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CONVOLUTION STRUCTURE OF (GENERALIZED) HERMITE TRANSFORMS

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1. Introduction. In this paper we present a survey on the investigation of the convolution structure of the Hermite transform

$$\mathfrak{H}[f](n) = \int_{-\infty}^{\infty} f(x)\tilde{H}_n(x)e^{-x^2}dx, \tag{1}$$

where $\tilde{H}_n$ are the standardized Hermite polynomials, defined by

$$\tilde{H}_{2n}(x) = H_{2n}(x)/H_{2n}(0), \quad \tilde{H}_{2n+1}(x) = H_{2n+1}(x)/DH_{2n+1}(0), \quad n \in \mathbb{N}_0, \tag{2}$$

D is the operator of differentiation and

$$H_n(x) = (-1)^n e^{x^2} D^n(e^{-x^2}), \quad n \in \mathbb{N}_0, \tag{3}$$

are the (ordinary) Hermite polynomials ($\mathbb{N}_0$ is the set of nonnegative integers). They are connected with the standardized Laguerre polynomials of order α, namely

$$R_n^{\alpha}(x) = L_n^{\alpha}(x)/L_n^{\alpha}(0), \quad n \in \mathbb{N}_0, \quad \alpha > -1 \tag{4}$$

(L_n^{α} are the Laguerre polynomials of order α) by means of

$$\begin{cases} \tilde{H}_{2n}(x) = R_n^{-1/2}(x^2) \\ \tilde{H}_{2n+1}(x) = xR_n^{1/2}(x^2), \quad n \in \mathbb{N}_0. \end{cases} \tag{5}$$

A basis for the construction of a convolution of the Hermite transform are product formulas for Hermite polynomials and they are based because of (5) on product formulas for Laguerre polynomials derived first by Watson [13].

In form of the standardized Laguerre polynomials R_n^{α} the result of Watson can be

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written as follows:

$$
(6)\quad \begin{aligned} R_n^\alpha(x)\,R_n^\alpha(y) &= \frac{2^\alpha\Gamma(\alpha+1)}{\sqrt{2\pi}}\int_0^\pi R_n^\alpha(x+y+2\sqrt{xy}\cos\vartheta)e^{-\sqrt{xy}\cos\vartheta}\,\cdot \\ &\quad \cdot\,\frac{J_{\alpha-1/2}(\sqrt{xy}\sin\vartheta)}{(\sqrt{xy}\sin\vartheta)^{\alpha-1/2}}\sin^{2\alpha}\vartheta\,d\vartheta,\quad \alpha>-1/2. \end{aligned}
$$

Since it was established only for $\alpha > -1/2$ one can derive product formulas only for Hermite polynomials of odd degree (see (5)). It was used in 1968 by Debnath [3] to define a convolution for the Hermite transform of odd functions. For a long time there was no progress.

First attempts were given in 1983, see [5], where a product formula for Hermite functions

$$H_z(t) = 2^z G(-z/2, 1/2; t^2),\quad t\in\mathbb{R}^+,\quad z\in\mathbb{C}$$

was proved. Here $\mathbb{R}^+$ denotes the set of positive real numbers, $\mathbb{C}$ the set of complex numbers. G is the confluent hypergeometric function of the second kind. It is identical with the Hermite polynomials if $z = n \in \mathbb{N}_0$. The product formula could be proved only for $Re(z) < 0$ and so the specialization to this case was not possible. In 1984 (see [6]) the problem was solved for all classical orthogonal polynomials, but only in suitable chosen spaces of distributions.

A first solution of the problem was given by Dimovski and Kalla in 1988, see [4] in a somewhat artifical construction (see section 2).

The solution of the problem in final form was given by Markett [11]. At first he established a convolution theorem for a generalized Hermite transform, with the generalized Hermite polynomials $H_n^{(\mu)}$ in the kernel, using their connection with the Laguerre polynomials and formula (6), see sections 3 to 6. Then by means of the extension of the product formula (6) to the critical case $\alpha = -1/2$ in [10] he succeeded to obtain also a convolution for the (ordinary) Hermite transform (1), see section 7.

