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EQUIVANISHING SEQUENCES OF MAPPINGS

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Abstract. Utilizing elementary properties of convergence of numerical sequences we prove Nikodym, Banach, Orlicz-Pettis type theorems.

1. Introduction. Nikodym type theorems on set functions; Banach and Banach-Steinhaus type theorems on families and sequences of mappings; Orlicz-Pettis type theorems on subseries convergence of series are basic theorems in set function theory and functional analysis. In modern text books they are usually proved by using the Baire category method. The method requires knowledge of elements of topology and functional analysis, which makes the theory difficult for average users.

A more recent approach consists in using sequential methods based on properties of convergence of sequences and matrices (double sequences) ([14], [1], [3], [4], [5], [19]). The sequential approach is not only simpler, but also more in agreement with the intuition of physicists, chemists, engineers and average users. The method is accessible for students at early stage of their studies. The simplicity of the method and its generality make it suitable for teaching mathematics at engineers type of schools. It should be also a good introduction to deeper study of mathematics.

The way of presenting the results goes from particularities to generalities. We do not pretend to formulate results in their possible generalities. The first part of the paper consists of the basic theorems. The second part is devoted to applications of the results in the first part. In the following subsection we give extensive summary of the paper.

1.1. *Summary.* We observe that a family M of countably additive set functions on a σ-field X to the real number system $\mathbb{R}$ is uniformly countably additive iff, for each sequence $\{f_i\}$ in M, $f_i(x_j) \to 0$ uniformly for $i \in \mathbb{N}$ as $j \to \infty$ whenever $\{x_j\}$ is a pairwise disjoint sequence of sets in X. Similarly, a family M of mappings on a topological group

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X is sequentially equicontinuous at zero iff, for each sequence $\{f_i\}$ in M, $f_i(x_j) \to 0$ uniformly for $i \in \mathbb{N}$ as $j \to \infty$ whenever $x_j \to 0$. Following these two examples we adopt the following definition.

Assume that $\{f_i\}$ is a sequence of mappings on a set X to $\mathbb{R}$ and $\{x_i\}$ is a sequence in X. We say that sequence $\{f_i\}$ is equivanishing on $\{x_i\}$ if $f_i(x_j) \to 0$ uniformly for $i \in \mathbb{N}$ as $j \to \infty$.

A family M of countably additive set functions on a σ-field X is uniformly countably additive iff each sequence $\{f_i\}$ in M is equivanishing on pairwise disjoint sequences in X; a family M of mappings on a topological group X is sequentially equicontinuous at zero iff, each sequence in M is equivanishing on sequences converging to zero in X. In view of the above examples we are interested in conditions under which a sequence $\{f_i\}$ of mappings on a set X to $\mathbb{R}$ is equivanishing on a sequence $\{x_i\}$ in X.

We note that if $\{f_i\}$ is a sequence of linear mappings on $\mathbb{R}$ to $\mathbb{R}$ and $x_i \to 0$, then the following condition holds:

($\mathcal{K}\{f_i\}$) For each subsequence $\{x_{m_i}\}$ of $\{x_i\}$ there exists a further subsequence $\{x_{n_i}\}$ of $\{x_{m_i}\}$ and x in X such that

$$\sum_{j=1}^{\infty} f_i(x_{n_j}) = f(x)$$

for each i in $\mathbb{N}$.

Assume that $\{f_i\}$ is a sequence of mappings on a set X to $\mathbb{R}$. We say that $\{x_i\}$ is a $K\{f_i\}$-null sequence in X if the condition ($\mathcal{K}\{f_i\}$) holds.

In general, the sequence $\{f_i\}$ is not equivanishing on $K\{f_i\}$-null sequences in X. For instance, sequence $\{f_i\}$ such that $f_i(x) = ix$ is not equivanishing on $K\{f_i\}$-null sequence $\{i^{-1}\}$. It appears that pointwise precompactness of $\{f_i\}$ and ($K\{f_i\}$) property of a sequence $\{x_i\}$ in X stand for sufficient conditions for $\{f_i\}$ to be equivanishing on $\{x_i\}$. More exactly, let $\{f_i\}$ be a sequence of mappings on a set X to $\mathbb{R}$. The sequence is said to be X-precompact if each its subsequence $\{f_{m_i}\}$ has an X-Cauchy subsequence $\{f_{n_i}\}$, i.e. for each subsequence $\{f_{v_i}\}$ of $\{f_{n_i}\}$ we have $f_{v_{i+1}}(x) - f_{v_i}(x) \to 0$ for each x in X as $i \to \infty$. In section 4.1 we prove the following

Equivanishing Theorem I. *Each X-precompact sequence $\{f_i\}$ of mappings on a set X to $\mathbb{R}$ is equivanishing on $K\{f_i\}$-null sequences.*

If $\mathsf{S}(X, \mathbb{R})$ is an admissible topological group of mappings on a set X to $\mathbb{R}$ (i.e., group operations in $\mathsf{S}(X, \mathbb{R})$ are defined pointwise and the topology for $\mathsf{S}(X, \mathbb{R})$ is finer than the topology of pointwise convergence), then sequentially precompact sequences in S are X-precompact. However, relatively countably compact sequences in $\mathsf{S}(X, \mathbb{R})$ may not be X-precompact, and conversely, X-precompact sequences of $\mathsf{S}(X, \mathbb{R})$ may not be relatively countably compact. A sequence $\{f_i\}$ is relatively countably compact in $\mathsf{S}(X, \mathbb{R})$ if each its subsequence $\{f_{m_i}\}$ has an adherent point in X, i.e., there exists f in S such that each neighbourhood of f contains a subsequence $\{f_{n_i}\}$ of $\{f_{m_i}\}$ ([12], [21]).

We observe that if $\mathsf{S}(X, \mathbb{R})$ is a topological space, $\{f_i\}$ is a relatively sequentially compact sequence in S, then the following condition holds:

($*$) For each subsequence $\{f_{m_i}\}$ of $\{f_i\}$ and for each countable subset Q of X there exists a subsequence $\{f_{n_i}\}$ of $\{f_{m_i}\}$ and f in $\mathsf{S}(X,\mathbb{R})$ such that

$$f_{n_i}(x) \to f(x)$$

for each x in Q.

A sequence $\{f_i\}$ in a set $\mathsf{S}(X,\mathbb{R})$ is said to be relatively X-countably compact whenever the condition ($*$) holds.

In section 3.5 we show that relatively countably compact sequences in a topological space $\mathsf{S}(X,\mathbb{R})$ are relatively X-countably compact.

