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HANKEL TYPE INTEGRAL TRANSFORMS CONNECTED WITH THE HYPER-BESSEL DIFFERENTIAL OPERATORS

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Abstract. In this paper we give a solution of a problem posed by the second author in her book, namely, to find symmetrical integral transforms of Fourier type, generalizing the cos-Fourier (sin-Fourier) transform and the Hankel transform, and suitable for dealing with the hyper-Bessel differential operators of order $m > 1$

$$B := x^{-\beta} \prod_{j=1}^{m} \left(x \frac{d}{dx} + \beta\gamma_j \right), \quad \beta > 0, \ \gamma_j \in \mathbf{R}, \ j = 1, \dots, m.$$

We obtain such integral transforms corresponding to hyper-Bessel operators of even order $2m$ and belonging to the class of the Mellin convolution type transforms with the Meijer G-function as kernels. Inversion formulas and some operational relations for these transforms are found.

1. Introduction. It is well known that the cos-Fourier transform

$$(1) \qquad (\mathcal{F}_c f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \cos(xt) f(t) dt$$

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and the sin-Fourier transform

$$(\mathcal{F}_s f)(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \sin(xt) f(t) dt \tag{2}$$

belong to the class of the symmetrical Fourier transforms having inversion formulas of the same kind. They are used for operational calculi related to the operator of differentiation and for treating some classes of differential equations (with constant coefficients).

Another case of a symmetrical Fourier type integral transform is the Hankel transform

$$(\mathcal{J}_\nu f)(x) = \int_0^\infty \sqrt{xt} \mathrm{J}_\nu(xt) f(t) dt \tag{3}$$

with $\mathrm{J}_\nu(z)$ being the Bessel function. This transform is closely related to the Bessel differential operator $B_\nu = x^{-2}\left(x\frac{d}{dx}+\nu\right)\left(x\frac{d}{dx}-\nu\right)$ and can be used to treat operationally differential equations involving B_ν.

In the book [4] a problem was posed to find symmetrical integral transforms of Fourier type, generalizing the cos-Fourier transform, the sin-Fourier transform and the Hankel transform, and suitable for dealing with the *hyper-Bessel differential operators of order* $m > 1$

$$B := x^{-\beta} \prod_{j=1}^{m} \left(x\frac{d}{dx} + \beta\gamma_j \right), \quad \beta > 0,\ \gamma_j \in \mathbf{R},\ j = 1, \dots, m, \tag{4}$$

called also *general Bessel type differential operators of arbitrary order*. These operators were introduced and operational calculi for them were developed by Dimovski [2] (see also details in [4, Ch. 3]). They appear quite often in equations of mathematical physics, and have as simplest examples the m-fold differentiation $D^m = \frac{d^m}{dx^m}$, the Bessel operator B_ν, the Bessel-Clifford operators and various 2nd and higher order singular differential operators with variable coefficients. As to the problem to find a Laplace type integral transform suitable for operational calculus for each such operator (4), it was solved by Dimovski [2] who proved that an integral transform proposed by Obrechkoff [7] serves well for these purposes. The theory of the Obrechkoff transform was further continued and related to the Meijer G-function (as a kernel-function) and to the generalized fractional calculus, by Kiryakova [4]. But the problem for Hankel type integral transforms corresponding to (4), remained open.

In this paper we give an answer to this problem in the case of hyper-Bessel differential operators of even orders $2m$. In some sense, i.e. in the considered class of integral operators, our solution is a complete one, because the proposed construction is not possible for hyper-Bessel operators of odd orders. This fact can be illustrated by noting that the generalized Hankel transforms given by us as Mellin convolution type transforms with the Meijer G-function as kernels, can be represented as compositions of m integral transforms of Hankel type with different indices. Each of these transforms is related with a differential operator of Bessel type of second order, i.e., their composition should be related with a hyper-Bessel differential operator of an even order.

2. Preliminaries. In this paper we deal with integral transforms in a special space of functions $\mathcal{M}_\gamma^{-1}(L)$. The theory of this space can be found in [9], [11]–[13]. In particular, it turned out that this space is very convenient for working with the Mellin convolution type transforms.

