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MEAN-PERIODIC OPERATIONAL CALCULI

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Abstract. Elements of operational calculi for mean-periodic functions with respect to a given linear functional in the space of continuous functions are developed. Application for explicit determining of such solutions of linear ordinary differential equations with constant coefficients is given.

1. Introduction. In 1935 J. Delsarte [1] introduced the notion of mean-periodic function with respect to a linear functional.

Let Φ be an arbitrary nonzero linear functional in $\mathcal{C}(-\infty;+\infty)$. The well known Riesz-Markov theorem implies the existence of a unique complex Radon measure α with compact support such that

$$\Phi\{f\} = \int_a^b f(t)d\alpha(t),$$

where $-\infty < a < b < +\infty$.

DEFINITION 1. A function $f \in \mathcal{C}(-\infty;+\infty)$ is said to be *mean-periodic* with respect to Φ, if

$$\Phi_x\{f(t+x)\} := \int_a^b f(t+x)d\alpha(x) = 0$$

for each $t \in (-\infty;+\infty)$.

The class of mean-periodic functions with respect to the functional Φ will be denoted by $\mathcal{M}_\Phi$.

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Later L. Schwartz [2], L. Ehrenpreis [3] and many others made exhaustible studies on different aspects of the mean-periodic functions. For more information we refer to C. Berenstein and B. A. Taylor [4].

It seems that till now there are no studies on these classes of functions from the point of view of the operational calculus and especially from J. Mikusiński's operational calculi viewpoint. Our aim here is to develop an operational calculus for an arbitrary class of mean-periodic functions.

Here we will include not only the classes considered by Delsarte [1, 5] and Schwartz [2], but also the classes of mean-periodic functions considered by A. Leont'ev [6].

Definition 2. A function $f \in \mathcal{C}([a_1; b_1])$ where $a_1 < a < b < b_1$ is said to be *mean-periodic* if

$$\int_a^b f(x+t)d\alpha(x) = 0$$

for $t \in (a_1 - a; b_1 - b)$.

Without any restriction of generality we may assume that $0 \in [a; b]$.

Futher we denote by $\mathcal{C}$ either $\mathcal{C}(-\infty; +\infty)$ or $\mathcal{C}([a_1; b_1])$.

In Mikusiński's operational calculi [7] the basic role is played by the integration operator

$$lf(t) = \int_0^t f(\tau)d\tau$$

in $\mathcal{C}[0; +\infty)$.

Here instead of l we consider a right inverse operator L of $\frac{d}{dt}$ which maps $\mathcal{M}_\Phi$ into $\mathcal{M}_\Phi$. The value Lf of L on f is defined as the solution of the following elementary non-local (in general) boundary value problem

$$y' = f, \quad \Phi\{y\} = 0 \quad \text{for} \quad f \in \mathcal{M}_\Phi.$$

It is easy to see that in order for such a solution y to exist it is necessary and sufficient that $\Phi\{1\} \neq 0$.

Then $y = lf + c$, $c = \text{const}$. Hence $\Phi\{y\} = \Phi\{lf\} + c\Phi\{1\} = 0$.

If $\Phi\{1\} \neq 0$ we can assume without loss of generality that $\Phi\{1\} = 1$. Further we consider only this case. The case $\Phi\{1\} = 0$ we will postpone for another publication.

If $\Phi\{1\} = 1$ then the right inverse operator of $\frac{d}{dt}$ we are interested in has the form

$$Lf = lf - \Phi\{lf\}. \tag{1}$$

Lemma 1. *The right inverse operator L is a mapping of $\mathcal{M}_\Phi$ into $\mathcal{M}_\Phi$.*

Proof. By (1) we have

$$Lf(t) = \int_0^t f(\tau)d\tau - \Phi\{lf\}.$$

We should prove that if $f \in \mathcal{M}_\Phi$, then

$$\phi(t) := \Phi_x\{(Lf)(t+x)\} = 0$$

for each $t \in (-\infty; +\infty)$.