In section 4.2, we prove the following

Equivanishing Theorem II. *Relatively X-countably compact sequences in a set $\mathsf{S}(X,\mathbb{R})$ of mappings on a set X to $\mathbb{R}$ are equivanishing on KS-null sequences in X.*

We explain that a sequence $\{x_i\}$ is a *KS*-null sequence in X if for each subsequence $\{x_{m_i}\}$ of $\{i\}$ there exists a subsequence $\{x_{n_i}\}$ of $\{x_{m_i}\}$ and x in X such that

$$\sum_{i=1}^{\infty} f(x_{n_i}) = f(x)$$

for each f in S.

Suppose that $\{f_i\}$ is a sequence of additive and continuous mappings on X to $\mathbb{R}$, ($X = \mathbb{R}$) and $x_j \to 0$. Then the following condition holds:

($KM\{f_i\}$) There exists a scalar sequence $\{\alpha_j\}$ such that $\alpha_j \to \infty$ and for each subsequence $\{m_j\}$ of $\{j\}$ there exists a subsequence $\{n_j\}$ of $\{m_j\}$ and x in X such that

$$\sum_{j=1}^{\infty} \alpha_{n_j} f_i(x_{n_j}) = f_i(x)$$

for each i in $\mathbb{N}$.

Following this example we adopt the definition. Let $\{f_i\}$ be a sequence of mappings on a set X to $\mathbb{R}$ and let $\{x_i\}$ be a sequence in X. We say that $\{x_i\}$ is a $KM\{f_i\}$-null sequence in X if condition ($KM\{f_i\}$) holds.

In section 4.3 we prove the following

Equivanishing Theorem III. *Each X-bounded sequence $\{f_i\}$ of mappings on a set X to $\mathbb{R}$ is equivanishing on $KM\{f_i\}$-null sequences.*

We explain that $\{f_i\}$ is X-bounded if $\{f_i(x)\}$ is a bounded sequence for each x in X.

As direct corollaries of the theorems on equivanishing of sequences we get: theorems on equivanishing of families of mappings; generalizations of theorems of uniform countable additivity; Banach type theorems on equicontinuity; Orlicz-Pettis type theorems on subseries convergent series.

2. The basic tools. As it was mentioned the basic tools for this paper are to be properties of convergence of numerical sequences and numerical matrices. By $\mathbb{R}$ is denoted the set of real numbers with the convergence of sequences induced by the absolute value. These properties are:

(F) (Fréchet) *Limits of sequences are limits of their subsequences;*

(L) (Linearity) *Sums and products of sequences converge to sums and products of their limits.*

(U) (Urysohn) *If each subsequence of a given sequence has a subsequence converging to a given point, then the sequence converges to the point.*

(S) (Stability) *Constant sequences converge to their terms.*

(H) (Hausdorff) *Each sequence may have at most one limit.*

Now, two properties concerning matrices.

By a matrix or, equivalently, double sequence we mean a function defined on the Cartesian product $\mathbb{N} \times \mathbb{N}$. It is denoted by $\{x_{i,j}\}$. The sequence $\{x_{i,i}\}$ is called the principal diagonal of the matrix $\{x_{i,j}\}$. By a square submatrix of $\{x_{i,j}\}$ we mean a matrix $\left\{x_{m_i,m_j}\right\}$ where $\{m_i\}$ is a subsequence of $\{i\}$.

Assume that $\{x_{i,j}\}$ is a numerical matrix.

(A) *If $x_{i,j} \to 0$ for $i \in \mathbb{N}$ as $j \to \infty$, then there exists a square submatrix $\left\{x_{m_i,m_j}\right\}$ such that for each subsequence $\{n_i\}$ of $\{m_i\}$ with $m_i < n_i$ for $i \in \mathbb{N}$ we have $x_{m_i,n_i} \to 0$ as $i \to \infty$.*

(Y) *If for each square submatrix $\left\{x_{m_i,m_j}\right\}$ there exists a square submatrix $\left\{x_{n_i,n_j}\right\}$ of $\left\{x_{m_i,m_j}\right\}$ and a subsequence $\{v_i\}$ of $\{n_i\}$ such that*
(i) *$x_{v_i,n_j} \to 0$ for $j \in \mathbb{N}$ as $i \to \infty$,*
(ii) *$\sum_{j=1}^{\infty} x_{v_i,n_j} \to 0$ as $i \to \infty$,*
then $x_{i,i} \to 0$.

The properties FLUSH are well known and there is no need to prove them. The properties (A) and (Y) are to be proved. We precede their proofs by the following lemma.

LEMMA 1. *Assume that $x_{i,j} \to 0$ for $i \in \mathbb{N}$ as $j \to \infty$ and $x_{i,j} \to 0$ for $j \in \mathbb{N}$ as $i \to \infty$ and assume that $\epsilon_{i,j}$ is a matrix of positive numbers. Then there exists a square submatrix $\left\{x_{m_i,m_j}\right\}$ of $\{x_{i,j}\}$ such that*

$$|x_{m_i,m_j}| < \epsilon_{i,j}$$

for $i, j, \in \mathbb{N}$ and $i \neq j$.

PROOF. Suppose that $m_1 = 1$. Since $x_{m_1,j} \to 0$ as $j \to \infty$ and $x_{i,m_1} \to 0$ as $i \to \infty$, there exists $m_2 \in \mathbb{N}$ such that $m_1 < m_2$ and

$$|x_{m_i,m_j}| < \epsilon_{i,j} \tag{1}$$

for $i, j = 1, 2$ and $i \neq j$. Since $x_{m_i,j} \to 0$ for $i = 1, 2$ as $j \to \infty$ and $x_{i,m_j} \to 0$ for $j = 1, 2$ as $i \to \infty$, there exists $m_3 \in \mathbb{N}$ such that $m_2 < m_3$ and (1) holds fr $i, j = 1, 2, 3$ and $i \neq j$. By induction we select a subsequence $\{m_i\}$ of $\{i\}$ such that (1) holds for $i, j = 1, 2, \ldots$ and $i \neq j$. ∎

PROOF OF (A). Letting $z_{i,j} = x_{i,j}$ for $i, j \in \mathbb{N}$, $i \le j$ and $z_{i,j} = 0$ for $i, j \in \mathbb{N}$, $i > j$, we see that the matrix $\{z_{i,j}\}$ satisfies the conditions of Lemma 1. Therefore there exists a square submatrix $\{z_{m_i,m_j}\}$ of $\{z_{i,j}\}$ such that $|z_{m_i,m_j}| < 2^{-i-j}$ for $i, j \in \mathbb{N}$ and $i \ne j$. Hence, $x_{m_i,n_j} \to 0$ for each subsequence $\{n_i\}$ of $\{m_i\}$ such that $m_i < n_i$ for $i \in \mathbb{R}$. ∎