Definition 1. Let $\gamma \geq 0$. We denote by $\mathcal{M}_\gamma^{-1}(L)$ the space of functions representable by the inverse Mellin integral transforms

$$f(x) = \frac{1}{2\pi i}\int_\sigma f^*(s)x^{-s}ds,\ x > 0 \tag{5}$$

with functions f^* satisfying the condition $f^*(s)|s|^\gamma \in L(\sigma)$. Here $\sigma = \{s \in \mathbf{C} : \Re(s) = 1/2\}$ and $L(\sigma)$ denotes the space of functions Lebesgue-integrable on σ.

The space $\mathcal{M}_\gamma^{-1}(L)$ equipped with the norm

$$\|f\|_{\mathcal{M}_\gamma^{-1}(L)} := \frac{1}{2\pi}\int_\sigma |s^\gamma f^*(s)ds|$$

is a Banach space.

In the case $\gamma = 0$ the space $\mathcal{M}_\gamma^{-1}(L)$ will be denoted by $\mathcal{M}^{-1}(L)$.

In the further discussions, we use a special subspace of $\mathcal{M}_\gamma^{-1}(L)$ introduced and investigated in [5] (see also [1] for similar space of functions).

Definition 2. Let $\lambda \geq 0$, $\gamma \geq 0$ and $S_\lambda := \{s \in \mathbf{C} : |\Re(s) - \frac{1}{2}| \leq \lambda\}$. Denote by ${}_\lambda\mathcal{M}_\gamma^{-1}(L)$ the space of functions which can be represented in the form (5) with functions f^* analytic in the strip S_λ and satisfying the additional conditions:

1) The norms $\|f^*(s)\left(\frac{1}{2} + i\Im(s)\right)^\gamma\|_{L(\sigma(\tau))}$ are uniformly bounded by a constant $C > 0$ for all straight lines $\sigma(\tau) := \{s \in S_\lambda : \Re(s) = \tau\}$ from the strip S_λ;

2) $f^*(s)\left(\frac{1}{2} + i\Im(s)\right)^\gamma \to 0$, if $\Im(s) \to \infty$ uniformly with respect to $\Re(s)$.

Obviously, the space ${}_\lambda\mathcal{M}_\gamma^{-1}(L)$ is a subspace of $\mathcal{M}_\gamma^{-1}(L)$ for all $\gamma \geq 0$.

We use also the following definition introduced in [11] and considered in details in [13] (see also [10]).

Definition 3. Denote by $\mathcal{K}$ the set of kernels $k : (0,\infty) \to \mathbf{R}$ for which the following conditions are fulfilled:

1) $k \in L(\epsilon, E)$ for any $\epsilon,\ E$, such that $0 < \epsilon < E < \infty$;

2) The integral

$$k^*(s) = \int_0^\infty k(u)u^{s-1}du,\ \Re(s) = \frac{1}{2} \tag{6}$$

is convergent and for almost all $\epsilon,\ E > 0$ and $t \in \mathbf{R}$ there exists a constant $C > 0$ such that

$$\left|\int_\epsilon^E k(u)u^{it-1/2}du\right| < C.$$

If for the kernel $k \in \mathcal{K}$ there exists the conjugate kernel $\hat{k} \in \mathcal{K}$ such that the equality

$$k^*(s)\hat{k}^*(1-s) = 1$$

holds almost everywhere on the line $\Re(s) = 1/2$, then we say that $k, \hat{k} \in \mathcal{K}^* \subset \mathcal{K}$.

It is easy to check that if $x^{-1/2}k(x) \in L(0,\infty)$, then $k \in \mathcal{K}$ and $k \notin \mathcal{K}^*$.

To derive and prove our results, we use the following two theorems from [11] (see also [13]).