It is easy to see that

$$\begin{aligned}(Lf)(t+x) &= \int_0^{x+t} f(\tau)d\tau - \Phi\{lf\}\\ &= \int_0^{x} f(\tau)d\tau + \int_x^{x+t} f(\tau)d\tau - \Phi\{lf\} = lf(x) + \int_x^{x+t} f(\tau)d\tau - \Phi\{lf\}.\end{aligned}$$

From this we have

$$\begin{aligned}\phi(t) = \Phi_x\{(Lf)(t+x)\} &= \Phi_x\left\{\int_0^{t+x} f(\tau)d\tau\right\} - \Phi\{lf\}\Phi\{1\}\\ &= \Phi\{lf\} + \Phi_x\left\{\int_x^{x+t} f(\tau)d\tau\right\} - \Phi\{lf\} = \Phi_x\left\{\int_x^{x+t} f(\tau)d\tau\right\}.\end{aligned}$$

Differentiating ϕ we obtain

$$\phi'(t) = \Phi_x\{f(t+x)\} = 0$$

which implies $\phi(t) = c = const$. Since $\phi(0) = \Phi_x\{\int_x^x f(\tau)d\tau\} = 0$, we get $c = 0$. This means that

$$\phi(t) = \Phi_x\{(Lf)(t+x)\} = 0 \qquad \text{for each} \quad t \in (-\infty; +\infty).$$

This statement completes the proof.

The next step is to find a convolution of L in $\mathcal{M}_\Phi$. By (1) the operator L is defined in the whole space $\mathcal{C}(-\infty; +\infty)$. In this case (see [8]) the operation

$$(f * g)(t) = \Phi_x\left\{\int_x^t f(t+x-\tau)g(\tau)d\tau\right\} \tag{2}$$

is a convolution of L in $\mathcal{C}(-\infty; +\infty)$ such that

$$Lf = \{1\} * f. \tag{3}$$

Now we will show that (2) is a convolution of L in $\mathcal{M}_\Phi$, too. This follows from the next more general result

THEOREM 1. *If $f \in \mathcal{M}_\Phi$ and $g \in \mathcal{C}$, then $f * g \in \mathcal{M}_\Phi$.*

PROOF. We should prove that

$$\phi(t) := \Phi_\tau\{(f*g)(\tau+t)\} = \Phi_\tau\{h(t+\tau)\} = 0 \quad \text{if} \quad h(t) = \Phi_x\left\{\int_x^t f(x+t-\tau)g(\tau)d\tau\right\}.$$

We obtain

$$\begin{aligned}\Phi_\tau\{h(t+\tau)\} &= \Phi_\tau\Phi_x\left\{\int_x^{t+\tau} f(x+t+\tau-\xi)g(\xi)d\xi\right\}\\ &= \Phi_\tau\Phi_x\left\{\int_x^{\tau} f(x+t+\tau-\xi)g(\xi)d\xi\right\}\\ &\quad + \Phi_\tau\Phi_x\left\{\int_\tau^{\tau+t} f(x+t+\tau-\xi)g(\xi)d\xi\right\}.\end{aligned}$$

The interchanging of x and τ gives

$$\Phi_\tau\Phi_x\left\{\int_x^{\tau} f(x+t+\tau-\xi)g(\xi)d\xi\right\} = \Phi_x\Phi_\tau\left\{-\int_x^{\tau} f(x+t+\tau-\xi)g(\xi)d\xi\right\}.$$

By changing the order of integration we get

$$\Phi_\tau \Phi_x \left\{ \int_x^\tau f(x+t+\tau-\xi) g(\xi) d\xi \right\} = 0.$$

On the other hand we have

$$\Phi_\tau \Phi_x \left\{ \int_\tau^{t+\tau} f(x+t+\tau-\xi) g(\xi) d\xi \right\} = \Phi_\tau \left\{ \int_\tau^{t+\tau} \Phi_x \{ f(x+t+\tau-\xi) \} g(\xi) d\xi \right\} = 0,$$

since $\Phi_x\{f(x+t+\tau-\xi)\} = 0$ due to $f \in \mathcal{M}_\Phi$.

Thus the theorem is proved.

Corollary. *The operation (2) is a convolution of L in $\mathcal{M}_\Phi$.*

It is necessary to note that L cannot be represented as a convolution operator in $\mathcal{M}_\Phi$. But the representation

$$Lf = \{1\} * f$$

is still true, despite the fact that the function constant $\{1\}$ is not an element of $\mathcal{M}_\Phi$.