PROOF OF (Y). Let $\{m_i\}$ be a subsequence of $\{i\}$. Let $\{n_i\}$ be a subsequence of $\{m_i\}$ and let $\{v_i\}$ be a subsequence of $\{n_i\}$ such that the conditions (i) and (ii) hold. Then the matrix $\{x_{v_i,v_j}\}$ satisfies the conditions of Lemma 1. Therefore, there exists a square submatrix $\{x_{r_i,r_j}\}$ of $\{x_{v_i,v_j}\}$ such that $|x_{r_i,r_j}| < 2^{-i-j}$ for $i, j \in \mathbb{N}$ and $i \ne j$. By the assumption, there exists a square submatrix $\{x_{p_i,p_j}\}$ of $\{x_{r_i,r_j}\}$ and a subsequence $\{p_{s_i}\}$ of $\{p_i\}$

$$\sum_{j=1} x_{p_{s_i},p_j} \to 0$$

as $i \to \infty$. By the triangle inequality, we have

$$|x_{p_{s_i}p_{s_i}}| \le \Big|\sum_{j=1}^{\infty} x_{p_{s_i}p_j}\Big| + \sum_{j=1,\, j\ne s_i}^{\infty} |x_{p_{s_i}p_j}|.$$

Consequently, $x_{p_{s_i}p_{s_i}} \to 0$. By (U), $x_{ii} \to 0$. ∎

In the sequel we shall refer to the following

LEMMA 2. *If for each square submatrix $\{x_{m_im_j}\}$ of a matrix $\{x_{ij}\}$ there exists a further square submatrix $\{x_{n_in_j}\}$ of $\{x_{m_im_j}\}$ and a subsequence $\{v_i\}$ of $\{n_i\}$ such that*

$$(i) \qquad \lim_{i\to\infty} \sum_{j=1}^{\infty} x_{v_in_j} = \sum_{j=1}^{\infty} \lim_{i\to\infty} x_{v_in_j}$$

then $x_{ii} \to 0$.

PROOF. Let $\{m_i\}$ be a subsequence of $\{i\}$, let $\{n_i\}$ be a subsequence of $\{m_i\}$ and let $\{v_i\}$ be a subsequence of $\{n_i\}$ such that (i) holds. We put

$$\lim_{i\to\infty} x_{v_i,v_j} = x_{v_j}$$

for $j \in \mathbb{N}$. We note that $x_{v_j} \to 0$ and we may assume that series

$$\sum_{j=1}^{\infty} x_{v_j}$$

is subseries convergent in $\mathbb{R}$. Consider matrix $\{x_{v_i,v_j} - x_{v_j}\}$. We claim that the matrix satisfies conditions of property (Y). In fact, let $\{p_i\}$ be a subsequence of $\{v_i\}$. By the assumption there exists a subsequence $\{q_i\}$ of $\{p_i\}$ and a subsequence $\{r_i\}$ of $\{q_i\}$ such that

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} x_{r_i,q_j} = \sum_{j=1}^{\infty} \lim_{i\to\infty} x_{r_i,q_j} = \sum_{j=1}^{\infty} x_{q_j}$$

Hence,

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} (x_{r_i,q_j} - x_{q_j}) = \sum_{j=1}^{\infty} \lim_{i\to\infty} (x_{r_i,q_j} - x_{q_j}) = 0,$$

moreover, $x_{r_i,q_j} - x_{q_j} \to 0$ for $j \in \mathbb{N}$ as $i \to \infty$. Hence, by (Y), $x_{v_i,v_i} - x_{v_i} \to 0$. Since $x_{v_j} \to 0$, we have $x_{v_i,v_i} \to 0$. By (U), $x_{ii} \to 0$. ■

3. Basic definitions. We recall the definitions of: $\mathcal{K}\{f_i\}$-null sequence; *KS*-null sequence; *KM*$\{f_i\}$-null sequence; X-precompact sequence; relatively X-countably compact sequence. We define: K-null sequence; M-null sequence and *KM*-null sequence. We point out some of their properties and give examples.

3.1. *Null sequences, K, M and KM-null sequences.* Assume that X is a topological group (vector space). A sequence $\{x_j\}$ in X is said to be:

(a) a null sequence if $x_j \to 0$;

(b) a K-null sequence if for each subsequence $\{m_j\}$ of $\{j\}$ there exists a further subsequence $\{n_j\}$ of $\{m_j\}$ and x in X such that
$$\sum_{j=1}^{\infty} x_{n_j} = x;$$

(c) an M-null sequence if there exists a scalar sequence $\{\alpha_j\}$ such that $\alpha_j \to \infty$ and $\{\alpha_j x_j\}$ is a K-null sequence.

(d) a KM-null sequence if there exists a scalar null sequence $\{\alpha_i\}$ such that $\alpha_i \to \infty$ and $\{\alpha_i x_i\}$ is a K-null sequence.

We say that a topological group (vector space) X is a K, M or *KM*-topological group (vector space) if null sequences in X are K, M or *KM*-null sequences, respectively.

Quasi-normed groups have property M, complete quasi-normed groups have property *KM*.

3.2. *$\{f_i\}$-null sequences, $K\{f_i\}$, $M\{f_i\}$ and KM$\{f_i\}$-null sequences.* Assume that $\{f_i\}$ is a sequence of mappings on a set X to $\mathbb{R}$. A sequence $\{x_j\}$ is said to be:

(a) an $\{f_i\}$-null sequence if $f_i(x_j) \to 0$ for each $i \in \mathbb{N}$ as $j \to \infty$;

(b) a $K\{f_i\}$-null sequence if for each subsequence $\{x_{m_j}\}$ of $\{x_j\}$ there exists a further subsequence $\{x_{n_j}\}$ of $\{x_{m_j}\}$ and x in X such that
$$\sum_{j=1}^{\infty} f_i(x_{n_j}) = f_i(x)$$
for each i in $\mathbb{N}$;

(c) an $M\{f_i\}$-null sequence if there exists a scalar sequence $\{\alpha_j\}$ such that $\alpha_j \to \infty$ and $\alpha_j f_i(x_j) \to 0$ for $i \in \mathbb{N}$ as $j \to \infty$;

(d) a *KM*$\{f_i\}$-null sequence if there exists a scalar null sequence $\{\alpha_j\}$ such that $\alpha_j \to \infty$ and for each subsequence $\{x_{m_j}\}$ of $\{x_j\}$ there exist a further subsequence $\{x_{n_j}\}$ of $\{x_{m_j}\}$ and x in X such that
$$\sum_{j=1}^{\infty} \alpha_{n_j} f_i(x_{n_j}) = f_i(x)$$
for $i \in \mathbb{N}$.

