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HILBERT TRANSFORM AND SINGULAR INTEGRALS ON THE SPACES OF TEMPERED ULTRADISTRIBUTIONS

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Abstract. The Hilbert transform on the spaces $\mathcal{S}'^*(\mathbf{R}^d)$ of tempered ultradistributions is defined, uniquely in the sense of hyperfunctions, as the composition of the classical Hilbert transform with the operators of multiplying and dividing a function by a certain elliptic ultrapolynomial. We show that the Hilbert transform of tempered ultradistributions defined in this way preserves important properties of the classical Hilbert transform. We also give definitions and prove properties of singular integral operators with odd and even kernels on the spaces $\mathcal{S}'^*(\mathbf{R}^d)$, whose special cases are the Hilbert transform and Riesz operators.

1. Introduction. The Hilbert transform on distribution and ultradistribution spaces has been studied by many mathematicians, see e.g. Tillmann [17], Beltrami and Wohlers [1], Vladimirov [18], Singh and Pandey [15], Ishikawa [2], Ziemian [20] and Pilipović [11]. In all these papers the Hilbert transform is defined by one of the two methods: by the method of adjoints or by considering a generalized function on the kernel which belongs to the corresponding test function space.

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Ziemian [20] defined the right and left Hilbert transform of a tempered distribution $T \in \mathcal{S}'(\mathbf{R})$ as an element of $\mathcal{S}'(\mathbf{R})$, using the kernel

$$G(z) = \int_0^\infty \chi(x) x^{-z-1}\, dx, \quad z \in \mathbf{C} \setminus \{0\},$$

where $\chi \in C_0^\infty(\mathbf{R})$ and $\chi = 1$ in a neighbourhood of zero.

Koizumi in [4] and [5] considered the generalized Hilbert transform H defined by

$$Hf(x) := \lim_{\epsilon \to 0+} \frac{x+i}{\pi} \int_{|t|>\epsilon} \frac{f(x-t)}{t(x-t+i)}\, dt$$

for $f \in W^2(\mathbf{R})$, where $W^2(\mathbf{R})$ denotes the space of all functions f such that $\tilde{f} \in L^2(\mathbf{R})$ with the corresponding function $\tilde{f}$ given by

$$\tilde{f}(x) := \frac{f(x)}{1+|x|}, \qquad x \in \mathbf{R}.$$

A generalization of the same type was given by Ishikawa [2] who extended the definition of the Hilbert transform to the space of tempered distributions.

We follow this way and generalize Ishikawa's Hilbert transform to the spaces of tempered ultradistributions first in the one-dimensional and then in the d-dimensional case. Our generalization is defined as the composition of the classical Hilbert transform with the operators $\varphi \mapsto P\varphi$ and $\varphi \mapsto (1/P)\varphi$, where P is an elliptic ultrapolynomial.

The first section is devoted to the generalized Hilbert transform defined on the spaces $\mathcal{S}'^{(M_p)}(\mathbf{R})$ and $\mathcal{S}'^{\{M_p\}}(\mathbf{R})$. Structural properties of the basic spaces imply that the Hilbert transform of a tempered ultradistribution is defined uniquely in the sense of hyperfunctions.

In the second section we define the Hilbert transform on the spaces $\mathcal{S}'^*(\mathbf{R}^d)$. The simple structure of the kernel enables us to define this transform as the iterations of the one-dimensional Hilbert transform.

In the third section we consider the general singular integrals with odd and even kernels of tempered ultradistributions. For the L^2-theory of singular integrals we refer to [16]. We use the classical results of Pandey [9] and follow his ideas in our definition of singular integral operators on the spaces of tempered ultradistributions with values in certain spaces of ultradistributions which contain $\mathcal{S}'^{(M_p)}(\mathbf{R}^d)$ and $\mathcal{S}'^{\{M_p\}}(\mathbf{R}^d)$, respectively.

This paper is a continuation of the paper [3] and we will not recall the notation, referring the reader to [3] also for the basic definitions and the main results. As in [3], we will assume throughout the paper that a given sequence (M_p) of positive numbers, defining the spaces of ultradifferentiable functions and their duals, satisfies only conditions $(M.1)$ and $(M.3')$ from the list of the standard conditions (see [6], [7] and [3]); if we will need additional conditions, it will be marked every time in the text.

2. Hilbert transform, one-dimensional case. In this section, all functions and ultradistributions are considered on the real line.

We represent the space $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$) as the projective limit of the spaces $\mathcal{D}_a^{(M_p)}$, where $a > 0$ (resp. $\mathcal{D}_{(a_p)}^{\{M_p\}}$, where (a_p) is a sequence of positive numbers increasing to ∞) in order to define the Hilbert transform on $\mathcal{S}'^*$.

We follow here Ishikawa's way of notation (see [2]) using the symbols $\mathcal{D}_a^{(M_p)}$ with $a > 0$ (resp. $\mathcal{D}_{(a_p)}^{\{M_p\}}$ with (a_p) being a sequence of positive numbers increasing to ∞) instead of the symbol $\mathcal{S}$ with respective indices to avoid too many spaces denoted by the symbol S in the paper. Consequently, it should be remembered that elements of the space $\mathcal{D}_a^{(M_p)}$, (resp. $\mathcal{D}_{(a_p)}^{\{M_p\}}$) do not have compact supports, in general .

In contrast to the case of tempered distributions, the space $\mathcal{S}^{(M_p)}$ (resp. $\mathcal{S}^{\{M_p\}}$) is not dense in the space $\mathcal{D}_a^{(M_p)}$ (resp. $\mathcal{D}_{(a_p)}^{\{M_p\}}$), in general. We overcome this difficulty due to the parts 4 and 5 of Theorem 1.4.

Let us give now the definitions and a structural characterization of basic spaces which will be used in further investigations of the Hilbert transform.

By $\mathcal{R}$ we denote the family of all sequences of positive numbers increasing to ∞. For a given $b > 0$ (resp. $(b_p) \in \mathcal{R}$), we denote by P_b (resp. by $P_{(b_p)}$) an arbitrary entire function such that there exist positive constants $\mathcal{C}$ and L such that the following inequalities are satisfied

$$|P_b(\zeta)| \le \mathcal{C} \exp[M(L|\zeta|)] \quad \big(\text{resp. } |P_{(b_p)}(\zeta)| \le \mathcal{C} \exp[N_{(b_p)}(L|\zeta|)]\big) \tag{1}$$

for all $\zeta \in \mathbf{C}$ and

$$\exp[M(b|\zeta|)] \le P_b(\zeta) \quad \big(\text{resp. } \exp[N_{(b_p)}(|\zeta|)] \le P_{(b_p)}(\zeta)\big) \tag{2}$$

for all $\zeta \in \mathbf{C}$ such that $|\operatorname{Re}\zeta| \ge |\operatorname{Im}\zeta|$.

We recall that M means here the so-called associated function for the sequence (M_p) and $N_{(b_p)}$ is the associated function for the corresponding sequence (N_p), given by $N_p := M_p \prod_{j=1}^{p} b_j$ for $p \in \mathbf{N}$, i.e.

$$M(t) := \sup_{p \in \mathbf{N}_0} \log_+(t^p/M_p), \quad N_{(b_p)}(t) := \sup_{p \in \mathbf{N}_0} \log_+(t^p/N_p)$$

for $t > 0$, where $\log_+ t := \max(\log t, 0)$. Since we assume that the sequence (M_p) satisfies conditions $(M.1)$ and $(M.3')$ (clearly, (N_p) also fulfills these conditions), it is easy to see that the associated function M (and so $N_{(b_p)}$) is a nondecreasing function on $[0, \infty)$, equal to 0 in a right neighbourhood of 0.

