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ON THE SOLUTION OF OPTIMAL PERFORMANCE OF PAGE STORAGE HIERARCHIES WITH AN INDEPENDENT REFERENCE STRING

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Introduction

The topic of this paper is the statistical analysis of demand paging algorithms for two-level memory hierarchies. The paged *virtual memory* has become a common feature of large modern operating systems. Memory in such systems is divided into fixed-size blocks called *pages*. The memory consists of at least two levels; a four or five level memory hierarchy including fast buffer storage, main memory, drums, disks and tapes is common. In our paper we assume, for simplicity, a two-level system. If the pages are transferred from secondary storage to main storage only when needed, then the process is called *demand paging*. If access to a page which is not in main memory is necessary a *page fault* is said to occur and this page is brought into memory. All accesses to the hierarchy must be served from the first level.

In connection with the work of operating systems or interactive data base systems or large program systems on large computers with virtual memory the optimal page replacement problem arises. This means that the effective operation of a computer system with virtual memory requires that the amount of processor time wasted owing to page faults should be minimized. The choice of a good page replacement algorithm is very important for minimizing the number of page faults.

We assume that the program consists of n pages and only m $(< n)$ of them may be in the first level. A model for program behaviour that has been used most frequently for its simplicity and its convenience of analysis is the *independent reference model*, in which the probability of referring to the page i at time t is p_i for all t $(i = 1, 2, \ldots, n)$. Let $\eta_1, \eta_2, \ldots, \eta_t, \ldots$ denote the reference string; then by our assumption this sequence of random variables is independent and identically distributed

$$P\{\eta_t = i\} = p_i, \quad i = 1, 2, \ldots, n.$$

In many cases, for the performance index for any demand of page replacement algorithm the expected long-run page fault rate is used (see e.g. Aven, Boguslavsky, Kogan [5], Gelenbe [9]). In our earlier papers (Arató [2], [3], Benczúr, Krámli, Pergel [6]) we used for the performance index the expected number of faults when a program is paged. When the distribution $\{p_i\}$ is known, the optimal demand paging algorithm for the independent reference model keeps in memory the m pages with the largest values of p_i. Most papers devoted to this problem (see e.g. Aho *et al.* [1], Franaszek, Wagner [8], Aven *et al.* [5]) assume that the distribution $\{p_i\}$ is known. Therefore the algorithms proposed by them are only known and have to be estimated in the course of the execution of the program.

The problem of minimizing expected number of page faults when the distribution $\{p_i\}$ is unknown was first formulated by Arató [2]. He used the Bayesian method and the solution of the so-called "two-armed bandit problem" (see De Groot [7]). In the Bayesian case he proved later, [3], [4], for the simple case of two pages, that the least frequently used (LFU) strategy is optimal on every finite time interval when the reference string is an independent, identically distributed sequence of random variables with unknown probability distribution. Benczúr, Krámli, Pergel [6] proved a similar theorem in two extreme cases of loss function for arbitrary n and m. They also used the Bayesian approach and generalized the solution of the "two-armed bandit problem".

In this paper (§ 1) we first give a new proof of the optimality of the LFU algorithm, using the Bayesian approach for the independent reference string.

In Section 2 we prove the optimality of the LFU strategy without the Bayesian assumption. This means that we get a result similar in sequential hypothesis testing, where Wald's theorem on the optimality of the likelihood sequential procedure has a more complicated proof than the Bayesian one.

The "absolute" optimality of the LFU strategy was solved during the authors' stay at the Banach Center in Warsaw in the course of the statistical semester in 1976.

1. The Bayesian case

In this part we present a short summary of our earlier results and models.