(1) *If $\{f_i\}$ is a sequence of strongly additive set functions on a σ-field X to $\mathbb{R}$, then pairwise disjoint sequences in X are $K\{f_i\}$-null sequences in X.*

We recall that an additive set function f on a σ-field X is strongly additive if it is vanishing on pairwise disjoint sequences in X, i.e., $f(x_i) \to 0$ for each pairwise disjoint sequence $\{x_i\}$ in X. By Drewnowski's lemma ([9]) for each sequence $\{f_i\}$ of strongly additive set functions on a σ-field X to $\mathbb{R}$ and for each pairwise disjoint sequence $\{x_j\}$ in X, there exists a subsequence $\{x_{n_j}\}$ such that for each $i \in \mathbb{N}$, f_i is a countably additive set function on a σ-field X' generated by members of the subsequence $\{x_{n_j}\}$. Hence we get (1).

(2) *If $\{f_i\}$ is a sequence of additive (linear) sequentially continuous mappings on a topological group (vector space) X, then null sequences, K, M and KM-null sequences in X are, respectively, $\{f_i\}$, $K\{f_i\}$, $M\{f_i\}$, and $KM\{f_i\}$-null sequences.*

3.3. *S, KS, MS and KMS-null sequences.* Assume that $\mathsf{S}(X,\mathbb{R})$ is a set of mappings on a set X to $\mathbb{R}$. A sequence $\{x_j\}$ is said to be:

(a) an S-null sequence if $f(x_j) \to 0$ for each f in S as $j \to \infty$;

(b) a *KS*-null sequence if for each subsequence $\{x_{m_j}\}$ of $\{x_j\}$ there exists a further subsequence $\{x_{n_j}\}$ of $\{x_{m_j}\}$ and x in X such that

$$\sum_{j=1}^{\infty} f(x_{n_j}) = f(x)$$

for each f in S;

(c) an *MS*-null sequence if there exists a scalar sequence $\{\alpha_j\}$ such that $\alpha_j \to \infty$ and $\alpha_j f(x_j) \to 0$ for each f in S as $j \to \infty$;

(d) a *KMS*-null sequence if there exists a scalar sequence $\{\alpha_j\}$ such that $\alpha_j \to \infty$ and for each subsequence $\{m_j\}$ of $\{j\}$ there exists a further subsequence $\{n_j\}$ of $\{m_j\}$ and x in X such that

$$\sum_{j=1}^{\infty} \alpha_{n_j} f(x_{n_j}) = f(x)$$

for each f in S.

(1) *S, KS, MS and MKS-null sequences in S are $\{f_i\}$, $K\{f_i\}$, $M\{f_i\}$ and KM$\{f_i\}$-null sequences in S.*

(2) *If $\mathsf{S}(X,\mathbb{R})$ is a countable set of strongly additive set functions on a σ-field X, then pairwise disjoint sequences in X are KS-null sequences.*

(3) *If $\mathsf{S}(X,\mathbb{R})$ is a set of countably additive set functions on a σ-field X to $\mathbb{R}$, then pairwise disjoint sequences in X are KS-null sequences in S.*

(4) *If $\mathsf{S}(X,\mathbb{R})$ is a set of additive (linear) sequentially continuous mappings on a topological group (vector space) X, then null sequences, K, M, KM-null sequences in X are, respectively, S, KS, MS and KMS-null sequences.*

3.4. *X-precompact sequences.* A sequence $\{f_i\}$ of mappings on a set X to $\mathbb{R}$ is an X-null sequence if $f_i(x) \to 0$ for each x in X as $i \to \infty$. If for each subsequence $\{f_{m_i}\}$

of $\{f_i\}$, $\{f_{m_{i+1}} - f_{m_i}\}$ is an X-null sequence, i.e., $f_{m_{i+1}}(x) - f_{m_i}(x) \to 0$ for each x in X, then we say that $\{f_i\}$ is an X-Cauchy sequence ([15]).

We say that a sequence $\{f_i\}$ of mappings on a set X to $\mathbb{R}$ is X-precompact if each its subsequence $\{f_{m_i}\}$ has an X-Cauchy subsequence $\{f_{n_i}\}$.

Subsequences of X-precompact squences are X-precompact.

(1) *If* $\mathsf{S}(X, \mathbb{R})$ *is an admissible topological space (group), then sequentially relatively compact (sequentially precompact) sequences in* $\mathsf{S}(X, \mathbb{R})$ *are* X*-precompact.*

3.5. *Relatively* X*-countably compact sequences in* $\mathsf{S}(X, \mathbb{R})$. Asume that $\mathsf{S}(X, \mathbb{R})$ is a sequence of mappings on a set X to $\mathbb{R}$. We recall that a sequence $\{f_i\}$ in $\mathsf{S}(X, \mathbb{R})$ is relatively X-countably compact in S if for each its subsequence $\{f_{m_i}\}$ and each countable subset Q of X there exists a further subsequence $\{f_{n_i}\}$ of $\{f_{m_i}\}$ and a mapping f in S such that

$$f_{n_i}(x) \to f(x)$$

for each x in Q as $i \to \infty$.

Subsequences of relatively X-countably compact sequences are relatively X-countably compact.

(1) *Relatively compact sequences in a topological space* $\mathsf{S}(X, \mathbb{R})$ *are relatively* X*-countably compact.*
(2) *Relatively countably compact sequences in an admissible topological space* $\mathsf{S}(X, \mathbb{R})$ *are relatively* X*-countably compact.*