In case conditions $(M.1)$, $(M.2)$ and $(M.3)$ are satisfied, an example of such an entire function is given by

$$P_b(\zeta) := \prod_{p=1}^{\infty} \Big(1 + \frac{\zeta^2}{b^2 m_p^2}\Big) \quad \Big(\text{resp. } P_{(b_p)}(\zeta) := \prod_{p=1}^{\infty} \Big(1 + \frac{\zeta^2}{{b_p}^2 m_p^2}\Big)\Big)$$

for $\zeta \in \mathbf{C}$, where $m_p := M_p/M_{p-1}$ for $p \in \mathbf{N}$. It follows from (1) (see Proposition 4.5 in [6]) that $P_b(D)$ (resp. $P_{(b_p)}(D)$) is an ultradifferential operator of the class (M_p) (resp. $\{M_p\}$). It is easy to see that

$$|P_b(\zeta)| \le P_b(|\zeta|) \quad (\text{resp. } |P_{(b_p)}(\zeta)| \le P_{(b_p)}(|\zeta|)) \tag{3}$$

and the functions P_b and $P_{(b_p)}$ are non-decreasing on the positive halfline of the real axis.

In the sequel, we shall need some estimates of the derivatives of both P_b (resp. $P_{(b_p)}$) and $1/P_b$ (resp. $1/P_{(b_p)}$). We will formulate and prove these estimates in the lemma below for the functions $P_{(b_p)}$ and $1/P_{(b_p)}$, but the corresponding inequalities hold for P_b and $1/P_b$ can be proved in a similar way.

Lemma 2.1. *If $P_{(b_p)}$ fulfills* (1) *and* (2), *then*

(*a*) *for every $r > 0$ there is a positive constant $\mathcal{C}$ (depending on r) such that*

$$|\left(P_{(b_p)}(\zeta)\right)^{(\gamma)}| \le \mathcal{C}\,\frac{\gamma!}{r^\gamma} P_{(b_p/2)}(|\zeta|), \qquad \zeta \in \mathbf{C},\ \gamma \in \mathbf{N}_0, \tag{4}$$

in particular,

$$|\left(P_{(b_p)}(\xi)\right)^{(\gamma)}| \le \mathcal{C}\,\frac{\gamma!}{r^\gamma} |P_{(b_p/2)}(\xi)|, \qquad \xi \in \mathbf{R},\ \gamma \in \mathbf{N}_0, \tag{5}$$

(*b*) *there exist an $r > 0$ and a $\mathcal{C} > 0$ such that*

$$\left|\left(\frac{1}{P_{(b_p)}(\xi)}\right)^{(\gamma)}\right| \le \mathcal{C}\frac{\gamma!}{r^\gamma}\exp[-N_{(2b_p)}(|\xi|)], \qquad \xi \in \mathbf{R},\ \gamma \in \mathbf{N}_0. \tag{6}$$

The corresponding inequalities hold for P_b.

Proof. (a) Fix $r > 0$. Applying Cauchy's formula and (3), we get

$$\begin{aligned}\left|\left(P_{(b_p)}(\zeta)\right)^{(\gamma)}\right| &= \left|\frac{\gamma!}{2\pi i}\int_{|z-\zeta|=r}\frac{P_{(b_p)}(z)dz}{(z-\zeta)^{\gamma+1}}\right| \\ &\le \frac{\gamma!}{r^\gamma}P_{(b_p)}(|\zeta|+r) \le C_r\frac{\gamma!}{r^\gamma}P_{(b_p/2)}(|\zeta|)\end{aligned}$$

for $\zeta \in \mathbf{C}$, where $C_r := P_{(b_p/2)}(r)$, i.e. (4) holds true. To obtain (5) it is enough to notice that

$$0 \le P_{(b_p/2)}(|\xi|) = P_{(b_p/2)}(\xi)$$

for every $\xi \in \mathbf{R}$.

(b) Since $P_{(b_p)}(0) \neq 0$, there exist positive r and C_1 such that $|P_{(b_p)}(\zeta)| \ge C_1$ for $|\zeta| \le (1+\sqrt{2})r$. By the Cauchy formula,

$$\begin{aligned}\left|\left(\frac{1}{P_{(b_p)}(\xi)}\right)^{(\gamma)}\right| &\le \frac{\gamma!}{2\pi}\int_{|\zeta-\xi|=r}\frac{d\zeta}{|P_{(b_p)}(\zeta)|\cdot|\zeta-\xi|^{\gamma+1}} \\ &\le \frac{\gamma!}{C_1 r^\gamma} \le C_2\frac{\gamma!}{r^\gamma}\exp[-N_{(2b_p)}(|\xi|)]\end{aligned} \tag{7}$$

for $\xi \in \mathbf{R}$ such that $|\xi| \le \sqrt{2}r$, where $C_2 := C_1^{-1}\exp[N_{(2b_p)}(\sqrt{2}r)]$. Now let $\xi \in \mathbf{R}$, $|\xi| > \sqrt{2}r$ and let K_ξ be the circle with the radius $|\xi|/\sqrt{2}$ and the center at ξ. Evidently, every point ζ of K_ξ satisfies the inequality $|\mathrm{Re}\,\zeta| \ge |\mathrm{Im}\,\zeta|$. Applying the Cauchy formula for the circle K_ξ and the estimate (2), we obtain

$$\begin{aligned}\left|\left(\frac{1}{P_{(b_p)}(\xi)}\right)^{(\gamma)}\right| &\le \frac{\gamma!}{2\pi}\int_{K_\xi}\frac{d\zeta}{|P_{(b_p)}(\zeta)|\cdot|\zeta-\xi|^{\gamma+1}} \\ &\le \gamma! 2^{\gamma/2}|\xi|^{-\gamma}\sup_{\Theta\in[0,2\pi]}\exp[-N_{(b_p)}(|\xi+|\xi|e^{\Theta i}|/\sqrt{2})] \\ &\le \gamma! r^{-\gamma}\exp[-N_{(b_p)}(|\xi|/2)] \le \gamma! r^{-\gamma}\exp[-N_{(2b_p)}(|\xi|)]\end{aligned}$$

for every $\gamma \in \mathbf{N}_0$ and $\xi \in \mathbf{R}$, $|\xi| > \sqrt{2}r$. This and (7) imply (6) and the proof is finished.

For the sake of convenience, we will use in the sequel the following notation for given $(a_p),(b_p)\in\mathcal{R}$:

$$A_\alpha := \prod_{p=1}^{\alpha} a_p, \quad B_\beta := \prod_{p=1}^{\beta} b_p \qquad (\alpha,\beta\in\mathbf{N}), \qquad A_0 := B_0 := 1. \tag{8}$$

DEFINITION 2.1. Let $a,b>0$ and let $(a_p),(b_p)\in\mathcal{R}$. The spaces $\mathcal{D}_a^{(M_p),b}$, $\mathcal{D}_{a,b}^{(M_p)}$, $\mathcal{D}_{(a_p)}^{\{M_p\},(b_p)}$, and $\mathcal{D}_{(a_p),(b_p)}^{\{M_p\}}$ are defined to be the sets of all smooth functions φ on $\mathbf{R}$ such that

$$\begin{aligned} p_{a,b}(\varphi) &:= \sup_{\alpha\in\mathbf{N}_0}\frac{a^\alpha}{M_\alpha}\|(P_b\varphi)^{(\alpha)}\|_\infty<\infty,\\ q_{a,b}(\varphi) &:= \sup_{\alpha\in\mathbf{N}_0}\frac{a^\alpha}{M_\alpha}\|P_b\varphi^{(\alpha)}\|_\infty<\infty,\\ p_{(a_p),(b_p)}(\varphi) &:= \sup_{\alpha\in\mathbf{N}_0}\frac{\|(P_{(b_p)}\varphi)^{(\alpha)}\|_\infty}{M_\alpha A_\alpha}<\infty,\\ q_{(a_p),(b_p)}(\varphi) &:= \sup_{\alpha\in\mathbf{N}_0}\frac{\|P_{(b_p)}\varphi^{(\alpha)}\|_\infty}{M_\alpha A_\alpha}<\infty, \end{aligned}$$

respectively, where A_α, B_α are defined in (8), equipped with the topologies induced by the norms $p_{a,b}$, $q_{a,b}$, $p_{(a_p),(b_p)}$ and $q_{(a_p),(b_p)}$, respectively. Further, we define

$$\begin{aligned} \mathcal{D}_a^{(M_p)} &:= \operatorname{proj\,lim}_{b>0}\mathcal{D}_a^{(M_p),b}, \quad \mathcal{D}_{(a_p)}^{\{M_p\}} := \operatorname{proj\,lim}_{(b_p)\in\mathcal{R}}\mathcal{D}_{(a_p)}^{\{M_p\},(b_p)},\\ \mathcal{D}^{(M_p),b} &:= \operatorname{proj\,lim}_{a>0}\mathcal{D}_a^{(M_p),b}, \quad \mathcal{D}^{\{M_p\},(b_p)} := \operatorname{proj\,lim}_{(a_p)\in\mathcal{R}}\mathcal{D}_{(a_p)}^{\{M_p\},(b_p)}. \end{aligned}$$