Let us consider a program which consists of n pages, numbered by $1, 2, \ldots, n$, and m of them can be stored in the high speed memory and $n-m$ are stored on the second level. The reference string $\eta_1, \eta_2, \ldots, \eta_t$ forms, from the probabilistic point of view, a sequence of independent identically distributed random variables; the common probability distribution

$$P_{i,w} = P_w(\eta_t = i)$$

of the random variables η_t depends on a parameter w whose value is unknown. The dependence on w is given as follows: the range of parameter w is the set W of all permutations of the natural numbers $1, \ldots, n$; $w(i)$ denotes the one-to-one mapping of the set $\{1, \ldots, n\}$ realized by w. There is given a fixed decreasing sequence $p_1 > \ldots > p_n \geqslant 0$ of probabilities $(p_1 + \ldots + p_n = 1)$ and

$$\{P_{i,w}\} \rightleftharpoons \{P_w(\eta_t = i)\} = \{p_{w(i)}\}.$$

Following the Bayesian approach in decision theory, we assume that w itself is a random variable. As we have no preliminary information about the distribution $P_w(\eta_t = i)$, we assume that the prior distribution of parameter w is the uniform one.

Let us denote by $D_{t,N}$ the set of all possible sequential decision procedure $d_t, \ldots, d_{N-1}$ on a finite time interval $[t, N)$. By the decision d_t, which depends only on the initial decision d_0 and the observed reference string $\{\eta_1, \ldots, \eta_{t'}\}$ $t' \in [t, N)$, we mean the subset of pages being absent from the central memory after the observation of string $\{\eta_1, \ldots, \eta_{t'}\}$. The decision d_t consists of $n-m$ elements. By Arató's model [2] (case A) the memory can be rearranged without extra cost before each reference $\eta_t \in d_t$, but a page-fault, $\eta_t \in d_t$, increases the cost by 1 unit; i.e. the loss function has the following form (a page from the second level is delivered to the $(m+1)$st place and when its content is delivered the page must be moved back to the second level):

$$X_t^{d_{t-1}} = \begin{cases} 1 & \text{if} \quad \eta_t \in d_t, \\ 0 & \text{otherwise}. \end{cases} \tag{1}$$

The other extreme case (case B) is also investigated: each change of a page increases the cost at the moment t by 1 unit and η_t must be stored in the central memory; i.e. the loss function has the following form (the page must be replaced by the required page from the second level, i.e. the two pages are exchanged):

$$X_t^{d_t, d_{t-1}} = |d_t \setminus d_{t-1}|, \quad \text{and, for every } t, \quad X_t \notin d_t, \tag{2}$$

where $|\cdot|$ denotes the number of elements of a finite set. Notice that if $\eta_t \in d_{t-1}$ then $X_t^{d_t, d_{t-1}} \geqslant 1$.

Our aim is to find the set of sequential decision procedures, $\{d_0, \ldots, d_{N-1}\}$, which minimize the risk function

$$v(N) = E\Big(\sum_{t=1}^{N} X_t^{d_{t-1}}\Big) \qquad \Big(v(N) = E\Big(\sum_{t=1}^{N} X_t^{d_t, d_{t-1}}\Big)\Big)$$

in case A (resp. B).

Let us denote the conditional expectation under a given realization $\{y_1, \ldots, y_t\}$ of $\{\eta_1, \ldots, \eta_t\}$ by $E_{y_1, \ldots, y_t}$ and define the families of conditional risk functions

$$v(y_1, \ldots, y_t, N-t) = \min_{\{d_t, \ldots, d_{N-1}\} \in D_{t,N}} E_{y_1, \ldots, y_t} \sum_{\tau=t+1}^{N} X_\tau^{d_{\tau-1}}, \tag{3}$$

and

$$(4)\qquad v(y_1, \ldots, y_t, d_t, N-t) = \min_{\{d_{t+1}, \ldots, d_{N-1}\} \in D_{t+1,N}} E_{y_1, \ldots, y_t} \sum_{\tau=t+1}^{N} X_\tau^{d_{\tau+1}, d_\tau}$$

in case A and B, respectively. Families (3) and (4) satisfy the Bellman equations (see e.g. [7]):

$$(5)\qquad v(y_1, \ldots, y_{t-1}, N-t+1) = \min_{d_{t-1}} E_{y_1, \ldots, y_{t-1}} [X_t^{d_{t-1}} + v(y_1, \ldots, y_{t-1}, \eta_t, N-t)],$$

$$(6)\qquad v(y_1, \ldots, y_{t-1}, d_{t-1}, N-t+1) = \min_{d_t} E_{y_1, \ldots, y_{t-1}} [X_t^{d_t, d_{t-1}} + v(y_1, \ldots, y_{t-1}, \eta_{t-1}, \eta_t, d_t, N-t)].$$