THEOREM A. *Let $f \in \mathcal{M}^{-1}(L)$, $k \in \mathcal{K}$, k^* be given by* (6) *and f^* be determined by* (5). *Then the following Parseval formula takes place*

$$\int_0^\infty k(xy)f(y)dy = \frac{1}{2\pi i}\int_\sigma k^*(s)f^*(1-s)x^{-s}ds.$$

THEOREM B. *Let $k \in \mathcal{K}^*$ and let $\hat{k} \in \mathcal{K}^*$ be its conjugate kernel. Then the integral transform*

$$g(x) = (Kf)(x) := \int_0^\infty k(xt)f(t)dt,\ x > 0$$

is an automorphism in the space $\mathcal{M}^{-1}(L)$ and its inverse transform is given by

$$f(x) = (\hat{K}g)(x) := \int_0^\infty \hat{k}(xt)g(t)dt,\ x > 0.$$

3. Generalized Hankel transforms. First we introduce integral transforms of Hankel type with, in general, unsymmetrical forms of inverse transforms and find the connection of these transforms with the hyper-Bessel differential operators.

DEFINITION 4. The *generalized Hankel transforms*, corresponding to hyper-Bessel operators of form (4) of even order $2m$, are defined as Mellin convolution type integral transforms of the form

$$g(x) = (\mathcal{G}_\beta f)(x) := \int_0^\infty G(xt)f(t)dt,\ x > 0, \tag{7}$$

where the kernel-function

$$G(x) := \frac{\beta}{\sqrt{\eta}} G^{m,0}_{0,2m}\left(\left(\frac{x}{\eta}\right)^\beta \middle| \begin{matrix} - \\ (\gamma_j + 1 - \frac{1}{\beta})_1^{2m} \end{matrix}\right),\ \beta, \eta > 0,\ \gamma_j \in \mathbf{R},\ j = 1, \dots, 2m$$

is a particular case of the Meijer G-function.

The Meijer G-function is defined by the following Mellin-Barnes type contour integral:

$$G^{m,n}_{p,q}\left(z \middle| \begin{matrix} (\alpha_j)_1^p \\ (\beta_j)_1^q \end{matrix}\right) = \frac{1}{2\pi i}\int_L \Psi(s)z^{-s}ds, \tag{8}$$

$$\text{with} \quad \Psi(s) = \frac{\prod_{j=1}^m \Gamma(\beta_j + s)\prod_{j=1}^n \Gamma(1 - \alpha_j - s)}{\prod_{j=n+1}^p \Gamma(\alpha_j + s)\prod_{j=m+1}^q \Gamma(1 - \beta_j - s)}, \tag{9}$$

where $z \neq 0$, $0 \leq m \leq q$, $0 \leq n \leq p$, $\alpha_j \in \mathbf{C}$, $1 \leq j \leq p$, $\beta_j \in \mathbf{C}$, $1 \leq j \leq q$, an empty product, if it occurs, is taken to be one, and an infinite contour L separates all the left poles $s = -\beta_j - k$, $j = 1, 2, \dots, m$, $k = 0, 1, 2, \dots$ of the numerator from the right ones $s = 1 - \alpha_j + k$, $j = 1, 2, \dots, n$, $k = 0, 1, 2, \dots$ and under suitable conditions it may be one of the three types: $L_{-\infty}$, $L_{+\infty}$ or $L_{i\infty}$ (in particular, even a rectilinear line $L = (\gamma - i\infty, \gamma + i\infty)$). The description of the contours and detailed list of the properties and particular cases of the G-function can be found in [8], see also [4], [13].

We call integral transforms (7) generalized Hankel transforms because the classical Hankel transform (3) is a particular case of (7) (see Example 3 in the next section) and because these transforms play the same role for the hyper-Bessel differential operators as the Hankel transform for the Bessel operator. The inverse transforms having similar but not exactly the same forms as the generalized Hankel transforms (7) are given by the following theorem.