Proof. Assume that $\{f_i\}$ is a relatively countably compact sequence in a topological space $\mathsf{S}(X, \mathbb{R})$ and Q is a countable subset of X. Let $\{f_{m_i}\}$ be a subsequence of $\{f_i\}$ and let $\{x_i\}$ be a sequence of all members of Q. We note that for each $j \in \mathbb{N}$ $\{f_{m_i}(x_j)\}$ is a bounded sequence. Therefore, for $j = 1$, there exists a subsequence $\{m_{1i}\}$ of $\{m_i\}$ such that $\{f_{m_{1i}}(x_1)\}$ is a convergent sequence, so, it is a Cauchy sequence. For $j = 2$, there exists a subsequence $\{m_{2i}\}$ of $\{m_{1i}\}$ such that $\{f_{m_{2i}}(x_2)\}$ is a convergent sequence. By induction we select a sequence $\{m_{ji}\}_{i=1}^{\infty}$, of sequences such that $\{m_{j+1,i}\}$ is a subsequence of $\{m_{j,i}\}$ and $\{f_{m_{j+1,i}}(x_{j+1})\}$ is a convergent sequence as $i \to \infty$. Letting $m_{ii} = n_i$ for $i \in \mathbb{N}$ we see that $\{f_{n_i}(x_j)\}$ is a convergent sequence for each $j \in \mathbb{N}$. Therefore, it is a Cauchy sequence for each $j \in \mathbb{N}$. Let f be an adherent point of $\{f_{n_i}\}$. We claim that

$$f_{n_i}(x_j) \to f(x_j)$$

for each $j \in \mathbb{N}$ as $i \to \infty$. Otherwise, there exists $\epsilon > 0$, an index $j \in \mathbb{N}$ and a subsequence $\{f_{v_i}\}$ of $\{f_{n_i}\}$ such that

$$|f_{v_i}(x_j) - f(x_j)| > 2\epsilon \tag{1}$$

for $i \in \mathbb{N}$. On the other hand, let $V = \{h \in \mathsf{S} : |f(x_j) - h(x_j)| < \epsilon\}$. Since $\mathsf{S}(X, \mathbb{R})$ is an admissible topological space, V is a neighbourhood of f and since f is an adherent point for $\{f_{n_i}\}$, there exists a subsequence $\{f_{w_i}\}$ of $\{f_{n_i}\}$ such that $f_{w_i} \in V$ for $i \in \mathbb{N}$. In particular,

$$|f_{w_i}(x_j) - f(x_j)| < \epsilon \tag{2}$$

for $i \in \mathbb{N}$. By (1) and (2), we have

$$|f_{v_i}(x_j) - f_{w_i}(x_j)| > |f_{v_i}(x_j) - f(x_j)| - |f_{w_i}(x_j) - f(x_j)| > \epsilon$$

for each $i \in \mathbb{N}$. This contradicts the fact that $\{f_{n_i}(x_j)\}$ is a Cauchy sequence. ∎

4. Proofs of the equivanishing theorems. In the sequel we shall refer to the following.

PROPOSITION 1. *A sequence $\{f_i\}$ of mappings on a set X to $\mathbb{R}$ is equivanishing on a sequence $\{x_j\}$ in X, iff for any two subsequences $\{m_i\}$ and $\{n_i\}$ of $\{i\}$ $f_{m_i}(x_{n_i}) \to 0$.*

PROOF. We are to show that

$$f_i(x_j) \to 0$$

uniformly for $i \in \mathbb{N}$ as $j \to \infty$ or, equivalently, for each sequence $\{m_i\}$ in $\mathbb{N}$,

$$f_{m_i}(x_i) \to 0. \tag{1}$$

Let $\{m_i\}$ be a sequence in $\mathbb{N}$. We note that

$$f_{m_i}(x_j) \to 0 \tag{2}$$

for $i \in \mathbb{N}$ as $j \to \infty$. Let $\{n_i\}$ be a subsequence of $\{i\}$. If $\{m_{n_i}\}$ is a bounded sequence, then, by (2),

$$f_{m_{n_i}}(x_{n_i}) \to 0.$$

If $\{m_{n_i}\}$ is an unbounded sequence, then there exists a subsequence $\{r_i\}$ of $\{n_i\}$ such that $\{m_{r_i}\}$ is a subsequence of $\{i\}$. By the assumption

$$f_{m_{r_i}}(x_{r_i}) \to 0.$$

Hence, by (U), we get (1). ∎

4.1. *The proof of Equivanishing Theorem I.* We are to show that if $\{f_i\}$ is an X-precompact sequence of mappings on a set X to $\mathbb{R}$ and $\{x_j\}$ is a $K\{f_i\}$-null sequence in X, then $\{f_i\}$ is equivanishing on $\{x_j\}$ or, equivalently,

$$f_i(x_j) \to 0 \tag{1}$$

uniformly for $i \in \mathbb{N}$ as $j \to \infty$. By Proposition 4.1, (1) holds if for any two subsequences $\{m_i\}$ and $\{n_i\}$ of $\{i\}$ we have

$$f_{m_i}(x_{n_i}) \to 0 \tag{2}$$

as $i \to \infty$. Let $\{m_i\}$ and $\{n_i\}$ be subsequences of $\{i\}$ and let $\{s_i\}$ be a subsequence of $\{i\}$ such that $\{f_{m_{s_i}}\}$ is an X-Cauchy sequence. Consider a matrix $\{h_i(t_j)\}$ with

$$h_i = f_{m_{s_i}} \qquad \text{and} \qquad t_i = x_{n_{s_i}}$$

for $i \in \mathbb{N}$. Note that $h_i(t_j) \to 0$ as $j \to \infty$ for $i \in \mathbb{N}$. By (A), there exists a subsequence $\{v_i\}$ of $\{i\}$ such that

$$h_{v_i}(x_{v_{i+1}}) \to 0. \tag{3}$$

We have

$$h_{v_{i+1}}(t_{v_{i+1}}) = h_{v_{i+1}}(t_{v_{i+1}}) - h_{v_i}(t_{v_{i+1}}) + h_{v_i}(t_{v_{i+1}}). \tag{4}$$

We claim that the matrix

$$\{h_{v_{i+1}}(t_{v_{i+1}}) - h_{v_i}(t_{v_{i+1}})\}$$

satisfies conditions of (Y). In fact, let $\{m_i\}$ be a subsequence of $\{i\}$. Since $\{t_i\}$ is a KS-null sequence in X, there exists a subsequence $\{n_i\}$ of $\{m_i\}$ and t in X such that

$$\sum_{j=1}^{\infty} h(t_{v_{n_j+1}}) = h(t)$$

for each h in S. In particular,

$$\sum_{j=1}^{\infty} h_{v_{n_i+1}}(t_{v_{n_j+1}}) = h_{v_{n_i+1}}(t) \qquad \text{and} \qquad \sum_{j=1}^{\infty} h_{v_{n_i}}(t_{v_{n_j+1}}) = h_{v_{n_i}}(t)$$

for $i \in \mathbb{N}$. Therefore,

$$\sum_{j=1}^{\infty}(h_{v_{n_i+1}}(t_{v_{n_j+1}}) - h_{v_{n_i}}(t_{v_{n_j}})) = h_{v_{n_i+1}}(t) - h_{v_{n_i}}(t)$$

Since $\{h_i\}$ is an X-Cauchy sequence, we have

$$h_{v_{n_i+1}}(t_{v_{n_j+1}}) - h_{v_{n_i}}(t_{v_{n_j}}) \to 0 \qquad \text{and} \qquad h_{v_{n_i+1}}(t) - h_{v_{n_i}}(t) \to 0$$

for $i \in \mathbb{N}$ as $j \to \infty$. This proves our claim. Consequently, by (Y),

$$h_{v_{i+1}}(t_{v_{i+1}}) - h_{v_i}(t_{v_{i+1}}) \to 0.$$

Hence, by (4) and (3),

$$h_{v_{i+1}}(t_{v_{i+1}}) \to 0.$$

By (U), $h_i(t_i) \to 0$ or, equivalently, $f_{m_{s_i}}(x_{n_{s_i}}) \to 0$. Again, by (U), we get (2).