2. A first solution. In [3] Debnath defined a convolution for the Hermite transform (1), but only for odd functions. Let f, g be odd functions. Then Debnath defined the convolution for the transform (1) (after some corrections given in [4] and after some changes because of the use of standardized Hermite polynomials $\tilde{H}_n$) as

$$
(7)\quad \begin{aligned} (f \overset{o}{*} g)(t) &= t\int_{-\infty}^{\infty}\left(\int_0^\pi \frac{f([x^2+t^2+2xt\cos\vartheta]^{1/2})}{(x^2+t^2+2xt\cos\vartheta)^{1/2}}e^{-xt\cos\vartheta}\,\cdot\right. \\ &\quad \left.\cdot\, J_0(xt\sin\vartheta)\sin\vartheta\,d\vartheta\right)g(x)xe^{-x^2}\,dx, \end{aligned}
$$

where J_ν is the Bessel function of first kind and order ν.

Now let f, g be even functions. Then Dimovski and Kalla introduced in [4] the integral operator l by

$$(lf)(x) = \int_0^x f(t)dt$$

and defined the convolution for the Hermite transform (1) by

$$f \overset{e}{*} g = D(lf \overset{o}{*} lg).$$

In general case it is used that every function f defined on $\mathbb{R}$ can be written as a sum of an odd function f_0 and an even function f_e. Then the convolution is defined by means of

$$f * g = f_0 \overset{o}{*} g_0 + f_e \overset{e}{*} g_e, \tag{8}$$

in easily understandable notation. Then in [4] it is proved:

THEOREM 2.1. *Let f, g be locally integrable on $\mathbb{R}$ and of order $O(e^{ax^2})$ as $|x| \to +\infty$ with $a < 1$. Then there exists the Hermite transform (1) of $f * g$, defined by formula (8), and*

$$\mathfrak{H}[f * g] = \mathfrak{H}[f]\,\mathfrak{H}[g]. \tag{9}$$

3. Generalized Hermite Polynomials (GHP). In [12], Problems and Exercises 25, Szegö introduced generalized Hermite polynomials which was investigated by Chihara [1], see also [2]. By means of the Laguerre polynomials they can be defined as follows:

$$H_n^{(\mu)}(x) = \begin{cases} (-1)^k 2^{2k} k!\, L_k^{(\mu-1/2)}(x^2), & n = 2k \\ (-1)^k 2^{2k+1} k!\, x L_k^{(\mu+1/2)}(x^2), & n = 2k+1, \end{cases} \tag{10}$$

$$\mu > -1/2, \ k \in \mathbb{N}_0.$$

In another standardization, see [11], we write

$$\tilde{H}_n^{(\mu)}(x) = \begin{cases} H_{2k}^{(\mu)}(x)/H_{2k}^{(\mu)}(0), & n = 2k \\ H_{2k+1}^{(\mu)}(x)/DH_{2k+1}^{(\mu)}(0), & n = 2k+1 \end{cases} \tag{11}$$

and from (4) we obtain

$$\tilde{H}_n^{(\mu)}(x) = \begin{cases} R_k^{(\mu-1/2)}(x^2), & n = 2k \\ x R_k^{(\mu+1/2)}(x^2), & n = 2k+1. \end{cases} \tag{12}$$

Obviously $H_n^{(0)} = H_n$ and $\tilde{H}_n^{(0)} = \tilde{H}_n$, see (2), (3) and (5).