2. Operational calculus for mean-periodic functions with respect to a given linear functional. Convolution (2) can be used for building a *Mikusiński's* type operational calculus only when the set of the non-zero nondivisors of 0 is nonempty. Since we cannot assert this in the general case, we are to follow an alternative approach. To this end we consider multiplier fractions instead of convolution fractions (see [8]).

Definition 3. A linear operator $A : \mathcal{M}_\Phi \to \mathcal{M}_\Phi$ is said to be a *multiplier* of the convolution algebra $(\mathcal{M}_\Phi, *)$ iff the relation

$$A(f * g) = (Af) * g$$

holds for all $f, g \in \mathcal{M}_\Phi$.

Here we cannot give a complete characterization of the multiplier operators in the general case, but it is easy to see that the set $\mathfrak{M}$ of all the multipliers of $(\mathcal{M}_\Phi, *)$ is nonempty. Evidently, the identity operator I of $\mathcal{M}_\Phi$ belongs to $\mathfrak{M}$. But there are other multipliers too, e.g. the operator L defined by (1) is a multiplier since $Lf = \{1\} * f$. Indeed $\{1\} \notin \mathcal{M}_\Phi$ but the relation $L(f * g) = (Lf) * g$ is true in $\mathcal{M}_\Phi$, since it is true in $\mathcal{C}$.

For further applications we are to distinguish some of the non-divisors of 0.

Definition 4. The entire function of exponential type

$$E(\lambda) = \Phi_\tau\{e^{\lambda\tau}\} \tag{4}$$

is said to be the *indicatrix* of the functional Φ.

It is well known that if the support of Φ is not reduced to a single point, then $E(\lambda)$ has infinitely many (denumerable) zeros

$$\lambda_1, \lambda_2, \cdots, \lambda_n, \cdots$$

with corresponding multiplicities

$$\kappa_1, \kappa_2, \cdots, \kappa_n, \cdots$$

According to our initial assumption $E(0) = 1$, i.e. $\lambda = 0$ is not a zero of $E(\lambda)$.

Let $\mathfrak{N}$ be the multiplicative set of all non-divisors of 0 with the exception of the zero operator.

LEMMA 2. *If $\lambda \in \mathbb{C}$ is an arbitrary complex number, then the element $I - \lambda L$ is a divisor of* 0 *iff $E(\lambda) = 0$.*

PROOF. Let $I - \lambda L$ be a divisor of 0 of (2). Then there exists a multiplier $A \in \mathfrak{M}$, $A \neq 0$, such that $(I - \lambda L)A = 0$.

Let $f \in \mathcal{M}_\Phi$ be such that $u = Af \neq 0$. Then

$$0 = [(I - \lambda L)A]f = (I - \lambda L)(Af) = u - \lambda Lu,$$

i.e.

$$u - \lambda Lu = 0.$$

If we apply the functional Φ to this equation, we obtain

$$\Phi\{u\} - \lambda\Phi\{Lu\} = 0.$$

But $\Phi\{Lu\} = 0$ and hence $\Phi\{u\} = 0$.

Differentiating the equation $u - \lambda Lu = 0$, we obtain

$$u' - \lambda u = 0 \quad \text{with} \quad \Phi\{u\} = 0.$$

Since $u(t) = ce^{\lambda t}$ with $c \neq 0$, we obtain at once $\Phi_\tau\{e^{\lambda\tau}\} = 0$, i.e. $E(\lambda) = 0$.

Conversely, let $E(\lambda) = 0$. Then it is easy to convert the consideration and to arrive at the conclusion that $I - \lambda L$ is a divisor of 0 in $\mathfrak{M}$.

LEMMA 3. *The convolution operators $A = f*$ where $f \in \mathcal{C}$ are multipliers of $(\mathcal{M}_\Phi, *)$.*

The proof follows immediately from Theorem 1 and from the fact that the operation $*$ is associative in $\mathcal{C}$.

LEMMA 4. *An operator $A = f*$, with $f \in \mathcal{C}$ is a non-divisor of* 0 *in $\mathfrak{M}$ if f is non-divisor of* 0 *of the convolution algebra $(\mathcal{C}, *)$.*

PROOF. Let $f \neq 0$ be a divisor of 0 of $(\mathcal{C}, *)$, i.e. there exists a $g \in \mathcal{C}$, such that

$$f * g = 0.$$

But $(f * g)* = (f*)(g*) = 0$ (where 0 is the zero operator in $\mathfrak{M}$). Hence $(f*)$ is a divisor of 0 in $\mathfrak{M}$.