THEOREM 1. *Under the conditions*

$$m+\sum_{j=1}^{2m}\gamma_j=\frac{m}{\beta};\ \gamma_j>\frac{1}{2\beta}-1,\ j=1,\dots,m;\ \gamma_j<\frac{1}{2\beta},\ j=m+1,\dots,2m, \tag{10}$$

the generalized Hankel transform (7) *is an automorphism in the space* $\mathcal{M}^{-1}(L)$ *and its inverse transform is given by*

$$f(x)=(\mathcal{H}_\beta g)(x):=\int_0^\infty H(xt)g(t)dt,\ x>0, \tag{11}$$

$$\textit{with}\qquad H(x):=\frac{\beta}{\sqrt{\eta}}G^{m,0}_{0,2m}\left(\left(\frac{x}{\eta}\right)^\beta\middle|\begin{matrix}-\\(-\gamma_j)_{m+1}^{2m},(-\gamma_j)_1^m\end{matrix}\right).$$

PROOF. The conditions under which the Meijer G-function is in the class $\mathcal{K}^*$ of kernels were found in [11], [13] (see also [3]). From these results it follows that the function

$$G^{m,0}_{0,2m}\left(x\middle|\begin{matrix}-\\(\gamma_j+\frac{1}{2}-\frac{1}{2\beta})_1^{2m}\end{matrix}\right)$$

belongs to the class $\mathcal{K}^*$ under the conditions (10). Using the relation ($k\in\mathcal{K}$)

$$\int_\epsilon^E\beta\eta^{-\frac{\beta}{2}}x^{\frac{\beta}{2}-\frac{1}{2}}k((x/\eta)^\beta)x^{it-1/2}\,dx\ =\ \int_{(\frac{\epsilon}{\eta})^\beta}^{(\frac{E}{\eta})^\beta}k(u)u^{\frac{it}{\beta}-\frac{1}{2}}\eta^{it}\,du$$

and direct evaluations, we deduce that the function

$$\begin{aligned}G(x)&=\beta\eta^{-\frac{\beta}{2}}x^{\frac{\beta}{2}-\frac{1}{2}}G^{m,0}_{0,2m}\left(\left(\frac{x}{\eta}\right)^\beta\middle|\begin{matrix}-\\(\gamma_j+\frac{1}{2}-\frac{1}{2\beta})_1^{2m}\end{matrix}\right)\\&=\frac{\beta}{\sqrt{\eta}}G^{m,0}_{0,2m}\left(\left(\frac{x}{\eta}\right)^\beta\middle|\begin{matrix}-\\(\gamma_j+1-\frac{1}{\beta})_1^{2m}\end{matrix}\right)\end{aligned}$$

belongs to the class $\mathcal{K}^*$, too. Its Mellin transform is given by ([5], [13]):

$$G^*(s)=\eta^{s-\frac{1}{2}}\frac{\prod_{j=1}^m\Gamma(\gamma_j+1-\frac{1}{\beta}+\frac{s}{\beta})}{\prod_{j=m+1}^{2m}\Gamma(\frac{1}{\beta}-\gamma_j-\frac{s}{\beta})}. \tag{12}$$

Then we have

$$H^*(s)=\frac{1}{G^*(1-s)}=\eta^{s-\frac{1}{2}}\frac{\prod_{j=m+1}^{2m}\Gamma(-\gamma_j+\frac{s}{\beta})}{\prod_{j=1}^m\Gamma(1+\gamma_j-\frac{s}{\beta})}.$$

Using the last relation we can write the conjugate kernel in the form

$$H(x)=\frac{\beta}{\sqrt{\eta}}G^{m,0}_{0,2m}\left(\left(\frac{x}{\eta}\right)^\beta\middle|\begin{matrix}-\\(-\gamma_j)_{m+1}^{2m},(-\gamma_j)_1^m\end{matrix}\right).$$

Now, the statement of Theorem 1 follows from Theorem B. ∎

Now we consider the *connection between the generalized Hankel transforms and the hyper-Bessel differential operators*, given by the following operational relation.