The GHP are solutions of second order linear differential equations, which are different for polynomials of odd resp. even degree (if $\mu \neq 0$):

$$xy''(x) + (2\mu - x^2)y'(x) + (2nx - \Theta_n x^{-1}) \ \ y(x) = 0, \tag{13}$$

with

$$\Theta_n = \begin{cases} 0, & n = 2k \\ 2\mu, & n = 2k+1. \end{cases} \tag{14}$$

The spectral analysis of these polynomials was given by Krall in [9]. The following has been proved:

THEOREM 3.1. $\{\tilde{H}_n^{(\mu)}\}_{n=0}^{\infty}$, $\mu > -1/2$ *forms a complete orthogonal system (COS) on $\mathbb{R}$ with weight*

$$\varrho^{(\mu)}(x) = |x|^{2\mu} e^{-x^2}. \tag{15}$$

More precisely we have

$$\int_{-\infty}^{\infty} \tilde{H}_n^{(\mu)}(x) \tilde{H}_m^{(\mu)}(x) \varrho^{(\mu)}(x)\, dx = \tilde{h}_n^{(\mu)} \delta_{nm}, \tag{16}$$

where

$$\tilde{h}_n^{(\mu)} = \begin{cases} \Gamma(\mu+1/2)k!/(\mu+1/2)_k, & n=2k \\ \Gamma(\mu+3/2)k!/(\mu+3/2)_k, & n=2k+1, \end{cases} \tag{17}$$

and $(a)_0 = 1,\ (a)_k = a(a+1)\dots(a+k-1)$ *is the Pochhammer symbol.*

There exist recurrences, differentiation formulas etc., see Chihara [1], [2].

4. Generalized Hermite transform (GHT)

Definition 4.1. The GHT is defined by means of

$$\mathfrak{H}^{(\mu)}[f](n) = f^\wedge(n) = \int_{-\infty}^{\infty} f(x)\tilde{H}_n(x)\varrho^{(\mu)}(x)\,dx, \quad \mu > -1/2,\ n \in \mathbb{N}_0 \tag{18}$$

provided that the integral exists.

Remark. Move precisely we have to write $f_\mu^\wedge(n)$, but we omit the letter μ, since in this paper μ is some arbitrary fixed number.

As space of originals we choose

$$L_{p,(\mu)}(\mathbb{R}) = L_{p,(\mu)} = \{f:\ f \text{ measurable on } \mathbb{R}, \|f\|_{p,(\mu)} < \infty\}, \tag{19}$$

where

$$\|f\|_{p,(\mu)} = \begin{cases} \left\{\displaystyle\int_{-\infty}^{\infty} |f(x)e^{-x^2/2}|^p |x|^{2\mu} dx\right\}^{1/p}, & 1 \le p < \infty \\ \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)e^{x^2/2}|, & p=\infty. \end{cases} \tag{20}$$

Obviously we have:

Theorem 4.1. *If* $f \in L_{p,(\mu)}$ *then* $\mathfrak{H}^{(\mu)}[f]$ *exists and it is a linear transformation.*

Since $\{\tilde{H}_n^{(\mu)}\}_{n=0}^\infty$ form a COS on $L_{2,(\mu)}$ inversion by series expansion in $L_{2,(\mu)}$ with respect to this COS is possible. Many rules of operational calculus can be derived using properties of the GHP, such as differentiation rules and others. We consider here further only the convolution theorem.

5. Product formulas. The starting point for product formulas for GHP is the linearization formula for the product of Laguerre polynomials (6). Substituting

$$z(\vartheta) = x + y + 2\sqrt{xy}\cos\vartheta$$

after some calculations we obtain the kernel form of (6):

$$R_n^\alpha(x)R_n^\alpha(y) = \int_0^\infty R_n^\alpha(z)K_L^\alpha(x,y,z)z^\alpha e^{-z}dz, \quad \alpha > -1/2, \tag{21}$$