Conversely, let $(f*)$ with $f \in \mathcal{C}$ is a divisor of 0 in $\mathfrak{M}$, i.e. there exists an operator $A \in \mathfrak{M}$, $A \neq 0$, such that

$$(f*)A = 0.$$

Let $g \in \mathcal{M}_\phi$ be such that $Ag \neq 0$. Then

$$(f*)(Ag) = 0 \Longleftrightarrow f * (Ag) = 0,$$

i.e. f is a divisor of 0 in $\mathcal{C}$.

DEFINITION 5. $\mathfrak{R} = \mathfrak{N}^{-1}\mathfrak{M}$ is the ring of the multiplier quotients $\frac{P}{Q}$, where $P \in \mathfrak{M}$ and $Q \in \mathfrak{N}$, when the convolution algebra $(\mathcal{M}_\phi, *)$ is *annihilators-free.*

We give sufficient conditions for $(\mathcal{M}_\phi, *)$ to be annihilatots-free.

LEMMA 5. *If* $\mathcal{C} = \mathcal{C}([a_1; b_1])$ *or if* Φ *is a multipoint functional, i.e. a functional of the form*

$$\Phi\{f\} = \sum_{k=1}^{m} \alpha_k f(\beta_k) \tag{6}$$

with

$$\beta_1 < \beta_2 < \cdots < \beta_m,$$

then $(\mathcal{M}_\Phi, *)$ *is annihilators-free.*

PROOF. The first part follows from the Theorem of L. Schwartz and A. Leont'ev about the totality of the spectral projectors (see [9]).

The second part follows from the fact that for a multipoint functional (6) we can give a characterization of the elements of $\mathcal{M}_\Phi$ as unique continuations of functions on $(-\infty; +\infty)$, defined in $\mathcal{C}([\beta_1; \beta_m])$ (see [6]).

In other words in this special case, there is a bijective correspondence between $\mathcal{C}([\beta_1; \beta_m])$ and $\mathcal{M}_\Phi$, which is an algebra isomorphism, too. But $(\mathcal{C}[\beta_1; \beta_m], *)$ is annihilators-free and hence $(\mathcal{M}_\Phi, *)$ is annihilators-free, too.

EXAMPLE. The anti-periodic functions with anti-period 1, defined on $(-\infty; +\infty)$ and satisfying the condition

$$f(t+1) = -f(t)$$

are mean-periodic functions with respect to the functional

$$\Phi\{f\} = \frac{1}{2}[f(0) + f(1)].$$

The corresponding convolution has the form

$$(f * g)(t) = \frac{1}{2}\int_0^t f(t-\tau)g(\tau)d\tau - \frac{1}{2}\int_t^1 f(1+t-\tau)g(\tau)d\tau.$$

In order to obtain a continuous anti-periodic function from a given continuous function $f(x)$ on $[0; 1]$ one should impose the additional restriction $f(0) = f(1) = 0$. This restriction by no means is essential for our consideration. We could consider partially continuous instead of continuous functions as in [6].

For simplicity we denote the identity operator of $\mathfrak{M}$ by 1.

DEFINITION 6. $S = \frac{1}{L}$ is said to be the *algebraic differentiation operator.*

It is important to find the connection between the ordinary differentiation operator and the operator S.

THEOREM 2. *If* $f \in \mathcal{C}^1$, *then*

$$(f')* = S \cdot (f*) - \Phi\{f\}, \tag{7}$$

where $\Phi\{f\}$ *is to be understood as a numerical multiplier operator.*

PROOF. Equality (7) is equivalent to

$$Lf' = f - \Phi\{f\}$$

since $L = \{1\}$. Using the expression (1) it is easy to verify this relation.

Remark 1. The stars $(*)$ in (7) mean that the corresponding expressions are considered as elements of $\mathfrak{M}$, i.e. as operators. These stars can be avoided if we agree to use the notation fg for the convolution (2) as Mikusiński [7] did in his book. But he could not avoid considerations of a "numerical operator" $f(0)$ in the formula $f' = sf - f(0)$. The advantage of using the stars $(*)$ in (7) is the fact that all the terms are operators - elements of $\mathfrak{M}$.