THEOREM 2.2. *If condition $(M.2')$ is satisfied, then*

$$\mathcal{S}^{(M_p)} = \operatorname{proj\,lim}_{a>0}\mathcal{D}_a^{(M_p)} = \operatorname{proj\,lim}_{b>0}\mathcal{D}^{(M_p),b}$$

and

$$\mathcal{S}^{\{M_p\}} = \operatorname{proj\,lim}_{(a_p)\in\mathcal{R}}\mathcal{D}_{(a_p)}^{\{M_p\}} = \operatorname{proj\,lim}_{(b_p)\in\mathcal{R}}\mathcal{D}^{(M_p),(b_p)}.$$

PROOF. We shall prove the assertion only in the case $*=\{M_p\}$, which is more complicated than the other one, but the ideas of the proof in both cases are similar.

First recall that every sequence (M_p) satisfying conditions $(M.1)$ and $(M.3')$ tends very quickly to infinity. More precisely, it follows from these conditions that $pM_{p-1}/M_p\to 0$ as $p\to\infty$ (see [6], (4.6)) and this implies that

$$\frac{a^p p!}{M_p}\to 0 \qquad \text{as } p\to\infty \tag{9}$$

for an arbitrary $a>0$.

Define

$$\gamma_{(a_p),(b_p)}(\varphi) := \sup_{\alpha,\beta\in\mathbf{N}_0}\sup_{x\in\mathbf{R}}\frac{\langle t\rangle^\beta|\varphi^{(\alpha)}(t)|}{M_\alpha A_\alpha M_\beta B_\beta},$$

where $\langle t\rangle := (1+t^2)^{1/2}$ for $t\in\mathbf{R}$. Since $\langle t\rangle^\beta \le 2^{\beta/2}(1+|t|^\beta)$ for $t\in\mathbf{R}$, we get from (9) and the definition of the functions $N_{(b_p)}$ the following estimate:

$$\begin{aligned}\gamma_{(a_p),(b_p)}(\varphi) &\le \sup_{\alpha,\beta\in\mathbf{N}_0}\frac{2^{\beta/2}\|\varphi^{(\alpha)}\|_\infty}{M_\alpha A_\alpha M_\beta B_\beta} + \sup_{\alpha,\beta\in\mathbf{N}_0}\sup_{t\in\mathbf{R}}\frac{2^{\beta/2}|t|^\beta|\varphi^{(\alpha)}(t)|}{M_\alpha A_\alpha M_\beta B_\beta}\\
&\le \mathcal{C}\left(\sup_{\alpha\in\mathbf{N}_0}\frac{\|\varphi^{(\alpha)}\|_\infty}{M_\alpha A_\alpha} + \sup_{\alpha\in\mathbf{N}_0}\frac{\|\exp[N_{(b_p/\sqrt{2})}]\varphi^{(\alpha)}\|_\infty}{M_\alpha A_\alpha}\right)\\
&\le \mathcal{C}\sup_{\alpha\in\mathbf{N}_0}\frac{\|P_{(b_p/\sqrt{2})}\varphi^{(\alpha)}\|_\infty}{M_\alpha A_\alpha} = \mathcal{C}q_{(a_p),(b_p/\sqrt{2})}(\varphi)\end{aligned}$$

for each $\varphi\in C^\infty$, where the constants A_α, B_α are defined in (8). On the other hand, inequality (1) yields

$$\begin{aligned}&q_{(a_p),(b_p)}(\varphi) \le \mathcal{C}\sup_{\alpha\in\mathbf{N}_0}\sup_{t\in\mathbf{R}}\frac{\exp[N_{(b_p)}(L|t|)]|\varphi^{(\alpha)}(t)|}{M_\alpha A_\alpha}\\
&\le \mathcal{C}\sup_{\alpha\in\mathbf{N}_0}\sup_{\beta\in\mathbf{N}_0}\sup_{t\in\mathbf{R}}\frac{|\langle t\rangle^\beta\varphi^{(\alpha)}(t)|}{M_\alpha A_\alpha M_\beta(B_\beta/L^\beta)} \le \mathcal{C}\gamma_{(a_p),(b_p/L)}(\varphi)\end{aligned}$$

for every $\varphi\in C^\infty$. The two estimates just proved show that the families $\{q_{(a_p),(b_p)} : (a_p),(b_p)\in\mathcal{R}\}$ and $\{\gamma_{(a_p),(b_p)} : (a_p),(b_p)\in\mathcal{R}\}$ of seminorms are equivalent.

Let us prove now the equivalence of the families $\{p_{(a_p),(b_p)} : (a_p),(b_p)\in\mathcal{R}\}$ and $\{q_{(a_p),(b_p)} : (a_p),(b_p)\in\mathcal{R}\}$. First notice that for an arbitrary $(a_p)\in\mathcal{R}$ we have $A_pA_q\le A_{p+q}$ and, because of condition $(M.1)$, $M_pM_q\le M_{p+q}$ for $p,q\in\mathbf{N}$. Therefore, applying (5) for a fixed $r>0$ and (9), we have

$$\begin{aligned}&p_{(a_p),(b_p)}(\varphi) = \sup_{\alpha\in\mathbf{N}_0}\frac{\|\left(P_{(b_p)}\varphi\right)^{(\alpha)}\|_\infty}{M_\alpha A_\alpha}\\
&\le \sup_{\alpha\in\mathbf{N}_0}\frac{1}{M_\alpha A_\alpha}\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\|P^{(\gamma)}_{(b_p)}\varphi^{(\alpha-\gamma)}\|_\infty\\
&\le \mathcal{C}\sup_{\alpha\in\mathbf{N}_0}\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\frac{\gamma!}{M_{\alpha-\gamma}A_{\alpha-\gamma}M_\gamma A_\gamma r^\gamma}\|P_{(b_p/2)}\varphi^{(\alpha-\gamma)}\|_\infty\\
&\le \mathcal{C}\sup_{\alpha\in\mathbf{N}_0}\frac{1}{2^\alpha}\sup_{\gamma\le\alpha}\frac{\gamma!}{M_\gamma A'_\gamma}\frac{\|P_{(b_p/2)}\varphi^{(\alpha-\gamma)}\|_\infty}{M_{\alpha-\gamma}A'_{\alpha-\gamma}}\sum_{\gamma\le\alpha}\binom{\alpha}{\gamma}\\
&\le \mathcal{C}q_{(a_p/2),(b_p/2)}(\varphi),\end{aligned}$$

where A_α, B_α are defined in (8) and

$$A'_\gamma := \prod_{p=1}^{\gamma} a_p/2,\quad \gamma\in\mathbf{N};\qquad A'_0 := 1. \tag{10}$$

Let (a_p) and (b_p) be given sequences in $\mathcal{R}$ and choose $(b'_p)\in\mathcal{R}$ such that $b'_p\le b_p/(2L)$ for $p\in\mathbf{N}$, where L is the constant from (1). This implies that

$$\tilde{N}(t) := \exp[N_{(b_p)}(L|t|) - N_{(2b'_p)}(|t|)] \le 1,\qquad t\in\mathbf{R}. \tag{11}$$

