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A REMARK ON THE CONDITIONING IN LIMIT THEOREMS FOR DEPENDENT RANDOM VECTORS IN R^d

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A main problem of the classical theory of probability concerns limit distributions for sums of infinitesimal systems of independent random variables. There exists a complete solution of this problem given by necessary and sufficient conditions for the convergence in law of such systems to arbitrarily fixed, infinitely divisible probability measure.

Now the following problem is still open. Let there be given an infinitely divisible probability measure Q on $\boldsymbol{R}^d$ with the characteristic function

$$\varphi_Q(\vec{t}) = \exp\left\{i(\vec{t},\vec{a}) - \tfrac{1}{2}(\vec{t}, A\vec{t}) + \int_{R^d}\left(e^{i(\vec{t},\vec{x})} - 1 - \frac{i(\vec{t},\vec{x})}{1+\|\vec{x}\|^2}\right)\frac{1+\|\vec{x}\|^2}{\|\vec{x}\|^2}\,d\mu(\vec{x})\right\}, \quad \vec{t}\in\boldsymbol{R}^d,$$

where $\vec{a}\in\boldsymbol{R}^d$, A is nonnegative definite $d\times d$-matrix, μ is the finite measure on $\boldsymbol{R}^d$, $\mu(\{\vec{0}\}) = 0$. Describe double sequences of random vectors $\{\{\vec{X}_{nk}\}_{1\leqslant k\leqslant k_n}\}_{n\in N}$ which converge in law to Q, i.e. the distributions of sums $S_n = \sum_{k=1}^{k_n}\vec{X}_{nk}$, $n\in N$, are weakly convergent to Q. First, for row-wise independent systems their infinitesimality can be replaced by more general condition

$$\lim_{n\to\infty}\sum_{k=1}^{k_n}|E\,e^{i(\vec{t},\vec{X}_{nk}-\vec{a}_{nk})} - 1|^2 = 0, \quad \vec{t}\in\boldsymbol{R}^d,$$

where $\vec{a}_{nk} := E(\vec{X}_{nk} I(\|\vec{X}_{nk}\| < \varepsilon))$, $\varepsilon > 0$, and the above conditions remain necessary and sufficient.

Next, let us admit a dependence between vectors in the same row. One way to find sufficient conditions for the convergence in law to Q which generalise the classical case is the following: we replace all mean values in known necessary and sufficient conditions by conditional mean values with respect to suitably chosen σ-fields and the ordinary convergence we replace by the convergence in probability of such obtained random vectors. Now we give an example of such result.

THEOREM. *Let there be given a double row-wise increasing sequence of σ-fields* $\{\{F_{nk}\}_{0\leqslant k\leqslant k_n}\}_{n\in N}$ *such that every* $\vec{X}_{nk}$ *is* F_{nk}*-measurable. Let us denote*

$$\vec{A}_{nk} := E\big(\vec{X}_{nk} I(\vec{X}_{nk}\in V)|F_{n,k-1}\big), \quad 1\leqslant k\leqslant k_n,\ n\in N,$$

$$\vec{Y}_{nk} := \vec{X}_{nk}-\vec{A}_{nk}, \quad 1\leqslant k\leqslant k_n,\ n\in N,$$

for a certain fixed $V\subset R^d$, $\vec{0}\in \operatorname{Int} V$. *If*

$$\sum_{k=1}^{k}\left\{\vec{A}_{nk}+E\left(\frac{\vec{Y}_{nk}}{1+\|\vec{Y}_{nk}\|^2}\,\middle|\,F_{n,k-1}\right)\right\}\xrightarrow{P}\vec{a}, \tag{1}$$

$$\sum_{k=1}^{k_n} E\left(\frac{Y_{nki}Y_{nkj}}{1+\|\vec{Y}_{nk}\|^2}\,\middle|\,F_{n,k-1}\right)\xrightarrow{P} a_{ij}+\int_{R^d}\frac{x_i x_j}{\|\vec{x}\|^2}d\mu, \quad 1\leqslant i,j\leqslant d, \tag{2}$$

and

$$\sum_{k=1}^{k_n} E\left(\frac{\|\vec{Y}_{nk}\|^2}{1+\|\vec{Y}_{nk}\|^2} I(\vec{Y}_{nk}\in E)|F_{n,k-1}\right)\xrightarrow{P}\mu(E), \quad E\in \operatorname{Cont}\mu,\ \vec{0}\notin\bar{E}, \tag{3}$$