4.2. *The proof of Equivanishing Theorem II.* We are to show that a relatively X-countably compact sequence $\{f_i\}$ in a set $\mathsf{S}(X, \mathbb{R})$ of mappings on a set X to $\mathbb{R}$ is equivanishing on KS-null sequences in X. Assume that $\{x_i\}$ is a KS-null sequence in X and consider a matrix $\{f_i(x_j)\}$. We are to show that

$$f_i(x_j) \to 0 \tag{1}$$

uniformly for $i \in \mathbb{N}$ as $j \to \infty$. By Proposition 4.1, (1) holds if for any two subsequences $\{m_i\}$ and $\{n_i\}$ of $\{i\}$

$$f_{m_i}(x_{n_i}) \to 0. \tag{2}$$

as $i \to \infty$. Let $\{m_i\}$ and $\{n_i\}$ be subsequences of $\{i\}$. Consider a matrix $\{h_i(t_j)\}$ with

$$h_i = f_{m_i} \qquad \text{and} \qquad t_j = x_{n_j}$$

for $i, j \in \mathbb{N}$. We claim that the matrix satisfies conditions of Lemma 2.2. In fact, let $\{p_i\}$ be a subsequence of $\{i\}$. Since $\{x_j\}$ is a KS-null sequence, its subsequence $\{t_j\}$ is a KS-null sequence in X. Therefore, there exists a subsequence $\{q_i\}$ of $\{p_i\}$ and t in X such that

$$\sum_{j=1}^{\infty} f(t_{q_j}) = f(t) \tag{3}$$

for each f in S. In particular,

$$\sum_{j=1}^{\infty} h_{q_i}(t_{q_j}) = h_{q_i}(t).$$

Let $Q = \{t_{q_j} : j \in \mathbb{N}\} \cup \{t\}$. Since $\{f_i\}$ is a relatively X-countable compact sequence in S, $\{h_{q_i}\}$ is its subsequence and Q is a countable set, there exists a subsequence $\{u_i\}$ of $\{q_i\}$ and h in S such that

$$h_{u_i}(t_{q_j}) \to h(t_{q_j}) \qquad \text{and} \qquad h_{u_i}(t) \to h(t)$$

for $j \in \mathbb{N}$ as $i \to \mathbb{N}$. Hence, by (3), we get

$$\lim_{i\to\infty} \sum_{j=1}^{\infty} h_{u_i}(t_{q_j}) = \lim_{i\to\infty} h_{u_i}(t) = h(t) = \sum_{j=1}^{\infty} h(t_{q_j}) = \sum_{j=1}^{\infty} \lim_{i\to\infty} h_{u_i}(t_{q_i})$$

This proves the claim. Consequently, by Lemma 2.2, $h_i(t_i) \to 0$. By (U), we get (2) which was to be proved. ∎

4.3. *The proof of Equivanishing Theorem III.* Assume that $\{f_i\}$ is an X-bounded sequence of mappings on a set X to $\mathbb{R}$ and $\{x_i\}$ is a $KM\{f_i\}$-null sequence. We are to show that

$$(1) \qquad f_i(x_j) \to 0$$

uniformly for $i \in \mathbb{N}$ as $j \to \mathbb{N}$. By Proposition 4.1, (1) holds if for any two subsequences $\{m_i\}$ and $\{n_i\}$ of $\{i\}$

$$(2) \qquad f_{m_i}(x_{n_i}) \to 0$$

Assume that $\{m_i\}$ and $\{n_i\}$ are subsequences of $\{i\}$ and $\{\alpha_j\}$ is a scalar sequence such that the condition ($KM\{f_i\}$) in 3.2(d) holds. Consider matrix

$$(3) \qquad \{\alpha_{n_j}\alpha_{n_i}^{-1} f(x_{n_j})\}.$$

Let $\{p_i\}$ be a subsequence of $\{n_i\}$, let $\{q_i\}$ be a subsequence of $\{p_i\}$ and let x be a member of X such that

$$\sum_{j=1}^{\infty} \alpha_{n_{q_j}} f_i(x_{n_{q_j}}) = f_i(x)$$

for $i \in \mathbb{N}$. In particular

$$\sum_{j=1}^{\infty} \alpha_{n_{q_i}}^{-1} \alpha_{n_{q_j}} f_{m_{q_i}}(x_{n_{q_j}}) = \alpha_{n_{q_i}}^{-1} f_{m_{q_i}}(x)$$

for $i \in \mathbb{N}$. Since $\{f_i\}$ is X-bounded and $\alpha_{n_{q_i}}^{-1} \to 0$, we have

$$\alpha_{n_{q_i}}^{-1} \alpha_{n_{q_j}} f_{m_{q_i}}(x_{n_{q_j}}) \to 0 \qquad \text{and} \qquad \alpha_{n_{q_i}}^{-1} f_{m_{q_i}}(x) \to 0$$

as $i \to \infty$ for j in $\mathbb{N}$. Hence, by (Y) and (3), we get (2). ∎

5. Corollaries to the equivanishing theorems. First, we prove theorems on equivanishing of families of mappings.

Assume that $\mathcal{F}$ is a family of mappings on a set X to $\mathbb{R}$ and $\{x_i\}$ is a sequence in X. We say that $\mathcal{F}$ is equivanishing on $\{x_i\}$ if $f_i(x_i) \to 0$ uniformly for f in $\mathcal{F}$ as $i \to \infty$.

5.1. *X-precompact, relatively X-countably compact and X-bounded subfamilies of* $\mathsf{S}(X,\mathbb{R})$. Assume that $\mathsf{S}(X,\mathbb{R})$ is a set of mappings on a set X to $\mathbb{R}$. A subfamily $\mathcal{F}$ of S is said to be:

(a) X-precompact if sequences in $\mathcal{F}$ are X-precompact;
(b) relatively X-countably compact in S if sequences in $\mathcal{F}$ are relatively X-countably compact in S;
(c) X-bounded if sequences in $\mathcal{F}$ are X-bounded.