Inequalities (1), (6), (11) and the properties of the sequences (M_p) and (A_p) mentioned above imply that there exist constants $\mathcal{C} > 0$ and $r > 0$ such that, for every $\varphi \in C^\infty$, we have

$$\begin{aligned}
q_{(a_p),(b_p)}(\varphi) &= \sup_{\alpha\in\mathbf{N}_0} \frac{1}{M_\alpha A_\alpha} \|P_{(b_p)}\varphi^{(\alpha)}\|_\infty \\
&= \sup_{\alpha\in\mathbf{N}_0} \frac{1}{M_\alpha A_\alpha} \left\| P_{(b_p)} \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} (1/P_{(b'_p)})^{(\alpha-\gamma)} (P_{(b'_p)}\varphi)^{(\gamma)} \right\|_\infty \\
&\le \mathcal{C} \sup_{\alpha\in\mathbf{N}_0} \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} \frac{(\alpha-\gamma)!}{r^{\alpha-\gamma} M_\alpha A_\alpha} \|(P_{(b'_p)}\varphi)^{(\gamma)}\|_\infty \\
&\le \mathcal{C} \sup_{\alpha\in\mathbf{N}_0} \frac{1}{2^\alpha} \sum_{\gamma\le\alpha} \binom{\alpha}{\gamma} \frac{(\alpha-\gamma)!}{M_{\alpha-\gamma} A'_{\alpha-\gamma}} \frac{1}{M_\gamma A'_\gamma} \|(P_{(b'_p)}\varphi)^{(\gamma)}\|_\infty \\
&\le \mathcal{C} \sup_{\beta\in\mathbf{N}_0} \frac{1}{M_\beta A'_\beta} \|(P_{(b'_p)}\varphi)^{(\beta)}\|_\infty = \mathcal{C} p_{(a_p/2),(b'_p)}(\varphi),
\end{aligned}$$

where A_γ and A'_γ are defined in (8) and (10).

Remark 2.1. From the preceding proof it follows that for every $a > 0$ (resp. $(a_p) \in \mathcal{R}$) there exists a $b > 0$ (resp. $(b_p) \in \mathcal{R}$) with $a < b$ (resp. $(a_p) \preceq (b_p)$, i.e. $a_p \le b_p$ for sufficiently large $p \in \mathbf{N}$) such that $\mathcal{D}_b^{(M_p)} \subseteq \mathcal{D}_a^{(M_p)}$ (resp. $\mathcal{D}_{(b_p)}^{\{M_p\}} \subseteq \mathcal{D}_{(a_p)}^{\{M_p\}}$) and the inclusion mapping is continuous.

Definition 2.2. For given $a > 0$ and $(a_p) \in \mathcal{R}$, we define the Hilbert transforms $\mathcal{H}_a$ and $\mathcal{H}_{(a_p)}$ on $\mathcal{D}_a^{(M_p)}$ and $\mathcal{D}_{(a_p)}^{\{M_p\}}$, respectively, in the following way:

$$(\mathcal{H}_a\varphi)(x) := \frac{1}{\pi P_a(x)} \operatorname{pv} \int_{-\infty}^{\infty} \frac{P_a(x-t)\varphi(x-t)}{t}\, dt, \qquad \varphi \in \mathcal{D}_a^{(M_p)}, x \in \mathbf{R}$$

and

$$(\mathcal{H}_{(a_p)}\varphi)(x) := \frac{1}{\pi P_{(a_p)}(x)} \operatorname{pv} \int_{-\infty}^{\infty} \frac{P_{(a_p)}(x-t)\varphi(x-t)}{t}\, dt, \quad \varphi \in \mathcal{D}_{(a_p)}^{\{M_p\}}, x \in \mathbf{R}.$$

Theorem 2.3. *For given $a > 0$ and $(a_p) \in \mathcal{R}$, the Hilbert transforms $\mathcal{H}_a : \mathcal{D}_a^{(M_p)} \to \mathcal{D}_a^{(M_p)}$ and $\mathcal{H}_{(a_p)} : \mathcal{D}_{(a_p)}^{\{M_p\}} \to \mathcal{D}_{(a_p)}^{\{M_p\}}$ are linear continuous surjections such that*

$$\mathcal{H}_a\mathcal{H}_a\varphi = -\varphi, \qquad \varphi \in \mathcal{D}_a^{(M_p)};$$

$$\mathcal{H}_{(a_p)}\mathcal{H}_{(a_p)}\varphi = -\varphi, \qquad \varphi \in \mathcal{D}_{(a_p)}^{\{M_p\}}.$$

Proof. We give the proof only in the case $* = \{M_p\}$; the proof in the case $* = (M_p)$ is analogous. The linearity and the continuity of $\mathcal{H}_{(a_p)}$ follow immediately from the fact that $\mathcal{H}_{(a_p)}$ is a composition of the three linear and continuous mappings $T_{(a_p)} : \mathcal{D}_{(a_p)}^{\{M_p\}} \to \mathcal{D}_{L^2}^{\{M_p\}}$, $\mathcal{H} : \mathcal{D}_{L^2}^{\{M_p\}} \to \mathcal{D}_{L^2}^{\{M_p\}}$ and $T_{(a_p)}^{-1} : \mathcal{D}_{L^2}^{\{M_p\}} \to \mathcal{D}_{(a_p)}^{\{M_p\}}$, defined by the formulas:

$$T_{(a_p)}(\varphi)(x) := P_{(a_p)}(x)\varphi(x), \qquad \varphi \in \mathcal{D}_{(a_p)}^{\{M_p\}}; \tag{12}$$

$$(\mathcal{H}\varphi)(x) := \frac{1}{\pi} \operatorname{pv} \int_{-\infty}^{\infty} \frac{\varphi(t)\, dt}{t-x}, \qquad \varphi \in \mathcal{D}_{L^2}^{\{M_p\}}; \tag{13}$$

(14) $$T^{-1}_{(a_p)}(\varphi) := \varphi(x)/P_{(a_p)}(x), \qquad \varphi \in \mathcal{D}^{\{M_p\}}_{L^2}$$

for $x \in \mathbf{R}$. Note that the Hilbert transform is considered in [11] only on $\mathcal{D}^{(M_p)}_{L^2}$, i.e. in the Beurling case, but it can be examined in a similar way in the Roumieu case.

From the definition of $\mathcal{H}_{(a_p)}$ and the properties of the classical Hilbert transform on $\mathcal{D}^{\{M_p\}}_{L^2}$, it follows that

$$\mathcal{H}_{(a_p)}(\mathcal{H}_{(a_p)}\varphi) = T^{-1}_{(a_p)}(\mathcal{H}T_{(a_p)}(T^{-1}_{(a_p)}(\mathcal{H}(T_{(a_p)}\varphi))))$$
$$= T^{-1}_{(a_p)}(\mathcal{H}(\mathcal{H}(T_{(a_p)}\varphi))) = T^{-1}_{(a_p)}(-T_{(a_p)}\varphi) = -\varphi$$

for every $(a_p) \in \mathcal{R}$ and $\varphi \in \mathcal{D}^{\{M_p\}}_{(a_p)}$. This completes the proof.

Definition 2.3. The generalized Hilbert transforms $\mathbf{H}_a$ and $\mathbf{H}_{(a_p)}$ are defined for $f \in \mathcal{D}'^{(M_p)}_a$ and $f \in \mathcal{D}'^{\{M_p\}}_{(a_p)}$ by

$$\langle \mathbf{H}_a f, \varphi\rangle := -\langle f, \mathcal{H}_a\varphi\rangle, \qquad \varphi \in \mathcal{D}^{(M_p)}_a,$$

and

$$\langle \mathbf{H}_{(a_p)} f, \varphi\rangle := -\langle f, \mathcal{H}_{(a_p)}\varphi\rangle, \qquad \varphi \in \mathcal{D}^{\{M_p\}}_{(a_p)},$$

respectively.