Solving recursively systems of equations (5) and (6), we can find the optimal strategies. In case A, $v(y_1, \ldots, y_t, N-1)$ does not depend on d_{t-1}; therefore it is sufficient to minimize for every t the conditional expectation

$$E_{y_1, \ldots, y_{t-1}}(X_t^{d_{t-1}}).$$

The optimality of the LFU strategy in case A is a consequence of the following observation:

LEMMA 1. *If the prior probability distribution of the parameter w is uniform and the frequency f_i of page i in the string $\{y_1, \ldots, y_{t-1}\}$ is less than that of page j, f_j, then*

$$(7)\qquad P(\eta_t = i | y_1, \ldots, y_{t-1}) < P(\eta_t = j | y_1, \ldots, y_{t-1}),$$

i.e. the order of posterior probabilities of pages after observing the string $\{y_1, \ldots, y_{t-1}\}$ is the same as the order of their frequencies in this string.

Proof. Let i and j be two pages such that $f_i < f_j$. If w_1 and w_2 are two permutations with the properties

$$\text{(i)}\qquad w_1(i) = w_2(j),$$

$$\text{(ii)}\qquad w_2(i) = w_1(j),$$

$$\text{(iii)}\qquad w_1(i) < w_1(j) \Leftrightarrow w_2(j) < w_2(i),$$

$$\text{(iv)}\qquad w_1(k) = w_2(k) \quad \text{for} \quad k \neq i, j,$$

then, using Bayes' theorem,

$$P(w_1 | y_1, \ldots, y_t) = \frac{\prod_{k=1}^{n} p_{w_2(k)}^{f_k}}{\sum_{w \in W} \prod_{l=1}^{n} p_{w(l)}^{f_l}}$$

and

$$P(w_2 | y_1, \ldots, y_t) = \frac{\prod_{k=1}^{n} p_{w_2(k)}^{f_k}}{\sum_{w \in W} \prod_{l=1}^{n} p_{w(l)}^{f_l}},$$

we get

$$P(w_1 | y_1, \ldots, y_t) < P(w_2 | y_1, \ldots, y_t).$$

Summing these probabilities, we get the required result.

Using Lemma 1 and the uniformity of the prior distribution of parameter w, we get the following statement.

THEOREM 1. *The least frequently used strategy minimizes the expected loss $E\left(\sum_{t=1}^{N} X_t^{d_{t-1}}\right)$ in case A, where d_0 is arbitrary, and the initial distribution ξ of random variable w is uniform.*

In case B we must argue more carefully.

First we prove that the optimal strategies are among the demand paging algorithms, i.e. among the algorithms satisfying the conditions

$$d_t = d_{t-1} \quad \text{if} \quad \eta_t \notin d_{t-1},$$

$$d_{t-1} \setminus d_t = \{\eta_t\} \quad \text{if} \quad \eta_t \in d_{t-1}.$$

The optimality of the LFU strategy in case B follows from Theorem 2.

THEOREM 2. *If d_t and d_t' are two different decisions for which*

$$d_t \setminus d_t' = \{i\}, \quad d_t' \setminus d_t = \{j\}$$

and the frequency f_i of the page i in the string is less than the frequency f_j of the page j, then

$$v(y_1, \ldots, y_t, d_t, N-t) < v(y_1, \ldots, y_t, d_t', N-t).$$

The proof can be carried out by induction for $\theta = N-t$. The assertion of Theorem 2 for $\theta = 1$ is an obvious consequence of the observation used in case A (Lemma 1). The proof of the induction step — the comparison of conditional risk functions

$$v(y_1, \ldots, y_t, d_t, N-t) \quad \text{and} \quad v(y_1, \ldots, y_t, d_t', N-t)$$

for $t < N-1$ is not so simple as in case A. Here we essentially exploit the fact that $v(y_1, \ldots, y_t, d_t, N-t)$ depends only on the frequencies of the pages in the string $\{y_1, y_2, \ldots, y_t\}$ and on d_t. A detailed proof is given in [6].