If $\mathsf{S}(X,\mathbb{R})$ is an admissible tpological group (topological space), then sequentially precompact (relatively countably compact) subsets of S are X-precompact (relatively X-countably compact). If $\mathsf{S}(X,\mathbb{R})$ is a topogical vector space, then bounded subset of S are X-bounded.

5.2. Theorems on equivanishing of families of mappings. As simple corollaries of the above definitions and the equivanishing theorems we get:

THEOREM 1. *If $\mathsf{S}(X,\mathbb{R})$ is a set (countable set) of strongly additive set functions on a σ-field X to $\mathbb{R}$, then setwise precompact (relatively X-countably compact) subsets of S are equivanishing on pairwise disjoint sequences in X or, equivalently, they are uniformly strongly additive.*

PROOF. Let $\mathcal{F}$ be a setwise precompact (relatively X-countably compact) subset of S and let $\{x_i\}$ be a pairwise disjoint sequence in X. We are to show that each sequence $\{f_i\}$ in $\mathcal{F}$ is equivanishing on $\{x_i\}$. Let $\{f_i\}$ be a sequence in $\mathcal{F}$. By the definition of $\mathcal{F}$, $\{f_i\}$ is an X-precompact (relatively X-countably compact) sequence in S. Hence, by Equivanishing Theorem I and 3.2(1) (Equivanishing Theorem II and 3.3(2)) $\{f_i\}$ is equivanishing on $\{x_i\}$. ∎

THEOREM 2. *If $\mathsf{S}(X,\mathbb{R})$ is a set of mappings on a set X to $\mathbb{R}$. Then:*

(a) *X-precompact (relatively X-countably compact) subsets of S are equivanishing on KS-null sequences;*
(b) *X-bounded subsets of S are equivanishing on KMS-null sequences.*

PROOF. Let $\mathcal{F}$ be an X-precompact (relatively X-countably compact, X-bounded) subset of S and let $\{x_i\}$ be a *KS*-null sequence in X (*KMS*-null sequence in X). We are to show that each sequence $\{f_i\}$ in $\mathcal{F}$ is equivanishing on $\{x_i\}$. Let $\{f_i\}$ be a sequence in $\mathcal{F}$. By the definition of $\mathcal{F}$, $\{f_i\}$ is an X-precompact (relatively X-countably compact, X-bounded) sequence in S. Hence, by Equivanishing Theorem I (Equivanishing Theorem II, Equivanishing Theorem III) $\{f_i\}$ is equivanishing on $\{x_i\}$. ∎

5.3. *Subseries S-convergent series.* Assume that X is a set, $\sum x_i$ is a formal series of elements of X and M is a set of mappings on a set X to $\mathbb{R}$. Consider a series

$$\sum_{i=1}^{\infty} f(x_i) \tag{1}$$

for each f in M.

We say that the series $\sum x_i$ is subseries convergent uniformly on a subset M if series (1) is subseries convergent uniformly on M.

We are interested in conditions under which $\sum x_i$ is subseries convergent uniformly on M. To this end we assume that $\sum x_i$ is subseries Cauchy uniformly on M if for each pairwise disjoint sequence $\{\sigma_i\}$ of finite subsets of $\mathbb{N}$, there exists a sequence $\{t_i\}$ in X such that

$$\sum_{j\in\sigma_i} f(x_j) = f(t_i)$$

for each f in M and $f(t_i) \to 0$ uniformly for f in M.

Proposition 1. *Series in X which are subseries Cauchy uniformly on a set M are subseries convergent uniformly on M.*

A simple proof of Proposition 1 is omitted.

Theorem 1. *Assume that $\mathsf{S}(X,\mathbb{R})$ is a set (a countable set) of strongly additive set functions on a σ-field X to $\mathbb{R}$. Then series of pairwise disjoint sets in X are subseries convergent uniformly on X-precompact (relatively X-countably compact) subsets of S.*

Proof. Assume that $\sum x_i$ is a series of pairwise disjoint sets in X and M is an X precompact (relatively X-countably compact) subsets of S. We claim that $\sum x_i$ is subseries Cauchy uniformly on M. In fact, let $\{\sigma_i\}$ be a pairwise disjoint sequence of finite subsets of $\mathbb{N}$ and let

$$t_i = \bigcup_{j\in\sigma_i} x_j$$

for $i \in \mathbb{N}$. Then

$$f(t_i) = \sum_{j\in\sigma_i} f(x_j)$$

for each f in M and $\{t_i\}$ is a pairwise disjoint sequence. By Theorem 5.2.1 M is equivanishing on $\{t_i\}$ or, equivalently, $f(t_i) \to 0$ uniformly for f in M. Hence, by Proposition 1, we get the theorem. ∎

Assume that $\mathsf{S}(X,\mathbb{R})$ is a set of mappings on a set X to $\mathbb{R}$ and $\sum x_i$ is a series of members of X. We say the series is subseries S-convergent in X if for each (finite or infinite) subset $\sigma \in \mathbb{N}$ there exists an x in X such that

$$f(x) = \sum_{j\in\sigma} f(x_j)$$

for each f in S.

Theorem 2. *Series which are subseries S-convergent in X are subseries convergent uniformly on X-precompact (relatively X-countably compact) subsets of S.*

Proof. Assume that $\sum x_i$ is a subseries S-convergent series in X and M is an X-precompact (relatively X-countably compact) subset of S. Let $\{\sigma_i\}$ be a pairwise disjoint sequence of finite subsets of $\mathbb{N}$ and $\{t_i\}$ be a sequence in X such that

$$f(t_i) = \sum_{j\in\sigma_i} f(x_j)$$

for each f in S. Since $\sum x_i$ is subseries S-convergent in X, $\{t_i\}$ is a *KS*-null sequence in X. Hence, by Theorem 5.2.2(a), M is equivanishing on $\{t_i\}$ or, equivalently, $f(t_i) \to 0$ uniformly for f in M. Hence, by Proposition 1, $\sum x_i$ is subseries convergent uniformly on M. ■

5.4. *Uniform countable additivity, equicontinuity, subseries convergent series.* Combining Theorem 5.2.2(a) and Theorem 5.3.2 we get

THEOREM 1. *Assume that* $\mathsf{S}(X, \mathbb{R})$ *is a set of mappings on a set* X *to* $\mathbb{R}$. *Then:*

(a) *X-precompact (relatively X-countably compact) subsets of S are equivanishing on KS-null sequences in X;*

(b) *subseries S-convergent series in X are subseries convergent uniformly on X-precompact (relatively X-countably compact) subsets of S.*

Assume that $\mathsf{S}(X, \mathbb{R})$ is an admissible topological group. Then sequentially precompact subsets of S are X-precompact, by 3.5(2), relatively countably compact subsets of S are relatively X-countably compact. Hence, as a particular case of Theorem 1, we get:

THEOREM 2. *If* $\mathsf{S}(X, \mathbb{R})$ *is an admissible topological group, then:*

(a) *sequentially precompact (relatively countably compact) subsets of S are equivanishing on KS-null sequences;*

(b) *subseries S-convergent series in X are subseries convergent uniformly on sequentially precompact (relatively countably compact) subsets of S.*

As particular interpretations of Theorem 2 we get the following corollaries.