In the theorem below, we list several properties of the Hilbert transforms $\mathbf{H}_a$ and $\mathbf{H}_{(a_p)}$ defined above. In particular, we shall prove that

(15) $$\langle \mathcal{F}(\mathbf{H}_* f), \varphi\rangle = \begin{cases} -i\langle \mathcal{F}f, \varphi\rangle, & \text{if } \operatorname{supp}\varphi \subset (0,\infty), \\ i\langle \mathcal{F}f, \varphi\rangle, & \text{if } \operatorname{supp}\varphi \subset (-\infty,0) \end{cases}$$

for every $\varphi \in \mathcal{D}^{(M_p)}$ if $f \in \mathcal{D}'^{(M_p)}$ and for every $\varphi \in \mathcal{D}^{\{M_p\}}$ if $f \in \mathcal{D}'^{\{M_p\}}$, where the symbol $\mathbf{H}_*$ means $\mathbf{H}_{(M_p)}$ and $\mathbf{H}_{\{M_p\}}$ in the respective cases.

Theorem 2.4. *The defined above Hilbert transforms* $\mathbf{H}_a : \mathcal{D}'^{(M_p)}_a \to \mathcal{D}'^{(M_p)}_a$ *and* $\mathbf{H}_{(a_p)} : \mathcal{D}'^{\{M_p\}}_{(a_p)} \to \mathcal{D}'^{\{M_p\}}_{(a_p)}$ *have the following properties:*

1. $\mathbf{H}_a$ *and* $\mathbf{H}_{(a_p)}$ *are linear continuous surjections;*

2. $\mathbf{H}_a(\mathbf{H}_a f) = -f$ *for* $f \in \mathcal{D}'^{(M_p)}_a$ *and* $\mathbf{H}_{(a_p)}(\mathbf{H}_{(a_p)} f) = -f$ *for* $f \in \mathcal{D}'^{\{M_p\}}_{(a_p)}$*;*

3. If $f \in \mathcal{D}'^{(M_p)}_a$, *then formula (15) holds for all* $\varphi \in \mathcal{D}^{(M_p)}$ *and if* $f \in \mathcal{D}'^{\{M_p\}}_{(a_p)}$, *then formula (15) holds for all* $\varphi \in \mathcal{D}^{\{M_p\}}$*;*

4. Under conditions $(M.2)$ *and* $(M.3)$, *if* $f \in \mathcal{D}'^{(M_p)}_a$ (*resp.* $f \in \mathcal{D}'^{\{M_p\}}_{(a_p)}$) *with* $0 < a < b$ (*resp.* $(a_p) \preceq (b_p)$) *such that* $f|_{\mathcal{D}^{(M_p)}_b} \in \mathcal{D}'^{(M_p)}_b$ (*resp.* $f|_{\mathcal{D}^{\{M_p\}}_{(b_p)}} \in \mathcal{D}'^{\{M_p\}}_{(b_p)}$) (*see Remark 2.1*). *Then* $\mathbf{H}_a f - \mathbf{H}_b f|_{\mathcal{D}^{(M_p)}_b}$ (*resp.* $\mathbf{H}_{(a_p)} f - \mathbf{H}_{(b_p)} f|_{\mathcal{D}^{\{M_p\}}_{(b_p)}}$) *is an ultrapolynomial of the class* (M_p) (*resp.* $\{M_p\}$)*;*

5. If $f, g \in \mathcal{D}'^{(M_p)}_a$ (*resp.* $f, g \in \mathcal{D}'^{\{M_p\}}_{(a_p)}$) *and* $f|_{\mathcal{D}^{(M_p)}} = g|_{\mathcal{D}^{(M_p)}}$ (*resp.* $f|_{\mathcal{D}^{\{M_p\}}} = g|_{\mathcal{D}^{\{M_p\}}}$), *then* $\mathbf{H}_a f - \mathbf{H}_a g$ (*resp.* $\mathbf{H}_{(a_p)} f - \mathbf{H}_{(a_p)} g$) *is an ultrapolynomial of the class* (M_p) (*resp.* $\{M_p\}$).

Proof. We shall prove the assertion only in the case $* = \{M_p\}$. Parts 1 and 2 follow immediately from the previous theorem. Let us prove part 3. Let $\varphi \in \mathcal{D}^{\{M_p\}}$ and $\operatorname{supp} \varphi \subset (0,\infty)$. From the properties of the classical Fourier and Hilbert transforms of functions in L^2, we have

$$\begin{aligned}
&\langle \mathcal{F}(\mathbf{H}_{(a_p)} f), \varphi\rangle = \langle \mathbf{H}_{(a_p)} f, \mathcal{F}\varphi\rangle = \langle f, T^{-1}_{(a_p)} \mathcal{H} T_{(a_p)} \mathcal{F}\varphi\rangle \\
&= \langle f, T^{-1}_{(a_p)} \mathcal{H}\mathcal{F}(P_{(a_p)}(D)\varphi)\rangle = \langle f, T^{-1}_{(a_p)} \mathcal{F}(-iP_{(a_p)}(D)\varphi)\rangle \\
&= -i\langle f, T^{-1}_{(a_p)} T_{(a_p)} \mathcal{F}\varphi\rangle = -i\langle f, \mathcal{F}\varphi\rangle = -i\langle \mathcal{F}f, \varphi\rangle.
\end{aligned}$$

In a similar way we can prove part 3 in the case $\varphi \in \mathcal{D}^{\{M_p\}}$ and $\operatorname{supp} \varphi \subset (-\infty, 0)$.

Let us prove now part 4. For each $\varphi \in \mathcal{D}^{(M_p)}$ with $\operatorname{supp} \varphi \subset (0,\infty)$, we have

$$\begin{aligned}
\langle \mathcal{F}(\mathbf{H}_{(a_p)} f - \mathbf{H}_{(b_p)} f), \varphi\rangle &= \langle \mathcal{F}(\mathbf{H}_{(a_p)} f), \varphi\rangle - \langle \mathcal{F}(\mathbf{H}_{(b_p)} f), \varphi\rangle \\
&= -i\langle \mathcal{F}f, \varphi\rangle - (-i)\langle \mathcal{F}f, \varphi\rangle = 0.
\end{aligned}$$

Analogously, we have

$$\langle \mathcal{F}(\mathbf{H}_{(a_p)} f - \mathbf{H}_{(b_p)} f), \varphi\rangle = 0$$

for $\varphi \in \mathcal{D}^{\{M_p\}}$ with $\operatorname{supp} \varphi \subset (-\infty, 0)$. Therefore $\operatorname{supp} \mathcal{F}(\mathbf{H}_{(a_p)} f - \mathbf{H}_{(b_p)} f) \subseteq \{0\}$. Theorem 3.1 in [7] implies the existence of an ultradifferential operator $P(D)$ such that

$$\mathcal{F}(\mathbf{H}_{(a_p)} f - \mathbf{H}_{(b_p)} f) = P(D)\,\delta. \tag{16}$$

Applying the inverse Fourier transform on (16), we obtain

$$(\mathbf{H}_{(a_p)} f - \mathbf{H}_{(b_p)} f)(x) = P(x), \quad x \in \mathbf{R},$$

i.e. property 4 holds.

Assertion 5 follows from the fact that

$$\operatorname{supp} \mathcal{F}(\mathbf{H}_{(a_p)} f - \mathbf{H}_{(a_p)} g) \subseteq \{0\},$$

which can be proved analogously as part 4.

Definition 2.4. Using the fact that for every $f \in \mathcal{S}'^{(M_p)}$ (resp. $f \in \mathcal{S}'^{\{M_p\}}$) there is an $a > 0$ (resp. $(a_p) \in \mathcal{R}$) such that f has a linear and continuous extension F on $\mathcal{D}^{(M_p)}_a$ (resp. $\mathcal{D}^{\{M_p\}}_{(a_p)}$), we define the Hilbert transform $\mathbf{H}^{(M_a)} f$ (resp. $\mathbf{H}^{\{M_a\}} f$) of $f \in \mathcal{S}'^{(M_p)}$ (resp. $f \in \mathcal{S}'^{\{M_p\}}$) by

$$\mathbf{H}^{(M_p)} f := \mathbf{H}_a F \quad (\text{resp. } \mathbf{H}^{\{M_p\}} f := \mathbf{H}_{(a_p)} F).$$

The above theorem shows that the Hilbert transform of an element of the space $\mathcal{S}'^*$ is defined uniquely up to an ultrapolynomial of class $*$.