THEOREM 2. *Let $f \in {}_{\beta}\mathcal{M}_{\gamma}^{-1}(L)$. Then under the conditions*

$$\gamma \geq 2m;\ \frac{m}{\beta} \geq m + \sum_{j=1}^{2m} \gamma_j;\ \gamma_j > \frac{1}{2\beta} - 1,\ j = 1, \dots, m,$$

we have for the generalized Hankel transform (7) *the operational relation*

$$(\mathcal{G}_\beta B_\beta f)(x) = (-1)^m x^\beta (\mathcal{G}_\beta f)(x). \tag{13}$$

Here the differential operator of even order $2m$,

$$B_\beta := \left(\frac{x}{\eta}\right)^{-\beta} \frac{1}{\beta^{2m}} \prod_{j=1}^{2m} \left(x \frac{d}{dx} + \beta\gamma_j\right) \tag{14}$$

is a particular case of the hyper-Bessel differential operators (4) *up to a constant factor.*

PROOF. Using Theorem A, the fact that the kernel of the transform (7) belongs to the class $\mathcal{K}^*$ of kernels and the table of the Mellin integral transforms (see [6], [8]), we can rewrite the generalized Hankel transform (7) in the form

$$g(x) = (\mathcal{G}_\beta f)(x) = \frac{1}{2\pi i} \int_\sigma G^*(s) f^*(1-s) x^{-s} ds \tag{15}$$

with $G^*(s)$ given by (12).

The hyper-Bessel differential operator (14) can be represented in the space ${}_{\beta}\mathcal{M}_{\gamma}^{-1}(L)$, $\gamma \geq 2m$ as (see [5], [13])

$$\begin{aligned}(B_\beta f)(x) &= \left(\frac{x}{\eta}\right)^{-\beta} \frac{(-1)^m}{2\pi i} \int_\sigma B^*(s) f^*(s) x^{-s} ds \\ &= \eta^\beta \frac{(-1)^m}{2\pi i} \int_\sigma B^*(s-\beta) f^*(s-\beta) x^{-s} ds,\end{aligned} \tag{16}$$

where

$$B^*(s) = \prod_{j=1}^{m} \left(\gamma_j - \frac{s}{\beta}\right) \prod_{j=m+1}^{2m} \left(-\gamma_j + \frac{s}{\beta}\right).$$

Both (15) and (16) are particular cases of the generalized H-transforms given by

$$(H_{p,q}^{m,n} f)(x) := \frac{1}{2\pi i} \int_\sigma \Phi(s) f(1-s) x^{-s} ds,\ x > 0, \tag{17}$$

$$\text{with} \qquad \Phi(s) = \frac{\prod_{j=1}^{m} \Gamma(\beta_j + b_j s) \prod_{j=1}^{n} \Gamma(1 - \alpha_j - a_j s)}{\prod_{j=n+1}^{p} \Gamma(\alpha_j + a_j s) \prod_{j=m+1}^{q} \Gamma(1 - \beta_j - b_j s)}, \tag{18}$$

where $0 \leq m \leq q$, $0 \leq n \leq p$, $\alpha_j \in \mathbf{C}$, $a_j > 0$, $1 \leq j \leq p$, $\beta_j \in \mathbf{C}$, $b_j > 0$, $1 \leq j \leq q$. The theory of transformations (17) in the space ${}_{\lambda}\mathcal{M}_{\gamma}^{-1}(L)$ was developed in [5], [13]; we use some results from these papers in our proof. The composition property of the generalized H-transform (see [5], [13]) gives us:

$$(\mathcal{G}_\beta B_\beta f)(x) = \eta^\beta \frac{(-1)^m}{2\pi i} \int_\sigma G^*(s) B^*(1-s-\beta) f^*(1-s-\beta) x^{-s} ds. \tag{19}$$

By direct evaluations we arrive at the relation

$$(-1)^m \eta^\beta G^*(s) B^*(1-s-\beta) = (-1)^m G^*(s+\beta).$$

This relation and the shift property of the generalized H-transforms in the space ${}_\lambda\mathcal{M}_\gamma^{-1}(L)$ (see [5], [13]) gives us

$$(\mathcal{G}_\beta B_\beta f)(x) = \frac{(-1)^m}{2\pi i}\int_\sigma G^*(s+\beta)f^*(1-s-\beta)x^{-s}ds$$
$$= (-1)^m x^\beta \frac{1}{2\pi i}\int_\sigma G^*(s)f^*(1-s)x^{-s}ds = (-1)^m x^\beta(\mathcal{G}_\beta f)(x).$$

Thus, Theorem 2 is proved. ∎

4. Generalized symmetrical Hankel transforms. In this section we consider the most interesting case of a pair of integral transforms (7), (11), namely, when these transforms have the same form. This situation takes place if and only if $\gamma_j + \gamma_{j+m} = \frac{1}{\beta} - 1,\ j = 1,\dots,m$. Then, Theorems 1, 2 have the following simpler forms.