where

$$K_L^\alpha(x,y,z) = \begin{cases} \dfrac{2^{\alpha-1}\Gamma(\alpha+1)}{\sqrt{2\pi}(xyz)^\alpha}\exp\left(\dfrac{x+y+z}{2}\right)J_{\alpha-1/2}(\varrho(x,y,z)[\varrho(x,y,z)]^{\alpha-1/2} \\ \qquad\qquad \text{if } z \in ([\sqrt{x}-\sqrt{y}]^2,\ [\sqrt{x}+\sqrt{y}]^2) \\ 0 \qquad \text{elsewhere} \end{cases} \tag{22}$$

and

$$\varrho(x,y,z) = \frac{1}{2}[2(xy+xz+yz) - x^2 - y^2 - z^2]^{1/2}. \tag{23}$$

Setting

$$\tilde{J}_\nu(z) = 2^\nu \Gamma(\nu+1) J_\nu(z), \quad \nu > -1 \tag{24}$$

$$\Delta(x,y,z) = \varrho(x^2,y^2,z^2) \tag{25}$$

and

$$S(x,y) = \Big(-|x|-|y|, -\Big||x|-|y|\Big|\Big) \cup \Big(\Big||x|-|y|\Big|, |x|+|y|\Big), \tag{26}$$

using the connection (12) between GHP and Laguerre polynomials, Markett in [11] derived a linearization formula for GHP.

THEOREM 5.1. *Let* $\mu \in \mathbb{R}^+,\ n \in \mathbb{N}_0,\ x,y \in \mathbb{R}\setminus\{0\}$. *Then*

$$\tilde{H}_n^{(\mu)}(x)\tilde{H}_n^{(\mu)}(y) = \int_{-\infty}^{\infty} \tilde{H}_n^{(\mu)}(z) K_H^{(\mu)}(x,y,z)\varrho^{(\mu)}(z)dz, \tag{27}$$

where

$$K_H^{(\mu)}(x,y,z) = \begin{cases} \mathfrak{K}_H^{(\mu)}(x,y,z)e^{(x^2+y^2+z^2)/2}, & z \in S(x,y) \\ 0\ , & z \in \mathbb{R}\setminus S(x,y) \end{cases} \tag{28}$$

and

$$\mathfrak{K}_H^{(\mu)}(x,y,z) = \frac{\Gamma(\mu+1/2)\Delta^{2\mu-2}}{2\sqrt{\pi}\Gamma(\mu)|xyz|^{2\mu-1}}\left[\tilde{J}_{\mu-1}(\Delta) + \frac{2\mu+1}{2\mu}\,\frac{\Delta^2}{xyz}\tilde{J}_\mu(\Delta)\right]. \tag{29}$$

REMARKS. 1. Important for the following is the symmetrization of the support of the kernel $K_H^{(\mu)}$.

2. Obviously the product on the left of (27) equals the GHT of $K_H^{(\mu)}$.

From (28) resp. (27) with $n=0$ we have immediately

CONCLUSION. (a) The kernel $K_H^{(\mu)}$ is symmetrical with respect to x,y,z.

(b) We have

$$\int_{-\infty}^{\infty} K_H^{(\mu)}(x,y,z)\varrho^{(\mu)}(z)dz = 1. \tag{30}$$

Moreover Markett proved:

THEOREM 5.2. *There exists no value of* $\mu,\ \mu \in \mathbb{R}^+$ *such that* $K_H^{(\mu)}$ *is nonnegative for all* $x,y \in \mathbb{R}\setminus\{0\},\ z \in S(x,y)$.

6. Convolution structure. According to a general method one defines a generalized translation operator (GTO) of the GHT by means of Theorem 5.1 in the following manner:

DEFINITION 6.1. As GTO for GHT we define the operator $T_y^{(\mu)}$, $y \in \mathbb{R},\ \mu \in \mathbb{R}^+$ by means of

$$(T_y^{(\mu)}f)(x) = \int_{-\infty}^{\infty} f(z)K_H^{(\mu)}(x,y,z)\varrho^{(\mu)}(z)dz, \quad xy \neq 0 \tag{31}$$

and

$$(T_0^{(\mu)}f)(x) = \frac{1}{2}[f(x)+f(-x)], \quad (T_y^{(\mu)}f)(0) = \frac{1}{2}[f(y)+f(-y)], \tag{32}$$

provided that the integral in (31) exists.

