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ON THE DYNAMIC STOCHASTIC APPROXIMATION

VÁCLAV DUPAČ

Charles University, Prague, Czechoslovakia

ASSUMPTIONS:

(i) For $n \in N$, $R^0(n, x)$ is a Borel ($= \mathscr{B}_k$) measurable mapping from E_k into E_k.

(ii) $\{\mathscr{F}_n, n \in N\}$ is a non-decreasing sequence of σ-fields of events.

(iii) For $n \in N$, $G^0(n, x, \omega)$ is a $\mathscr{B}_k \times \mathscr{F}_n$-measurable random mapping from E_k into E_k, independent of $\mathscr{F}_{n-1}$ and such that $EG^0(n, x, \omega) = 0$, $x \in E_k$.

(iv) There is a positive definite matrix C and a $\lambda > 0$ such that $(CR^0(n, x), x) \leqslant -\lambda(Cx, x)$ for all $x \in E_k$, $n \in N$.

(v) $|R^0(n, x)|^2 + E|G^0(n, x)|^2 \leqslant K(1+|x|^2)$ for all $x \in E_k$, $n \in N$ and some $K > 0$.

(vi) For $n \in N$, $R^0(n, x) = Bx + \delta(n, x)$, where B is a $k \times k$ matrix (of constants) such that all its eigenvalues have negative real parts, and $\delta(n, x) = o(|x|)$ uniformly in $n \in N$ for $x \to 0$.

(vii) $\lim_{n\to\infty, x\to 0} E\big(G^0(n, x, \omega)\, G^{0T}(n, x, \omega)\big) = S_0$ exists; S_0 non-singular.

(viii) For some $\varepsilon > 0$,

$$\lim_{r\to\infty} \sup_{|x|<\varepsilon} \sup_{n\in N} E\big(|G^0(n, x, \omega)|^2 \chi_{\{|G^0(n,x,\omega)|>r\}}\big) = 0.$$

(ix) For $n \in N$, $Q(n)$ is a known $k \times k$ matrix and $q(n)$ a k-dimensional vector (unknown in general) such that

$$\lim_{n\to\infty} n^{\alpha} Q(n) = 0, \qquad \lim_{n\to\infty} n^{3\alpha/2} q(n) = q_\infty,$$

for some $\frac{1}{2} < \alpha < 1$ and $0 \leqslant q_\infty < +\infty$.

(x) $\theta(n) \in E_k$, $n \in N$, satisfy the difference equation

$$\theta(n+1) - \theta(n) = Q(n)\theta(n) + q(n), \qquad n \in N.$$

(xi) $R(n, x) = R^0\big(n, x-\theta(n)\big)$; $G(n, x, \omega) = G^0\big(n, x-\theta(n), \omega\big)$, $n \in N$, $x \in E_k$.

(xii) For fixed $a > 0$, $x \in E_k$ and α from (ix) define the sequence $\{X(n), n \in N\}$ by the recursive formula

$$X(1) = x,$$

$$X^*(n) = (I+Q(n))X(n),$$
$$X(n+1) = X^*(n)+an^{-\alpha}\{R(n+1, X^*(n))+G(n+1, X^*(n), \omega)\}.$$

(This is the dynamic Robbins–Monro procedure for tracking $\theta(n)$, the unique root of $R(n, x)$.)

THEOREM. *For $n \to \infty$ and every $x \in E_k$, the distribution of $n^{\alpha/2}(X(n)-\theta(n))$ tends to the normal distribution with mean value $a^{-1}B^{-1}q_\infty$ and the covariance matrix*

$$S = a\int_0^\infty e^{Bv}S_0 e^{B^Tv}dv.$$

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ON A GENERALIZATION OF A THEOREM OF W. SUDDERTH AND SOME APPLICATIONS

A. ENGELBERT and H. J. ENGELBERT
Friedrich-Schiller-Universität, Sektion Mathematik, Universitätschochhaus, 69 Jena, D.D.R.

Introduction and basic definitions

1. Let $(\Omega, \mathscr{F}, \mathbf{P})$ be a probability space and $(\mathscr{F}_n)_{n\geqslant 0}$ an increasing family of sub-σ-algebras of $\mathscr{F}$. We shall consider a sequence $X = (X_n)_{n\geqslant 0}$ of (real-valued) random variables which always is assumed to be adapted to the family $(\mathscr{F}_n)_{n\geqslant 0}$. A nonnegative (possibly, infinite) random variable T is called a *stopping time* (of the family $(\mathscr{F}_n)_{n\geqslant 0}$ of sub-σ-algebras of $\mathscr{F}$) if for all n the event $\{T = n\}$ belongs to $\mathscr{F}_n$. By $\overline{\mathfrak{M}}$ we shall denote the set of all stopping times and by $\mathfrak{M}$ the set of all a.s. finite stopping times.

Let $T \in \overline{\mathfrak{M}}$. Define the random variable X_T by

$$X_T(\omega) = \begin{cases} X_n(\omega) & \text{if} \quad T(\omega) = n, \\ \limsup_n X_n(\omega) & \text{if} \quad T(\omega) = \infty. \end{cases}$$

Let us introduce the class $\overline{\mathfrak{M}}(X)$ of all stopping times T satisfying the condition that the integral $\mathbf{E}X_T$ exists, i.e. $\mathbf{E}X_T^+ < \infty$ or $\mathbf{E}X_T^- < \infty$.(1) Finally, we set $\mathfrak{M}(X) = \overline{\mathfrak{M}}(X) \cap \mathfrak{M}$.

2. In the problem of optimal stopping (cf. Shiryaev [6] or Chow, Robbins, and Siegmund [4]) one considers the value(2)

$$V = \sup_{T\in\mathfrak{M}(X)} \mathbf{E}X_T$$

which is interpreted as the maximal gain that can be obtained by stopping the reward sequence $(X_n)_{n\geqslant 0}$ in an optimal way. Analogously, for any stopping time $S \in \mathfrak{M}(X)$ the value

$$V_S = \sup_{\substack{T\in\overline{\mathfrak{M}}(X)\\ T\geqslant S}} \mathbf{E}X_T$$

(1) For any real number x, we set $x^+ = \max(0, x)$ and $x^- = \max(0, -x)$.
(2) Of course, $\sup\emptyset = -\infty$ and $\inf\emptyset = +\infty$.