Theorem 3. *If* $\lambda \in \mathbb{C}$ *and* $E(\lambda) \neq 0$*, then*

$$\frac{1}{S-\lambda} = \left\{ \frac{e^{\lambda t}}{E(\lambda)} \right\} * . \tag{8}$$

Proof. Using (7) we get

$$(S-\lambda)\{e^{\lambda t}\}* = S\{e^{\lambda t}\} * -\lambda\{e^{\lambda t}\}* = \{\lambda e^{\lambda t}\} * +\Phi_\tau\{e^{\lambda\tau}\} - \lambda\{e^{\lambda t}\}* = E(\lambda),$$

i.e.

$$(S-\lambda)\{e^{\lambda t}\}* = E(\lambda).$$

The last equality is equivalent to (8).

Corollary. *If* $E(\lambda) \neq 0$*, then*

$$\frac{1}{(S-\lambda)^k} = \left\{ \frac{1}{(k-1)!} \frac{\partial^{k-1}}{\partial \lambda^{k-1}} \left(\frac{e^{\lambda t}}{E(\lambda)} \right) \right\} * .$$

The proof of the above relation may be carried out by induction.

3. Applications. We will use the operational calculus just developed to find mean-periodic solutions of ordinary linear differential equations with constant coefficients, i.e. of equations of the form

$$P\left(\frac{d}{dt}\right) u = f, \tag{9}$$

where $P(\lambda) = a_o\lambda^n + a_1\lambda^{n-1} + \cdots + a_n$ is a given polynomial and f is a mean-periodic function with respect to a given functional Φ.

Theorem 4. *If none of the zeros of the polynomial* $P(\lambda)$ *is a zero of the indicatrix* $E(\lambda) = \Phi_\tau\{e^{\lambda\tau}\}$ *of the functional* Φ*, then the equation (9) has a unique solution as mean-periodic function with respect to the functional* Φ*.*

Proof. Let us assume that there exists a mean-periodic solution u of (9), i.e. such that

$$\Phi_\tau\{u(\tau+t)\} = 0.$$

Differentiating $(n-1)$ times the last equation, we obtain

$$\Phi_\tau\{u^{(k)}(\tau+t)\} = 0, \qquad k = 1, 2, \cdots, n-1.$$

If we take $t = 0$, we get

$$\Phi_\tau\{u^{(k)}(\tau)\} = 0, \qquad k = 1, 2, \cdots, n-1. \tag{10}$$

In [11], p. 63 it is shown that the unique solution of (9) with the non-local boundary value conditions (10) can be obtained by the operational calculus, developed in the previous Section 2. In our notations the problem can be reduced to

$$P(S)(u*) = f * . \tag{11}$$

The solution of (11) formally can be written as

$$u* = \frac{1}{P(S)} \cdot (f*).$$

It is easy to interpret it in a usual way. We can develop $\frac{1}{P(S)}$ into partial fractions:

$$\frac{1}{P(S)} = \sum_{k=1}^{m} \sum_{l=1}^{\kappa_k} \frac{A_{kl}}{(S-\mu_k)^l},$$

where $\mu_1, \cdots, \mu_m$ are the different zeros of $P(\lambda)$ with their corresponding multiplicities $\kappa_k, \quad k = 1, \cdots, m.$

It remains only to use (8) and (9) in order to obtain

$$u = \sum_{k=1}^{m} \sum_{l=1}^{\kappa_m} A_{kl} \Big\{ \frac{1}{(l-1)!} \frac{\partial^{l-1}}{\partial \lambda^{l-1}} \Big(\frac{e^{\lambda t}}{E(\lambda)} \Big) \Big\} * f. \tag{12}$$

We proved that if there is a mean-periodic solution of (11) with respect to a functional Φ, then it has the form (12). Conversely, due to Theorem 1, each term of (12) is a mean-periodic function with respect to Φ.

Thus we proved the existence and uniquness of a mean-periodic solution.

The case when some of the roots of $P(\lambda)$ are zeros of $E(\lambda)$ is the so called resonance case. Then in order to ensure the existence of a mean-periodic solution of (9) some restrictions on the right-hand side of (9) should be imposed (see [10], or [8]).

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