In [11] it was proved:

THEOREM 6.1. *(a) If $\mu \geq 1/2$ then the GTO is a bounded linear operator from $L_{p,(\mu)}$ into itself with*

$$\|T_y^{(\mu)} f\|_{p,(\mu)} \leq M_\mu e^{y^2/2} \|f\|_{p,(\mu)}, \quad y \in \mathbb{R}.$$

(b) If $0 < \mu < 1/2$ one has for any $f \in L_{1,(\mu)} \cap L_{1,(\mu/2+1/4)}$

$$\|T_y^{(\mu)} f\|_{p,(\mu)} \leq M_\mu e^{y^2/2} \left[\|f\|_{1,(\mu)} + |y|^{1/2-\mu} \|f\|_{1,(\mu/2+1/4)} \right]$$

From Definition 6.1 and by means of the results of section 5 one proves straightforwardly:

PROPOSITION 6.1. *The GTO has the proporties*

$$(T_y^{(\mu)} f)(x) = (T_x^{(\mu)} f)(y), \tag{33}$$

$$(T_y^{(\mu)} \tilde{H}_n^{(\mu)})(x) = \tilde{H}_n^{(\mu)}(x) \tilde{H}_n^{(\mu)}(y) \tag{34}$$

and

$$(T_y^{(\mu)} f)^\wedge(n) = f^\wedge(n) \tilde{H}_n^{(\mu)}(y). \tag{35}$$

REMARK 1. The GTO is not a positive one, see Theorem 5.2.

As usual the convolution of the GHT is defined as follows

DEFINITION 6.2. As convolution of the GHT one defines

$$(f * g)(y) = \int_{-\infty}^{\infty} (T_y^{(\mu)} f)(x) g(x) \varrho^{(\mu)}(x) dx, \tag{36}$$

provided that the integral exists.

By means of Theorem 6.1 Markett proved in [11]:

THEOREM 6.2. *(a) Let $\mu \geq 1/2$, $f \in L_{p,(\mu)}$, $g \in L_{1,(\mu)}$. Then $f * g \in L_{p,(\mu)}$.*

(b) Let $0 < \mu < 1/2$, $f \in L_{1,(\mu)}$, $g \in L_{1,(\mu)} \cap L_{1,(\mu/2+1/4)}$. Then $f * g \in L_{1,(\mu)}$.

(c) Under the conditions of (a) or (b) we have

$$(f * g)^\wedge = f^\wedge g^\wedge. \tag{37}$$

REMARKS. 2. The convolution appears to be commutative and associative.

3. The convolution is for no value of $\mu \in \mathbb{R}^+$ a positive one, because the GTO is also not a positive one for any value of $\mu \in \mathbb{R}^+$, see Remark 1.

7. Limit case. In case $\mu = 0$ of the ordinary Hermite transform the product formula (27) of GHP is not correct in case of even degree n of the polynomials, since they are connected with the Laguerre polynomials of order $-1/2$, see (5), and the product formula (21) for Laguerre polynomials works only for polynomials of order $\alpha > -1/2$. The extension of formula (21) to the limit case $\alpha = 1/2$ was obtained by Markett in [10]. His proof uses a proof of J. Boersma in 1961, which was never published before. The result is

$$\begin{aligned} R_n^{-1/2}(x) R_n^{-1/2}(y) &= \tfrac{1}{2} \Big\{ e^{-\sqrt{xy}} R_n^{-1/2}([\sqrt{x} + \sqrt{y}]^2) + e^{\sqrt{xy}} R_n^{-1/2}([\sqrt{x} - \sqrt{y}]^2) \\ &\quad - \int_0^\infty R_n^{-1/2}(z) K_L^{-1/2}(x,y,z) e^{-z} z^{-1/2} dz \Big\}, \\ &\qquad x, y \in \mathbb{R}^+, \quad n \in \mathbb{N}_0, \end{aligned} \tag{38}$$