or, equivalently,

$$\sum_{k=1}^{k_n} P(\vec{Y}_{nk}\in E|F_{n,k-1})\xrightarrow{P}\int_E\frac{1+\|\vec{x}\|^2}{\|\vec{x}\|^2}d\mu, \quad E\in\operatorname{Cont}\mu,\ \vec{0}\notin\bar{E}, \tag{3'}$$

then each of the following conditions

$$\exp\left\{i\sum_{k=1}^{k_n}(\vec{t},\vec{Y}_{nk})-\sum_{k=1}^{k_n}E\big(e^{i(\vec{t},\vec{Y}_{nk})}-1|F_{n,k-1}\big)\right\}\xrightarrow{P}1, \quad \vec{t}\in R^d, \tag{4}$$

$$\sum_{k=1}^{k_n}|E(e^{i(\vec{t},\vec{Y}_{nk})}-1|F_{n,k-1})|^2\xrightarrow{P}0, \quad \vec{t}\in R^d, \tag{5}$$

$$\max_{1\leqslant k\leqslant k_n} P(\|\vec{X}_{nk}\|>\varepsilon|F_{n,k-1})\xrightarrow{P}0, \quad \varepsilon>0, \tag{6}$$

(7)

(a) $$\max_{1\leqslant k\leqslant k_n} P(\|\vec{Y}_{nk}\|>\varepsilon|F_{n,k-1})\xrightarrow{P}0, \quad \varepsilon>0,$$

(b) $$\sum_{k=1}^{k_n}\left[E\left(\frac{\|\vec{Y}_{nk}\|}{1+\|\vec{Y}_{nk}\|^2}\,\middle|\,F_{n,k-1}\right)\right]^2\xrightarrow{P}0,$$

is sufficient for the convergence in law of $\{\{\vec{X}_{nk}\}\}$ *to* Q.

In a similar way we can generalise the classical limit theorems induced by Levy's and Kolmogorov's representations for Q (see [3]).

There is an open question if there exist other reasonable methods of choosing $\{\{F_{nk}\}\}$. Dvoretzky in [1] and [2] has proposed the conditioning with respect to the former row sum, i.e.

$$F_{n,0} := \{\emptyset,\Omega\}, \quad F_{nk} := B\Big(\sum_{i=1}^{k}\vec{X}_{ni}\Big), \quad 1\leqslant k\leqslant k_n,\ n\in N.$$

In fact, he has also used the conditioning with respect to "at least all past", because he has passed to another "good" probability space, where row sums of new random variables have the same distributions as previously, but in addition they form a Markov sequence. In [3] we have present our doubts if such a transition preserves all conditions (1)–(3) and, for example, (6). Now we can give the following simple example that the main theorems of [1] and [2] are false.

Let us take two independent random variables Y_1, Y_2 defined as follows:

$$P[Y_1=+4]=P[Y_1=-4]=P[Y_2=+3]=P[Y_2=-3]=\tfrac{1}{2}.$$

We define a double sequence of dependent random variables $\{\{X_{nk}\}_{1\leqslant k\leqslant 8n}\}_{n\in N}$:

$$X_{n,8s+1}=-X_{n,8s+4}=-X_{n,8s+5}=X_{n,8s+8}=\frac{Y_1}{5\sqrt{n}},$$

$$X_{n,8s+2}=-X_{n,8s+3}=-X_{n,8s+6}=X_{n,8s+7}=\frac{Y_2}{5\sqrt{n}},$$

for $s=0,1,\ldots,n-1$. It is easy to verify that this double sequence satisfies the following conditions:

(i) $\sum_{k=1}^{8n}E\big(X_{nk}|\sum_{i=1}^{k-1}X_{ni}\big)=0,\ n\in N,$

(ii) $\sum_{k=1}^{8n}\big\{E\big(X_{nk}^2|\sum_{i=1}^{k-1}X_{ni}\big)-\big[E\big(X_{nk}|\sum_{i=1}^{k-1}X_{ni}\big)\big]^2\big\}=1,\ n\in N,$

(iii) $\sum_{k=1}^{8n}E\big(X_{nk}^2 I(|X_{nk}|>\varepsilon)|\sum_{i=1}^{k-1}X_{ni}\big)=0,$

for every $\varepsilon>0$ and sufficiently large $n\in N$.

Theorem 2.2 of [2] (a generalisation of the Lindeberg–Feller theorem) asserts that (i)–(iii) imply the convergence in distribution of S_n, $n\in N$, to the normal law $N(0,1)$, but it is obvious that $S_n=0$ for all $n\in N$.

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