Theorem 3. *The generalized symmetrical Hankel transform*

$$g(x) = (\mathcal{G}_\beta f)(x) = \int_0^\infty G(xt)f(t)dt,\ x>0, \tag{20}$$

with $$G(x) = \frac{\beta}{\sqrt{\eta}}G^{m,0}_{0,2m}\left(\left(\frac{x}{\eta}\right)^\beta \middle| \begin{matrix} - \\ (\gamma_j + 1 - \frac{1}{\beta})_1^m, (-\gamma_j)_1^m\end{matrix}\right),\ \beta>0,$$

is an automorphism in the space $\mathcal{M}^{-1}(L)$ under the conditions

$$\gamma_j > \frac{1}{2\beta} - 1,\ j=1,\dots,m,$$

with the inverse transform of the same form.

Theorem 4. *Let $f \in {}_\beta\mathcal{M}_\gamma^{-1}(L)$. Then, under the conditions*

$$\gamma \ge 2m;\ \gamma_j > \frac{1}{2\beta} - 1,\ j=1,\dots,m,$$

we have for the integral transform (20) *the following operational relation*:

$$(\mathcal{G}_\beta B_\beta f)(x) = (-1)^m x^\beta(\mathcal{G}_\beta f)(x), \tag{21}$$

where

$$B_\beta := \left(\frac{x}{\eta}\right)^{-\beta}\frac{1}{\beta^{2m}}\prod_{j=1}^m\left(x\frac{d}{dx} + \beta\gamma_j\right)\prod_{j=1}^m\left(x\frac{d}{dx} + \beta\left(\frac{1}{\beta} - 1 - \gamma_j\right)\right) \tag{22}$$

is a particular case of the hyper-Bessel differential operator (4).

Now let us consider *some examples of the generalized symmetrical Hankel transforms* (20) and the corresponding operational relations.

Example 1. The case $m=1,\ \eta=2,\ \beta=2,\ \gamma_1=0$ corresponds to the well known sin-Fourier transform (2) due to the relation ([8])

$$\sqrt{\frac{2}{\pi}}\sin x \to 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{2}+\frac{s}{2})}{\Gamma(1-\frac{s}{2})}\,.$$

Here we denote by $\to$ the correspondence between a function and its Mellin transform (6). In this case, operational relation (21) has the form

$$(\mathcal{F}_s B_s f)(x) = -x^2 (\mathcal{F}_s f)(x),$$

with a "hyper-Bessel operator"

$$B_s = x^{-2}\left(x\frac{d}{dx}\right)\left(x\frac{d}{dx} - 1\right) = \frac{d^2}{dx^2}\,.$$

EXAMPLE 2. Similarly, the case $m = 1,\ \eta = 2,\ \beta = 2,\ \gamma_1 = -\frac{1}{2}$ corresponds to the cos-Fourier transform (1), due to the formula

$$\sqrt{\frac{2}{\pi}}\cos x \to 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{s}{2})}{\Gamma(\frac{1}{2}-\frac{s}{2})}$$

and the operational relation (21) has now the form

$$(\mathcal{F}_c B_c f)(x) = -x^2 (\mathcal{F}_c f)(x)$$

with

$$B_c = x^{-2}\left(x\frac{d}{dx} - 1\right)\left(x\frac{d}{dx}\right) = \frac{d^2}{dx^2}.$$