3. Hilbert transform, multi-dimensional case. We now extend the definition of the Hilbert transform given in the preceding section to the d-dimensional case. We shall show that all the results of section 2, i.e. Lemma 2.1, Theorem 2.2, Theorem 2.3 and Theorem 2.4 remain true in the d-dimensional case.

If $a = (a^1, \ldots, a^d) \in \mathbf{R}^d$ and $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbf{N}^d_0$, we denote

$$a^\alpha := (a^1)^{\alpha_1} \cdot \ldots \cdot (a^d)^{\alpha_d}; \qquad M_\alpha := M_{\alpha_1 + \ldots + \alpha_d}.$$

By $\mathcal{R}^d$ we denote the family of all sequences (a_p) of elements of $\mathbf{R}^d$ of the form

$$a_p = (a_p^1, \ldots, a_p^d), \qquad (a_p^j) \in \mathcal{R} \qquad (j = 1, \ldots, d). \tag{17}$$

For a given sequence $(a_p) \in \mathcal{R}^d$ of the form (17) and $\alpha = (\alpha^1, \ldots, \alpha^d) \in \mathbf{N}_0^d$, we shall use the following extension of the notation (8): $A_\alpha := A_{\alpha^1} \cdot \ldots \cdot A_{\alpha^d}$, where $A_0 := 1$ and

$$A_{\alpha^j} := \prod_{p=1}^{\alpha^j} a_p^j \qquad \text{whenever } \alpha^j \in \mathbf{N}$$

for $j = 1, \ldots, d$.

Let $a = (a^1, \ldots, a^d) \in \mathbf{R}_+^d$, i.e. $a^j > 0$ for $j = 1, \ldots, d$, and let $(a_p) \in \mathcal{R}^d$ with elements of the form (17). Then we define

$$P_a(\zeta) := P_{a_1}(\zeta^1) \cdot \ldots \cdot P_{a_d}(\zeta^d); \qquad P_{(a_p)}(\zeta) := P_{(a_p^1)}(\zeta^1) \cdot \ldots \cdot P_{(a_p^d)}(\zeta^d)$$

for $\zeta = (\zeta^1, \ldots, \zeta^d) \in \mathbf{C}^d$.

REMARK 3.1. The d-dimensional version of Lemma 2.1 is true and the estimates (4) and (6) in the multi-dimensional case follow easily from the one-dimensional case.

Now the definitions of the seminorms $p_{a,b}$ and $q_{a,b}$ for $a = (a^1, \ldots, a^d) \in \mathbf{R}_+^d$ and $b = (b^1, \ldots, b^d) \in \mathbf{R}_+^d$, and the definitions of the seminorms $p_{(a_p),(b_p)}$ and $q_{(a_p),(b_p)}$ for $(a_p) \in \mathcal{R}^d$ and $(b_p) \in \mathcal{R}^d$ are the obvious modifications of ones given in Definition 2.1 (all the least upper bounds should be taken over $\alpha \in \mathbf{N}_0^d$). The definitions of all spaces given in Definition 2.1 are modified accordingly.

REMARK 3.2. With the above conventions, the d-dimensional analogue of Theorem 2.2 is true and its proof goes in the same way as in the one-dimensional case.

DEFINITION 3.1. For given $a \in \mathbf{R}_+^d$ and $(a_p) \in \mathcal{R}^d$, we define the Hilbert transforms $\mathcal{H}_a$ and $\mathcal{H}_{(a_p)}$ on the spaces $\mathcal{D}_a^{(M_p)}$ and $\mathcal{D}_{a_p}^{\{M_p\}}$, respectively, by the formulas:

$$(\mathcal{H}_a\varphi)(x) := \frac{1}{\pi^d P_a(x)} \operatorname{pv} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{P_a(x-t)\varphi(x-t)}{\prod_{j=1}^d t^j} \, dt^1 \ldots dt^d$$

for $\varphi \in \mathcal{D}_a^{(M_p)}$ and

$$(\mathcal{H}_{(a_p)}\varphi)(x) := \frac{1}{\pi^d P_{(a_p)}(x)} \operatorname{pv} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{P_{(a_p)}(x-t)\varphi(x-t)}{\prod_{j=1}^d t^j} \, dt^1 \ldots dt^d$$

for $\varphi \in \mathcal{D}_{(a_p)}^{\{M_p\}}$, where $x = (x^1, \ldots, x^d), t = (t^1, \ldots, t^d) \in \mathbf{R}^d$.

REMARK 3.3. The above defined Hilbert transforms $\mathcal{H}_a$ and $\mathcal{H}_{(a_p)}$ have the same properties as those mentioned in Theorem 2.3 and their proof are analogous. In particular, $\mathcal{H}_{(a_p)}$ is the composition of the mappings given by formulas (12), (14) and the following extension of formula (13):

$$(\mathcal{H}\varphi)(x) = \frac{1}{\pi^d} \operatorname{pv} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \frac{\varphi(t^1, \ldots, t^d) \, dt^1 \ldots dt^d}{\prod_{j=1}^d (t^j - x^j)}$$

for $x = (x^1, \ldots, x^d) \in \mathbf{R}^d$. Though in [11] only the one-dimensional Hilbert transform is considered on $\mathcal{D}_{L^2}^{\{M_p\}}$, one can easily extend it to the d-dimensional case and prove that this is an isomorphism of the space $\mathcal{D}_{L^2}^{\{M_p\}}(\mathbf{R}^d)$, because there is a constant $C > 0$ such that

$$\|\mathcal{H}(\varphi^{(\alpha)})\|_{L^2} \le C\|\varphi^{(\alpha)}\|_{L^2}, \qquad \alpha \in \mathbf{N}^d, \quad \varphi \in \mathcal{D}_{L^2}^{\{M_p\}}.$$

Definition 3.2. For given $a \in \mathbf{R}_+^d$ and $(a_p) \in \mathcal{R}^d$, we define the d-dimensional Hilbert transforms $\mathbf{H}_a$ and $\mathbf{H}_{(a_p)}$ on the dual spaces $\mathcal{D}'^{(M_p)}_a$ and $\mathcal{D}'^{\{M_p\}}_{(a_p)}$ by

$$\langle \mathbf{H}_a f, \varphi\rangle := -\langle f, \mathcal{H}_a\varphi\rangle, \qquad \varphi \in \mathcal{D}_a^{(M_p)}$$

for $f \in \mathcal{D}'^{(M_p)}_a$ and

$$\langle \mathbf{H}_{(a_p)} f, \varphi\rangle := -\langle f, \mathcal{H}_{(a_p)}\varphi\rangle, \qquad \varphi \in \mathcal{D}_{(a_p)}^{\{M_p\}}$$

for $f \in \mathcal{D}'^{\{M_p\}}_{(a_p)}$.

Remark 3.4. Theorem 2.4 remains true in the d-dimensional case and the proof of parts 1 - 3 and 5 can be easily transferred to this case. We shall give below the proof of part 4 for the Hilbert transform $\mathbf{H}_a f$ of an arbitrary $f \in \mathcal{D}'^{(M_p)}_a$.

For given $a = (a^1, \dots, a^d) \in \mathbf{R}^d$ and $b = (b^1, \dots, b^d) \in \mathbf{R}^d$, we have

$$\mathbf{H}_a f - \mathbf{H}_b f := \sum_{j=1}^{d} (\mathbf{H}_{a_j} f - \mathbf{H}_{a_{j-1}} f),$$

where $a_0 := b$ and $a_j := (a^1, \dots, a^j, b^{j+1}, \dots, b^d)$ for $j = 1, \dots, d$. Moreover

$$\mathbf{H}_{a_j} f = \mathbf{H}_{\tilde{a}_j}(\mathbf{H}_{a^j} f), \tag{18}$$

where $\tilde{a}_j := (a^1, \dots, a^{j-1}, b^{j+1}, \dots, b^d) \in \mathbf{R}^{d-1}$ for $j = 1, \dots, d$. By part 4 of Theorem 2.4, it follows from (18) that $\mathbf{H}_{a_j} f - \mathbf{H}_{a_{j-1}} f$ is equal to

$$\mathbf{H}_{\tilde{a}_j}(1_{x^1} \otimes \ldots \otimes 1_{x^{j-1}} \otimes P(x^j) \otimes 1_{x^{j+1}} \ldots \otimes 1_{x^d}), \tag{19}$$

where $P(x^j)$ is an ultrapolynomial in x^j. Since we have

$$\|(\mathbf{H}\frac{1}{P_b})^{(\alpha)}\|_{L^2} \le C\|(\frac{1}{P_b})^{(\alpha)}\|_{L^2}$$

for some constant $C > 0$, Lemma 2.1 implies that (19) is an entire function.

