduction

In this paper, the authors obtained asymptotic expressions for the joint densities of the linear combinations of the roots as well as the joint densities of the ratios of the linear combinations of the roots of the correlation matrices when the distributions underlying the data are multivariate normal or complex multivariate normal. These expressions are in terms of linear combinations of multivariate normal densities and multivariate Hermite polynomials. The authors have earlier obtained analogous results for the joint densities of the linear combinations and ratios of the linear combinations of the roots of the sample covariance matrices. The results obtained in this paper are useful in the application of simultaneous test procedures for the inference on eigenvalues of the correlation matrices of real and complex multivariate normal populations.

2. Joint distribution of linear combinations of the roots of correlation matrix in the real case

Let the columns of X: $p \times m$ be distributed independently and identically as multivariate normal with zero mean vector and covariance matrix $\Sigma = (\sigma_{gh})$ and $S = (s_{gh}) = XX'$. Then S has the central real Wishart distribution $W_p(\Sigma, m)$.

Let $\Omega = (\varrho_{jh})$, where $\varrho_{jh} = \sigma_{jh}/(\sigma_{jj}\sigma_{hh})^{1/2}$. Since Ω is symmetric, there exists an orthogonal matrix $U = (u_{jh})$ such that

$$U'\Omega U = \Lambda, \tag{2.1}$$

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