5. In this note only a brief description of the problem has been given. Nothing has been said about the tests based on nonlinear statistics (as that given by (1)), or about tests for alternatives of a different type when, essentially, there is no asymptotic normality of the test statistics. A more detailed discussion of the problems in question will appear in Teor. Verojatnost. i Primenen. vol. 24.

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A NOTE ON CONFIDENCE INTERVALS FOR THE KIEFER-WOLFOWITZ PROBLEM

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Introduction

In this simple note we shall be concerned with the so-called Kiefer–Wolfowitz situation; i.e. with the problem of finding a point θ of minimum of a (regression) function $f\colon R \to R$, R denoting the real line, when the only information available is that we can observe unbiased estimates of function values of f. Namely, we shall apply the method of Farrell [2] to obtain a confidence interval for θ, of length not exceeding any predetermined positive number. Originally, the method was employed for finding a confidence interval for the zero of a regression function. It provides some suitable stopping rule for the experimentation process based upon one of the stochastic approximation procedures (depending on the situation at hand, either on the procedure of Kiefer–Wolfowitz or on that of Robbins–Monro).

Let the sequential procedure for estimating point θ of minimum of a regression function be of the form[1]:

(1) $$X_{n+1} = X_n - a_n Y_n,$$

where a_n are positive numbers, X_1 and Y_n are r.v.'s, $n \geqslant 1$. It is well known that under assumptions (A1)–(A5), as well as under (A4)–(A7) (see below), $\lim_{n\to\infty} X_n = \theta$. (Equation (1) with Y_n given by (A4) defines the original Kiefer–Wolfowitz procedure.)

Now, the method can loosely be described as follows. Let $\underline{\theta} \leqslant \theta \leqslant \overline{\theta}$, for some known $\underline{\theta}$ and $\overline{\theta}$. Let $\{X_n^{(i)}, n \geqslant 1\}$, $i = 1, \ldots, 2k$ be $2k$ sequences of r.v.'s obtained by the use of $2k$ simultaneous and independent Kiefer–Wolfowitz procedures starting, respectively, with $X_1^{(i)} = \underline{\theta}$ for $i = 1, \ldots, k$ and $X_1^{(i)} = \overline{\theta}$ for $i = k+1, \ldots, 2k$. These sequences are stopped at such random moments, say M and N, that $X'_M \equiv \min\{X_M^{(1)}, \ldots, X_M^{(k)}\}$ and $X''_N = \max\{X_N^{(k+1)}, \ldots, X_N^{(2k)}\}$ have enabled one to construct a confidence interval for θ, with the confidence level $1-\alpha$ and length $\leqslant L$, α and L given in advance.

(1) All random variables (r.v.'s) are assumed to be defined on a probability space $(\Omega, \mathcal{F}, P)$ and relations between r.v.'s are meant with probability one.

The result

We shall begin with the following assumptions:

(A1) a_n, c_n are positive numbers, $n \geq 1$; $a_n, c_n \downarrow 0$; $\sum_n a_n = \infty$, $\sum_n a_n c_n < \infty$, $\sum_n a_n^2 c_n^{-2} < \infty$.

(A2) $f\colon R \to R$ is twice continuously differentiable. The second order derivative of f is bounded on R, i.e. $|f_{xx}(x)| \leq K_0$ for all $x \in R$ and some $K_0 < \infty$.

(A3) $\inf\{|f_x(x)|;\ |x-\theta| > \varepsilon\} > 0$, $\inf\{f(x)-f(\theta);\ |x-\theta| > \varepsilon\} > 0$ for every $\varepsilon > 0$ and with $f_x(x)$ denoting the first order derivative of f at x.

(A4) $E_{\mathscr{F}_n} Y_n = (2c_n)^{-1}[f(X_n+c_n)-f(X_n-c_n)]$, where $\mathscr{F}_n$ is the smallest σ-field that measures $\{X_1, \ldots, X_n\}$ and $E_{\mathscr{F}_n}$ denotes the conditional expectation given $\mathscr{F}_n$.

(A5) $E_{\mathscr{F}_n} Z_n^2 \leq \frac{1}{4}c_n^{-2}\sigma^2$, $\sigma^2 < \infty$, where $Z_n = Y_n - E_{\mathscr{F}_n} Y_n$.