4. Singular integral operators. Let Ω be an arbitrary function on $\mathbf{R}^d$, homogeneous of degree zero, such that $\Omega \in C^\infty(\mathbf{R}^d \setminus \{0\})$. Clearly, this function is integrable and square integrable on $\Sigma^{d-1} = \{x : \ |x| = 1\}$. Put

$$K(t) := \frac{\Omega(t')}{|t|^d}, \quad t \in \mathbf{R}^d, \ t \ne 0, \ t' = t/|t|. \tag{20}$$

If the mapping $t' \mapsto \Omega(t')$ is an odd function on Σ^{d-1}, we say that K is an odd kernel. If $\int_{\Sigma^{d-1}} \Omega(t')\,dt' = 0$, we say that K is an even kernel. Clearly, an odd kernel is also an even kernel (see [16], Ch. IV and VI).

Let the symbol pv$K$ denote the principal value of K (clearly, it is a tempered distribution). Let

$$\tilde{K} := \mathcal{F}(\mathrm{pv}K), \tag{21}$$

where $\mathcal{F}$ is understood as the Fourier transform of a tempered distribution.

The convolution $\varphi * (\mathrm{pv}K)$, where $\varphi \in L^p$ $(1 < p < \infty)$, yields the singular integral operator $\mathcal{K}$ with the kernel K. More precisely, the singular integral operator $\mathcal{K}$ on L^p with the kernel K is defined by

$$(\mathcal{K}\varphi)(x) := \lim_{\substack{\varepsilon\to 0\\ \delta\to\infty}} \int_{\delta \ge |t| \ge \varepsilon > 0} \varphi(x-t)K(t)\,dt, \quad \varphi \in L^p. \tag{22}$$

If the dimension of the space is $d = 1$ and $\Omega(t) = \operatorname{sgn} t/\pi$ then the singular integral operator $\mathcal{K}$ defined by (22) is the Hilbert transform.

The operator $\mathcal{K}$ defined on L^p $(1 < p < \infty)$ by (22), where K is an even kernel, is called an integral operator with even kernel.

We denote by $\mathcal{S}^*(\mathring{\mathbf{R}}^d)$ the set of all elements of $\mathcal{S}^*(\mathbf{R}^d)$ whose supports are contained in $\mathbf{R}^d \setminus \{0\}$ with the topology induced by the space $\mathcal{S}^*(\mathbf{R}^d)$ and let

$$\mathcal{S}^*_{\mathcal{F}}(\mathring{\mathbf{R}}^d) := \mathcal{F}(\mathcal{S}^*(\mathring{\mathbf{R}}^d)),$$

where $\mathcal{F}$ denotes the classical Fourier transform of a function.

THEOREM 4.1. *Let K be of the form (20) with the corresponding $\tilde{K}$ and $\mathcal{K}\varphi$, given by (21) and (22), respectively.*

Then

1. $\tilde{K}$ is homogeneous of order zero on $\mathbf{R}^d$ and $\tilde{K} \in C^\infty(\mathbf{R}^d \setminus \{0\})$;

2. We have $\mathcal{K}\varphi \in \mathcal{S}^*(\mathbf{R}^d)$ for every $\varphi \in \mathcal{S}^*_{\mathcal{F}}(\mathring{\mathbf{R}}^d)$ and $\tilde{K}\psi \in \mathcal{S}^*(\mathring{\mathbf{R}}^d)$ for every $\psi \in \mathcal{S}^*(\mathring{\mathbf{R}}^d)$; moreover, the mappings

$$\mathcal{S}^*_{\mathcal{F}}(\mathring{\mathbf{R}}^d) \ni \psi \mapsto \mathcal{K}\psi \in \mathcal{S}^*(\mathbf{R}^d); \quad \mathcal{S}^*(\mathring{\mathbf{R}}^d) \ni \psi \mapsto \tilde{K}\psi \in \mathcal{S}^*(\mathring{\mathbf{R}}^d)$$

are continuous;

3. The space $\mathcal{S}^*_{\mathcal{F}}(\mathring{\mathbf{R}}^d)$ is dense in $\mathcal{S}^*(\mathbf{R}^d)$.

PROOF. 1. For this assertion we refer to Corollary 9.5 in [10], p. 108.

2. If $\varphi \in \mathcal{S}^*_{\mathcal{F}}(\mathring{\mathbf{R}}^d)$, then we have

$$\mathcal{K}\varphi = (\mathrm{pv}K) * \varphi = \mathcal{F}^{-1}(\mathcal{F}(\mathrm{pv}K) \cdot \mathcal{F}(\varphi)) = \mathcal{F}^{-1}(\tilde{K} \cdot \mathcal{F}(\varphi)),$$

so $\mathcal{K}\varphi \in \mathcal{S}^*(\mathbf{R}^d)$. Now, if $\psi \in \mathcal{S}^*(\mathring{\mathbf{R}}^d)$, then $\varphi := \mathcal{F}^{-1}(\psi) \in \mathcal{S}^*_{\mathcal{F}}(\mathring{\mathbf{R}}^d)$, so $\mathcal{K}\varphi \in \mathcal{S}^*(\mathbf{R}^d)$, as we have just shown. Hence

$$\mathcal{F}(\mathcal{K}\varphi) = \mathcal{F}((\mathrm{pv}K) * \varphi) = \tilde{K} \cdot \psi \in \mathcal{S}^*(\mathring{\mathbf{R}}^d).$$

The above and the continuity of the Fourier transform and its inverse imply the assertion.

3. Let ψ be an arbitrary function in $\mathcal{E}^*$ such that

$$\psi(x) = \begin{cases} 1, & \text{if } |x| \ge 1; \\ 0, & \text{if } |x| \le 1/2 \end{cases}$$

and let $\psi_k(x) := \psi(kx)$ for $x \in \mathbf{R}$.

Fix $\theta \in \mathcal{S}^*(\mathbf{R}^d)$ and put $\kappa := \mathcal{F}^{-1}(\theta)$. Then $\kappa\psi_k \in \mathcal{S}^*(\mathring{\mathbf{R}}^d)$ for $k \in \mathbf{N}$ and $\kappa\psi_k \to \kappa$ as $k \to \infty$ in $\mathcal{S}^*(\mathring{\mathbf{R}}^d)$. This implies that $\mathcal{F}(\kappa\psi_k) \in \mathcal{S}^*_{\mathcal{F}}(\mathring{\mathbf{R}}^d)$ for $k \in \mathbf{N}$ and $\mathcal{F}(\kappa\psi_k) \to \theta$ as $k \to \infty$ in $\mathcal{S}^*(\mathbf{R}^d)$, which completes the proof of Theorem 4.1.

We define now the singular integral with an even kernel of a given tempered ultradistribution $f \in \mathcal{S}'^*$ as an element of $(\mathcal{S}^*_{\mathcal{F}}(\overset{\circ}{\mathbf{R}}{}^d))'$ by

$$\langle \mathbf{K} f, \theta \rangle := \langle f, \mathcal{K}\theta \rangle, \quad \theta \in \mathcal{S}^*_{\mathcal{F}}(\overset{\circ}{\mathbf{R}}{}^d). \tag{23}$$

It is easy to prove the following theorem:

THEOREM 4.2. *Suppose that K is an even kernel of the form (20) with the mentioned properties. Then the mapping $\mathbf{K} : \mathcal{S}'^*(\mathbf{R}^d) \to (\mathcal{S}^*_{\mathcal{F}}(\overset{\circ}{\mathbf{R}}{}^d))'$ defined by (23) is linear, continuous and injective.*

Now consider the operator $\mathcal{K}$ on $\mathcal{D}^*_{L^2}(\mathbf{R}^d)$.