COROLLARY 1. *Assume that* $\mathsf{S}(X, \mathbb{R})$ *is an admissible topological group of countably additive set functions on a σ-field X to $\mathbb{R}$. Then:*

(a) *sequentially precompact (relatively countably compact) subsets of S are equivanishing on pairwise disjoint sequences in X or, equivalently, they are uniformly strongly additive;*

(b) *series of pairwise disjoint sets in X are subseries convergent uniformly on sequentially precompact (relatively countably compact) subsets of S or, equivalently, sequentially precompact (relatively countably compact) subsets of S are uniformly countably additive.*

To prove the Corollary note that pairwise disjoint sequences in X are *KS*-null sequences and that they are subseries S-convergent in X.

COROLLARY 2. *Suppose that* $\mathsf{S}(X, \mathbb{R})$ *is an admissible topological group of additive and sequentially continuous mappings on a topological group X. Then:*

(a) *sequentially precompact (relatively countably compact) subsets of S are equivanishing on K-null sequences in X or, equivalently, they are K-equicontinous;*

(b) *subseries S-convergent series are subseries convergent uniformly on a sequentially precompact (relatively countably compact) subsets of S.*

To prove Corollary 2 note that K-null sequences in X are *KS*-null sequences in X and apply Theorem 2.

Corollary 3. *Assume that X is a locally convex space and X' is the dual space of X with an admissible group topology. Then:*

(a) *sequentially precompact (relatively countably compact) subsets of X' are equivanishing on K-null sequences in X or, equivalently, they are K-equicontinuous at zero;*

(b) *subseries weakly convergent series in X are subseries convergent uniformly on sequentially precompact (relatively countably compact) subsets of X'.*

We recall that complete quasi normed groups, complete paranormed linear spaces are examples of *KM*-topological groups, i.e. for each null sequence $\{x_i\}$ in X there exists a scalar sequence $\{\alpha_i\}$ such that $\alpha_i \to \infty$ and $\{\alpha_i x_i\}$ is a K-null sequence. Hence, by Theorem 5.2.2(b), we get:

Corollary 4. *X-bounded subsets of a set $\mathsf{S}(X,\mathbb{R})$ of additive and sequentially continuous mappings on a KM-topological group are equivanishing on null sequences in X or, equivalently, they are sequentially equicontinuous at zero.*

If X is a complete metric linear space, then Corollary 4 reduces to the famous Banach theorem on equicontinuity of pointwise bounded families of linear and continuous mappings on a complete metric linear space X ([12], §15.3(2)).

Finally, we note that the definitions and theorems in this paper are expressed in terms of convergence of sequences. The proofs are based on FLYUSA – properties of convergence of sequences. These made possible to reformulate the results in terms of mappings on a set X with values in a group equipped with FLYUSA – convergence.

References

[1] P. Antosik, *On the Mikusiński diagonal theorem*, Bull. Acad. Polon. Sci. 19 (1971), 305–310.

[2] P. Antosik, *A diagonal theorem for nonnegative metrices and equicontinuous sequences of mappings*, Bull. Acad. Polon. Sci. 24 (1976), 855–859.

[3] P. Antosik, *A lemma on matrices and its applications*, Contemporary Math. 52 (1986), 89–95.

[4] P. Antosik, J. Mikusiński and R. Sikorski, *Theory of Distributions. Sequential Approach*, Elsevier, Amsterdam, 1973.

[5] P. Antosik and C. Schwartz, *Matrix methods in analysis*, Lecture Notes in Math. 1119, Springer, Heidelberg, 1985.

[6] S. Banach, *Théorie des opérations linéaires*, Monografie Mat., vol. 1, Warszawa, 1932.

[7] J. Distel, *Sequences and series in Banach spaces*, Springer Verlag, 1984.

[8] V. M. Doubrovsky (Dubrovskii), *On some properties of complete set functions and their application to a generalization of a theorem of Lebesgue*, Mat. Sb. 20 (62) (1947), 317–329.

[9] L. Drewnowski, *Equivalence of Brooks-Jewett, Vitali-Hahn-Saks and Nikodym Theorems*, Bull. Acad. Polon. Sci. 20 (1972), 725–731.

[10] N. J. Kalton, *Subseries convergence in topological groups and vector spaces*, Israel J. Math. 10 (1971), 402–412.

[11] C. Kliś, *An example of a non-complete normed K-space*, Bull. Acad. Polon. Sci. 26 (1978), 415–420.

[12] G. Köthe, *Topological vector spaces I*, Springer Verlag, Berlin 1969.

[13] I. Labuda and Z. Lipecki, *On subseries convergent series and m-quasi-bases in topological spaces*, Manuscr. Math. 38 (1981), 87–98.

[14] J. Mikusiński, *A theorem on vector matrices and its application in measure theory and functional analysis*, Bull. Polon. Acad. Sci. 18 (1970), 193–196.

[15] J. Mikusiński, *Aksjomatyczna teoria zbieżności*, preprint.

[16] O. M. Nikodym, *Sur les familles bornées de fonctions parfaitement additives d'ensembles abstraits*, Monatsh. Math. Phys., 1933, 40.

[17] W. Orlicz, *Beiträge zur Theorie der Orthogonalentwicklungen II*, Studia Math. 1 (1929), 241–255.

[18] B. J. Pettis, *Integration in vector spaces*, Trans. Amer. Math. Soc., 44, 277–304.

[19] C. Schwartz, *Infinite matrices and the gliding hump*, World Scientific, 1996.

[20] H. Weber, *Compactness in spaces of group valued contents, the Vitali-Hahn-Saks theorem and Nikodym boundedness theorem*, Rocky Mt. J. Math. 16 (1986), 253–275.

[21] A. Vilensky, *Modern methods in topological vector spaces*, McGraw-Hill, N.Y., 1978.