Thus, by the Taylor expansion with the remainder we have

$$X_{n+1} = X_n - a_n f_x(X_n) + \gamma_n, \tag{2}$$

where

$$\gamma_n = -a_n(B_n+Z_n) \quad \text{and} \quad B_n = \tfrac{1}{4}c_n[f_{xx}(X_n+\nu_n^{(1)}c_n)-f_{xx}(X_n-\nu_n^{(2)}c_n)],$$
$$\nu_n^{(i)} \in (0, 1), \qquad i = 1, 2.$$

LEMMA 1. *We have*

$$P\Big(\max_{m \leq j \leq n}\Big|\sum_{i=m}^{j}\gamma_i\Big| > \delta\Big) \leq 4\sigma^2\delta^{-2}\sum_{i=m}^{n}(a_i/c_i)^2$$

for every $\delta > 0$ *and any* m, n *such that* $m \leq n$ *and* $\sum_{i=m}^{\infty} a_i c_i \leq \delta/K_0$.

Proof.

$$\sum_{i=m}^{n} a_i|B_i| \leq \tfrac{1}{2}K_0\sum_{i=m}^{n} a_i c_i \leq \tfrac{1}{2}\delta. \tag{3}$$

The sequence of sums $\{\sum_{i=m}^{j} a_i Z_i,\ j = m, m+1, \ldots\}$ is a martingale (relative to $\{\mathscr{F}_{j+1}, j = m, m+1, \ldots\}$) and, hence, using one of the fundamental inequalities for non-negative submartingales yields (see e.g. [1], p. 317, Theorem 3.4)

$$P\Big(\max_{m \leq j \leq n}\Big|\sum_{i=m}^{j} a_i Z_i\Big| > \frac{\delta}{2}\Big) \leq (\delta/2)^{-2}\cdot\sigma^2\sum_{i=m}^{n}(a_i/c_i)^2.$$

So, by (3) and the definition of γ_i's, the proof is accomplished.

PROPOSITION 1. *Let, for any fixed integer* m *and positive* δ, $A_{m,\delta}$ *be the event that for some pair of integers* r *and* s, $m \leq r < s$, $X_r \leq \theta$ *and* $X_s > \theta + 2\delta$. *Assume* $a_m K_0 < 1$ *and* $\sum_{i=m}^{\infty} a_i c_i \leq \delta/K_0$. *Then*

$$P(A_{m,\delta}) \leq 4\sigma^2\delta^{-2}\sum_{i=m}^{\infty}(a_i/c_i)^2.$$

Proof. Without loss of generality we may assume that $\theta = 0$.

Suppose $P(A_{m,\delta}) > 4\sigma^2\delta^{-2}\sum_{i=m}^{\infty}(a_i/c_i)^2$. Then with positive probability there exists a pair of integers r and s with $m \leq r < s$ such that $X_r \leq 0$, $X_i > 0$ for $i = r+1, \ldots, s$, $X_s > 2\delta$ and $\big|\sum_{i=r}^{s-1}\gamma_i\big| \leq 2\delta$. By (2) we have

$$X_s = X_r - \sum_{i=r}^{s-1} a_i f_x(X_i) + \sum_{i=r}^{s-1}\gamma_i$$

and hence (the next to last inequality being implied by assumption (A2))

$$X_s \leq X_r - a_r f_x(X_r) + \sum_{i=r}^{s-1}\gamma_i \leq X_r - a_r f_x(X_r) + 2\delta \leq X_r + a_r K_0|X_r| + 2\delta \leq 2\delta,$$

a contradiction. This completes the proof.

Remark 1. Obviously, under the conditions identical to those of Proposition 1, the following inequality holds

$$P(B_{m,\delta}) \leq 4\sigma^2\delta^{-2}\sum_{i=m}^{\infty}(a_i/c_i)^2,$$

with $B_{m,\delta} = \bigcup_{m \leq r < s}\{X_r \geq \theta, X_s < \theta - 2\delta\}$.

We are now in a position to state the final result. Suppose the condition

$$\underline{\theta} \leq \theta \leq \overline{\theta}$$

is given. Let $\{X_n^{(i)}, n \geq 1\}$ $i = 1, \ldots, 2k$, be $2k$ sequences of r.v.'s obtained by the use of simultaneous Kiefer–Wolfowitz procedures with the same sequences $\{a_n\}$ and $\{c_n\}$ and starting, respectively, with $X_1^{(i)} = \underline{\theta}$, $i = 1, \ldots, k$, and $X_1^{(i)} = \overline{\theta}$, $i = k+1, \ldots, 2k$. Recall that $\lim_{n\to\infty} X_n^{(i)} = \theta$, $i = 1, \ldots, 2k$.