2. Optimality of the LFU strategy

Now we are in a position to obtain our main result concerning the non-Bayesian case. Before formulating it, we give two definitions:

DEFINITION 1. A sequence $\{\delta_0, \ldots, \delta_{N-1}\}$ of probability distributions on the space Δ of all possible subsets d consisting of $n-m$ elements of the set $\{1, \ldots, n\}$ is called a *randomized sequential decision procedure* if and only if for every $0 \leqslant t \leqslant N-1$ the probability distribution δ_t depends only on the prior distribution ξ and the reference string $\{\eta_1, \ldots, \eta_t\}$.

Following a procedure $\{\delta_1, \ldots, \delta_{N-1}\}$ at every moment $t \in [0, N-1]$, we choose with probability $P_{\delta_t}(d)$ the set d of pages being absent from the central memory. Let us define the "action" $w(\delta)$ of a permutation $w \in W$ on a distribution δ by relation $P_{w(\delta)}(d) = P_\delta(x^{-1}(d))$. The following statement is true:

LEMMA 2. *For both forms of the loss function (cases* A *and* B) *among the randomized decision procedures the corresponding* LFU *strategies are optimal, i.e. the procedures* $\{\delta_0, \ldots, \delta_{N-1}\}$ *for which the measure* δ_t *is concentrated on the subsets* d *of the set* $\{1, 2, \ldots, n\}$ $(\{1, 2, \ldots, n\} \setminus \{\eta_t\})$ *in case* A (B) *consisting of the least frequently used pages in the string* $\{\eta_1, \ldots, \eta_{t-1}\}$.

Notice that in the case of strictly different frequencies δ_t is concentrated on a unique subset d.

Proof. The assertion follows from the fact that in relation (7) we have the strict inequality if $f_i < f_j$.

DEFINITION 2. A randomized sequential decision procedure $\{\delta_0, \delta_1, \ldots, \delta_{N-1}\}$ is called *symmetric* if and only if for every moment $0 \leqslant t \leqslant N$, permutation $w \in W$ and realization $\{y_1, \ldots, y_t\}$ the following holds:

$$w(\delta_t(\xi, y_1, \ldots, y_t)) = \delta(\xi, w(y_1), \ldots, w(y_t)).$$

We had to extend the sequential decision procedure to the randomized case in order to preserve the symmetry for the LFU strategies. A nonrandomized LFU strategy cannot be symmetric.

THEOREM 3. *If the prior distribution* ξ *is concentrated on a unique permutation* $w \in W$, *then for both forms of the loss-function among the symmetric randomized sequential decision procedures the* LFU *strategies are optimal.*

Remark. The theorem states that the LFU strategies are optimal also in the non-Bayesian case.

Proof. Let us denote for a realization $\{y_1, \ldots, y_N\}$ by $\nu(y_1, \ldots, y_N, \delta_0, \ldots, \delta_{N-1})$ the expected number of page faults when the strategy $\{\delta_0, \ldots, \delta_{N-1}\}$ is used. The expectation is taken on the basis of a distribution defined by $\{\delta_0, \ldots, \delta_{N-1}\}$. If the strategy $\{\delta_0, \ldots, \delta_{N-1}\}$ is symmetric, then for every $w \in W$

$$\nu(y_1, \ldots, y_N, \delta_0, \ldots, \delta_{N-1}) = \nu(w(y_1), \ldots, w(y_n), \delta_0, \ldots, \delta_{N-1}). \tag{8}$$

For an arbitrary prior distribution ξ the Bayesian risk can be computed from the quantities

$$E\Big(\sum_{t=1}^{N} X_t^{d_t, d_{t-1}}\Big) = \sum_{w \in W} \sum_{\{y_1, \ldots, y_N\} \in \Omega} \nu(w(y_1), \ldots, w(y_N), \delta_0, \ldots, \delta_{N-1}) \prod_{l=1}^{N} P_{y_l} P_\xi(w). \tag{9}$$

From (8), (9) we can deduce that for symmetric randomized sequential decision procedures the Bayesian risk does not depend on the prior distribution. This observation together with Lemma 2 proves Theorem 3.

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