THEOREM 4.3. *If K is an even kernel, then the mapping*

$$\mathcal{D}^*_{L^2} \ni \varphi \mapsto \mathcal{K}\varphi \in \mathcal{D}^*_{L^2}$$

is continuous.

PROOF. By [16], Ch. VI (Theorem 3.1), the mapping

$$L^2 \ni \varphi \mapsto \mathcal{K}\varphi \in L^2$$

is well defined and there exists a constant $C > 0$, depending on the dimension d, but not on φ, such that

$$\|\mathcal{K}\varphi\|_{L^2} \le C\|\varphi\|_{L^2}.$$

If $\varphi \in \mathcal{D}^*_{L^2}$, then for each $\alpha \in \mathbf{N}_0^d$ we have

$$(\mathcal{K}\varphi)^{(\alpha)} = ((\mathrm{pv}K) * \varphi)^{(\alpha)} = (\mathrm{pv}K) * \varphi^{(\alpha)} = \mathcal{K}(\varphi^{(\alpha)}),$$

in the sense of distributions. Since the functions $((\mathrm{pv}K) * \varphi)^{(\alpha)}$ and $(\mathrm{pv}K) * \varphi^{(\alpha)}$ are smooth, they are equal in the space L^2. Therefore there exists a constant $C > 0$, which does not depend neither on α or on φ, such that

$$\|(\mathcal{K}\varphi)^{(\alpha)}\|_{L^2} \le C\|\varphi^{(\alpha)}\|_{L^2}.$$

This implies that

$$\|\mathcal{K}\varphi\|_{\mathcal{D}^*_{L^2}} \le C\|\varphi\|_{\mathcal{D}^*_{L^2}}.$$

Using relations (12) -(14) in the d-dimensional version (see Remark 3.3) and Theorem 4.2, we define, for $a = (a^1, \dots, a^d) \in \mathbf{R}^d_+$ and $(a_p) \in \mathcal{R}^d$ with $a_p = ({a_p}^1, \dots, {a_p}^d)$ for $p \in \mathbf{N}$ (where $(a^j_p) \in \mathcal{R}$ for $j = 1, \dots, d$), the transforms $\mathbf{K}_a$ and $\mathbf{K}_{(a_p)}$ which act from the spaces $\mathcal{S}'^{(M_p)}(\mathbf{R}^d)$ and $\mathcal{S}'^{\{M_p\}}(\mathbf{R}^d)$ into themselves, respectively, by the formulas:

$$\langle \mathbf{K}_a f, \varphi \rangle = \langle f, \mathcal{K}_a \varphi \rangle, \qquad \mathcal{K}_a \varphi = P_a^{-1} \mathcal{K} P_a \varphi$$

for $f \in \mathcal{S}'^{(M_p)}(\mathbf{R}^d)$ and $\varphi \in \mathcal{S}^{(M_p)}(\mathbf{R}^d)$, and

$$\langle \mathbf{K}_{(a_p)} f, \varphi \rangle = \langle f, \mathcal{K}_{(a_p)} \varphi \rangle, \qquad \mathcal{K}_{(a_p)} \varphi = P_{(a_p)}^{-1} \mathcal{K} P_{(a_p)} \varphi$$

for $f \in \mathcal{S}'^{\{M_p\}}(\mathbf{R}^d)$ and $\varphi \in \mathcal{S}^{\{M_p\}}(\mathbf{R}^d)$.

Theorem 4.3 implies the following property.

COROLLARY 4.4. *The transform $\mathbf{K}_a$ (the transform $\mathbf{K}_{(a_p)}$) is a continuous linear mapping of the space $\mathcal{S}^{(M_p)}(\mathbf{R}^d)$ (resp. the space $\mathcal{S}^{\{M_p\}}(\mathbf{R}^d)$) onto itself.*

Let us remark that, in general, we do not have unicity type result (up to ultrapolynomials) as in the case of Hilbert transform for different a, $b \in \mathbf{R}^d_+$ and (a_p), $(b_p) \in \mathcal{R}^d$, respectively.

Remark 4.1. If $d = 1$ and $\Omega(t) = \operatorname{sgn} t/t$, then formula (23) defines the Hilbert transform on $\mathcal{S}'^*$. Notice that for every $a > 0$ and $(a_p) \in \mathcal{R}$, we have

$$\mathbf{H}_a f|_{\mathcal{S}_{\mathcal{F}}^{(M_p)}(\overset{\circ}{\mathbf{R}}{}^d)} = \mathbf{K}f, \qquad \mathbf{H}_{(a_p)} f|_{\mathcal{S}_{\mathcal{F}}^{\{M_p\}}(\overset{\circ}{\mathbf{R}}{}^d)} = \mathbf{K}f$$

for $f \in \mathcal{S}'^*(\mathbf{R}^d)$, where $\mathbf{K}$ is defined via $K(t) := 1/\pi t$.

Remark 4.2. An important class of singular integral operators with odd kernels is the one consisting of the Riesz transforms. In d dimensions these are d singular integral operators $R_1, R_2, \ldots, R_d$ defined by the kernels

$$K_j(x) := \frac{\Gamma((d+1)/2)}{\pi^{(d+1)/2}} \frac{x_j}{|x|^{d+1}}$$

for $j = 1, 2, \ldots, d$, where $x = (x_1, \ldots, x_d) \in \mathbf{R}^d$.

It is well known that $\sum_{j=1}^d R_j^2 = -I$ on $L^2(\mathbf{R}^d)$. We can apply Corollary 4.3 to

$$\mathcal{K}_{a,j} = T_a^{-1} R_j T_a, \qquad \mathcal{K}_{(a_p),j} = T_{(a_p)}^{-1} R_j T_{(a_p)}$$

with $a \in \mathbf{R}^d_+$ and $(a_p) \in \mathcal{R}^d$, respectively, for $j = 1, 2, \ldots, d$. Put

$$\mathbf{K}^2_{a,j} := \mathbf{K}_{a,j} \circ \mathbf{K}_{a,j}, \qquad \mathbf{K}^2_{(a_p),j} := \mathbf{K}_{(a_p),j} \circ \mathbf{K}_{(a_p),j}$$

for $j = 1, 2, \ldots, d$. If $f \in \mathcal{S}'^{(M_p)}$ and $\varphi \in \mathcal{S}^{(M_p)}$ with $\operatorname{supp} \varphi \subset \mathbf{R}^d \setminus \{0\}$, then

$$\sum_{j=1}^d \langle \mathcal{F}\mathbf{K}^2_{a,j} f, \varphi\rangle = \sum_{j=1}^d \langle f, T_a^{-1} {R_j}^2 T_a \hat{\varphi}\rangle = -\langle f, \hat{\varphi}\rangle = \langle \hat{f}, \varphi\rangle.$$

Thus if $a, b \in \mathbf{R}^d_+$ and $a \neq b$, then

$$\sum_{j=1}^d (\mathbf{K}^2_{a,j} - \mathbf{K}^2_{b,j}) f = P,$$

where P is an ultrapolynomial of the class (M_p). Analogously if $(a_p), (b_p) \in \mathcal{R}^d$ and $(a_p) \neq (b_p)$, then

$$\sum_{j=1}^d (\mathbf{K}^2_{(a_p),j} - \mathbf{K}^2_{(b_p),j}) f = P,$$

where P is an ultrapolynomial of the class $\{M_p\}$.

Remark 4.3. Let

$$\Omega(x/|x|) := \sum_{k=1}^m Y_k(x/|x|),$$

where Y_k are spherical harmonics of degree k, the kernel $K(x) := \Omega(x')/|x|$ is an example of a singular integral with an even kernel.

Other examples can be deduced on the base of Theorem 4.7 in [19], Ch. IV.

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