PROPOSITION 2. *Suppose* $\{X_n^{(i)}\}$, $i = 1, \ldots, 2k$, *are given as above*, δ *is positive and fixed and*

$$4\sigma^2\delta^{-2}\sum_{i=1}^{\infty}(a_i/c_i)^2 \leq (\alpha/2)^{1/k}, \qquad a_1 K_0 < 1, \qquad \sum_{i=1}^{\infty} a_i c_i \leq \delta/K_0. \tag{4}$$

Then, with $X'_M \equiv \min\{X_M^{(1)}, \ldots, X_M^{(k)}\}$ and $X''_N \equiv \max\{X_N^{(k+1)}, \ldots, X_N^{(2k)}\}$,

$$P(\theta \in [-2\delta + X'_M, 2\delta + X''_N] \text{ for any positive integers } M, N) \geq 1-\alpha.$$

Indeed, by Proposition 1 and Remark 1 we get

$$P(\exists (M,N)\colon \theta \notin [-2\delta+X'_M, 2\delta+X''_N])$$
$$\leqslant P((\forall i=1,\ldots,k)(\exists M \geqslant 1): X_1^{(i)} \leqslant \theta, X_M^{(i)}-2\delta > \theta)+P((\forall i = k+1,\ldots,2k)$$
$$(\exists N \geqslant 1): X_1^{(i)} \geqslant \theta, X_N^{(i)}+2\delta < \theta) \leqslant \alpha.$$

Thus, for any ε positive and fixed and any integers M, N such that $|X''_N-X'_M| \leqslant \varepsilon$, the interval $[-2\delta+X'_M, 2\delta+X''_N]$ is the confidence interval required (provided that it is non-degenerate, i.e., that $X'_M-X''_N \leqslant \varepsilon < 4\delta$) and is of length $\leqslant \varepsilon+4\delta$. In particular, one can define M, N to be such integers n_1, n_2 that $|X''_{n_2}-X'_{n_1}| \leqslant \varepsilon$ and the sum n_1+n_2 assumes the least value.

Assume lastly (A4)–(A7) in lieu of (A1)–(A5):

(A6) $f\colon R \to R$ is continuous. There exist positive constants C_0, K_1, K_2 such that for every $0 < c \leqslant C_0$ (and $x \in R$)

$$(5) \qquad 2cK_1(x-\theta)^2 \leqslant [f(x+c)-f(x-c)](x-\theta) \leqslant 2cK_2(x-\theta)^2 \quad (^2)$$

(A7) $a_n, c_n > 0$, $n \geqslant 1$; $a_n, c_n \downarrow 0$; $\sum_n a_n = \infty$, $\sum_n a_n^2 c_n^{-2} < \infty$.

Under assumptions (A4)–(A7) Proposition 2 remains valid with

$$(6) \qquad \sigma^2\delta^{-2}\sum_{i=1}^{\infty}(a_i/c_i)^2 \leqslant (\alpha/2)^{1/k}, \quad a_1K_2 < 1, \quad c_1 \leqslant C_0,$$

replacing conditions (4); it suffices to observe that equation (2) may then be replaced by the following one

$$X_{n+1} = X_n - a_nK^{(n)}(X_n-\theta)+\gamma_n,$$

where $0 < K_1 \leqslant K^{(n)} \leqslant K_2 < \infty$ and $\gamma_n = -a_nZ_n$.

Remark 2. Our method requires that the estimates for K_0 (or, resp., K_2), σ^2, $\underline{\theta}$ and $\bar{\theta}$ be known.

Remark 3. Conditions (6) and — a fortiori — (4) are stringent and, in practice, make the method applicable only in the case of "small" σ^2.

Concluding remark

The problem has been posed in a non-asymptotic set-up and, consistently, we have not borrowed from the asymptotic theory of stochastic approximation when solving it (cf. e.g. [3]; here one may note that inference from the asymptotic theory about the confidence intervals in question is hardly justifiable, unless there are known some results about the speed with which the behaviour of experimentation process approaches its asymptotic character). We have confined ourselves to the sequential methods of stochastic approximation type and, there, we have followed

(2) Clearly, if f is differentiable, condition (5) implies (A3).

the Farrell's idea, as far as we are aware, the only one among those proposed to date of strictly non-asymptotic (and non-deterministic) nature.

Needless to say, some more investigations in this direction must be done, in view of apparent drawbacks of the solution presented (cf. Remarks 2 and 3).

References

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[2] R. H. Farrell, *Bounded length confidence intervals for the zero of a regression function*, Ann. Math. Statist. 33 (1962), pp. 237–247.
[3] R. L. Sielken, Jr., *Stopping times for stochastic approximation procedures*, Z. Wahrsch. verv. Geb. 26 (1973), pp. 67–75.

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