with

$$K_L^{-1/2}(x,y,z) = \begin{cases} \frac{1}{2}(xyz)^{1/2}\exp\left(\frac{x+y+z}{2}\right) J_1(\varrho(x,y,z))[\varrho(x,y,z)]^{-1} \\ \qquad\qquad \text{if } z \in \left([\sqrt{x}-\sqrt{y}]^2,\ [\sqrt{x}+\sqrt{y}]^2\right) \\ 0 \qquad \text{elsewhere} \end{cases} \tag{39}$$

where ϱ is taken from (23).

By means of (5) and of the product formula (21) in case $\alpha = 1/2$ for Hermite polynomials of odd degree resp. (38) for Hermite polynomials of even degree Markett proved in [11] the extension of the product formula (27) to the limit case $\mu = 0$:

$$\begin{aligned} \tilde{H}_n(x)\tilde{H}_n(y) &= \frac{1}{4}\{e^{-xy}[\tilde{H}_n(-x-y)+\tilde{H}_n(x+y)]+ \\ &\quad + e^{xy}[\tilde{H}_n(y-x)+\tilde{H}_n(x-y)]\} + \\ &\quad + \int_{-\infty}^{\infty}\tilde{H}_n(z)K_H^{(0)}(x,y,z)e^{-z^2}dz, \quad x,y\in\mathbb{R}, \quad n\in\mathbb{N}_0, \end{aligned} \tag{40}$$

where

$$K_H^{(0)}(x,y,z) = \begin{cases} -\frac{xyz}{8}\tilde{J}_1(\Delta) + \frac{1}{4}\tilde{J}_0(\Delta)sgn(xyz)e^{(x^2+y^2+z^2)/2}, & z\in S(x,y) \\ 0 \qquad \text{elsewhere.} \end{cases} \tag{41}$$

The notations Δ, S and $\tilde{J}_\nu$ can be taken from (25), (26) and (24) respectively.

The additional term $\{\}$ in (40) vanishes obviously if n is odd.

The conclusion of Theorem 5.1 and Theorem 5.2 are also valid in case $\mu = 0$.

The GTO $T_y^{(0)} = T_y$ is according to this product formula defined by

$$\begin{aligned} (T_yf)(x) &= \frac{1}{4}\{e^{-xy}[f(-x-y)+f(x+y)] + e^{xy}[f(y-x)+f(x-y)]\} + \\ &\quad + \int_{-\infty}^{\infty} f(z)K_H^{(0)}(x,y,z)e^{-z^2}\,dz. \end{aligned} \tag{42}$$

It has the same properties as in case $\mu > 0$, especially Theorem 6.1, (b) is valid also in case $\mu = 0$ and the GTO T_y is not a positive one. The definition of the convolution follows the same line as in case $\mu > 0$, especially Theorem 6.2, (b) is valid also in case $\mu = 0$ and (37) is fulfilled. The convolution is of course not a positive one.

8. Concluding remarks. The history of the investigation of the convolution structure of the Hermite transform, which was finished by the paper [11] of Markett in 1993, is in some sense typical for mathematics nowadays. People interested in operational calculus did not take notice of the extension of Watson's product formula for Laguerre polynomials of order $\alpha > -1/2$ to the limit case $\alpha = -1/2$ in 1982, see [10], since it appeared in connection with approximation theory. So the papers [5], [6] and [4] appeared. Astonishing enough that Markett only eleven years after his extension of Watson's product formula made his application to the Hermite transform. Moreover introducing at first a parameter μ and considering the GHP $H_n^{(\mu)}$, so a more general case, he succeeded to define the convolution of the limit case $\mu = 0$ of the (ordinary) Hermite polynomials in a quite natural manner.

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