EXAMPLE 3. As it is well known, the sin-Fourier and the cos-Fourier transforms are particular cases of the Hankel transform (3) when $\nu = \pm\frac{1}{2}$. We can obtain the Hankel transform putting the values $m = 1,\ \eta = 2,\ \beta = 2,\ \gamma_1 = \frac{\nu}{2} - \frac{1}{4}$ into formula (20) and using the formula

$$\sqrt{x}\mathrm{J}_\nu(x) \to 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{4}+\frac{\nu}{2}+\frac{s}{2})}{\Gamma(\frac{3}{4}+\frac{\nu}{2}-\frac{s}{2})}\,. \tag{23}$$

It follows from Theorem 4 that in this case the following operational relation holds:

$$(\mathcal{J}_\nu B f)(x) = -x^2(\mathcal{J}_\nu f)(x),\ \nu > -1,$$

where $\mathcal{J}_\nu$ is the Hankel transform (3) and

$$B = x^{-2}\left(x\frac{d}{dx} + \nu - \frac{1}{2}\right)\left(x\frac{d}{dx} - \nu - \frac{1}{2}\right) = \left(\frac{d^2}{dx^2} + \frac{\frac{1}{4}-\nu^2}{x^2}\right)\,.$$

EXAMPLE 4. Let us consider the integral transform of the form (see [10]):

$$(\mathcal{G}f)(x) = \int_0^\infty \sqrt{xt}\left[\frac{2}{\pi}\mathrm{K}_0(xt) - \mathrm{Y}_0(xt)\right] f(t)dt,$$

where $\mathrm{K}_0(x)$ and $\mathrm{Y}_0(x)$ are McDonald's and Neumann's functions of zero index, respectively. This integral transform corresponds to the case $m = 2,\ \beta = 4,\ \eta = 4,\ \gamma_1 = \gamma_2 = -\frac{5}{8}$ in Theorem 3, due to the formula

$$\sqrt{x}\left[\frac{2}{\pi}\mathrm{K}_0(x) - \mathrm{Y}_0(x)\right] \to 4^{s-\frac{1}{2}}\frac{\Gamma(\frac{1}{8}+\frac{s}{4})\Gamma(\frac{1}{8}+\frac{s}{4})}{\Gamma(\frac{3}{8}-\frac{s}{4})\Gamma(\frac{3}{8}-\frac{s}{4})}\,.$$

The inverse transform has the same form and we have the operational relation

$$(\mathcal{G}Bf)(x) = x^4(\mathcal{G}f)(x),$$

where

$$B = x^{-4}\left(x\frac{d}{dx}-\frac{5}{2}\right)^2\left(x\frac{d}{dx}-\frac{1}{2}\right)^2.$$

Finally, let us consider our integral transform corresponding to the Bessel differential operator

$$B_\nu := x^{-2}\left(x\frac{d}{dx}+\nu\right)\left(x\frac{d}{dx}-\nu\right),\ \nu > -1. \tag{24}$$

It is given as a composition of the Erdélyi-Kober fractional derivative and the classical Hankel transform (3).

Namely, we use the following definition (see [4, Ch. 1, p. 60]; [13]) of the *Erdélyi-Kober fractional derivatives*:

$$(D_\delta^{\tau,\alpha} f)(x) := \left(\prod_{j=1}^{n}\left(\tau+j+\frac{1}{\delta}x\frac{d}{dx}\right)\right)(I_\delta^{\tau+\alpha,n-\alpha} f)(x),\ x>0,\ \delta>0,\ \alpha>0, \tag{25}$$

$$n = \begin{cases} [\alpha]+1, & \alpha \notin \mathbf{N}, \\ \alpha, & \alpha \in \mathbf{N}, \end{cases}$$

where

$$(I_\delta^{\tau,\alpha} f)(x) := \begin{cases} \frac{1}{\Gamma(\alpha)}\int_0^1 (1-u)^{\alpha-1}u^\tau f(xu^{1/\delta})\,du, & \alpha > 0, \\ f(x), & \alpha = 0 \end{cases} \tag{26}$$

are the Erdélyi-Kober fractional integrals of order $\alpha > 0$ and weight τ (see [9], also [4, Ch. 1]).

Theorem 5. *Let $D_2^{\frac{\nu}{2}-\frac{1}{2},\frac{1}{2}}$ be the Erdélyi-Kober fractional derivative* (25) *of order $\frac{1}{2}$ and weight $\frac{\nu}{2}-\frac{1}{2}$ and $\mathcal{J}_{\nu+\frac{1}{2}}$ be the Hankel transform* (3) *of index $\nu+\frac{1}{2}$. Then, the integral transform*

$$g(x) = (\mathcal{W}f)(x) := (D_2^{\frac{\nu}{2}-\frac{1}{2},\frac{1}{2}}\mathcal{J}_{\nu+\frac{1}{2}} f)(x),\ x>0,\ \nu>-1 \tag{27}$$

maps isomorphically the space $\mathcal{M}_\gamma^{-1}(L)$, $\gamma > \frac{1}{2}$ onto $\mathcal{M}_{\gamma-\frac{1}{2}}^{-1}(L)$ and its inverse transform is given by

$$f(x) = (\mathcal{V}g)(x) := \sqrt{2}\int_0^\infty J_\nu(xt)g(t)dt,\ x>0. \tag{28}$$

Moreover, in the space ${}_2\mathcal{M}_\gamma^{-1}(L)$, $\gamma > \frac{5}{2}$ the operational relation

$$(\mathcal{W}B_\nu f)(x) = x^2(\mathcal{W}f)(x) \tag{29}$$

holds true for the Bessel differential operator B_ν, (24) *and the integral transform $\mathcal{W}$,* (27).

Proof. As it was shown in the proof of Theorem 2, the integral transform connected with the Bessel differential operator ($m=1,\ \beta=2,\ \eta=2,\ \gamma_1=\frac{\nu}{2},\ \gamma_2=-\frac{\nu}{2}$ in (14)) by an operational relation of the type (13) has as a kernel the function k with the property

$$k^*(s) = 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{\nu}{2}+\frac{1}{2}+\frac{s}{2})}{\Gamma(\frac{\nu}{2}+\frac{1}{2}-\frac{s}{2})}. \tag{30}$$

Since this kernel does not satisfy the conditions of Theorem 1, we use a slightly different approach to prove the relation (29). Let us represent k^* in the form

$$k^*(s) = k_1^*(s)k_2^*(s),$$

where

$$k_1^*(s) = \frac{\Gamma(\frac{\nu}{2}+1-\frac{s}{2})}{\Gamma(\frac{\nu}{2}+\frac{1}{2}-\frac{s}{2})},\ k_2^*(s) = 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{\nu}{2}+\frac{1}{2}+\frac{s}{2})}{\Gamma(\frac{\nu}{2}+1-\frac{s}{2})}.$$

The kernel k_2 does satisfy the conditions of Theorem 1 and corresponds to the Hankel transform of index $\nu+\frac{1}{2}$ (see formula (23)). As to the kernel k_1, in the space $\mathcal{M}_\gamma^{-1}(L)$, $\gamma > \frac{1}{2}$ it corresponds to the Erdélyi-Kober fractional derivative (25) of order $\frac{1}{2}$ and weight $\frac{\nu}{2}-\frac{1}{2}$. Finally, using the compositional properties of the Mellin convolution type transforms ([9], [11], [13]), we obtain that the integral transform with the kernel (30) can be rewritten in the form (27). This transform maps isomorphically the space $\mathcal{M}_\gamma^{-1}(L)$, $\gamma > \frac{1}{2}$ onto $\mathcal{M}_{\gamma-\frac{1}{2}}^{-1}(L)$ with the inverse transform having the kernel of the form

$$h^*(s) = \frac{1}{k^*(1-s)} = 2^{s-\frac{1}{2}}\frac{\Gamma(\frac{\nu}{2}+\frac{s}{2})}{\Gamma(\frac{\nu}{2}+1-\frac{s}{2})}.$$

Using again Theorem 1, we can represent this transform in the form (28).

The operational relation (29) is proved exactly as the corresponding relation in Theorem 2. ∎

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