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THE WEYL CORRESPONDENCE AS A FUNCTIONAL CALCULUS

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Abstract. The aim of this paper is to use an abstract realization of the Weyl correspondence to define functions of pseudo-differential operators. We consider operators that form a self-adjoint Banach algebra. We construct on this algebra a functional calculus with respect to functions which are defined on the Euclidean space and have a finite number of derivatives.

1. Introduction. The aim of this paper is to use an abstract realization of the Weyl correspondence to define functions of pseudo-differential operators. We will start with a brief discussion of the Weyl correspondence and its abstract realization, leaving for later to discuss the classes of pseudo-differential operators we are going to consider.

The Weyl correspondence was introduced by Hermann Weyl in [23]. An English translation of this book was published by Methuen, London in 1931 and reprinted by Dover, New York in 1950.

The Weyl correspondence was Hermann Weyl's very successful attempt to go from classical observables to the quantum observables introduced by John von Neumann around 1924 in his axiomatic formulation of quantum mechanics.

A classical observable is a real function defined on the phase space, typically $\mathbb{R}^{2n}$, where the position-momentum coordinates $(x_1, \ldots, x_n, \xi_1, \ldots, \xi_n)$ live. Using von Neumann's spectral theorem one can identify quantum observables with self-adjoint, in general unbounded, operators on some Hilbert space, usually $L^2(\mathbb{R}^n)$. For instance, the coordinate x_j of the position vector is identified with the operator X_j of multiplication by x_j. The coordinate ξ_j of the momentum operator is identified with the operator $D_j = \frac{h}{2\pi i}\partial_{x_j}$ where h is the Planck's constant.

To proceed with his correspondence, Weyl postulated that given $p, q \in \mathbb{R}^n$, the exponential function $e^{-2\pi i(q\bullet x+p\bullet\xi)}$ should be assigned to the operator $e^{-2\pi i(q\bullet X+p\bullet D)}$ defined by the spectral theorem.

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Given a function a defined on the phase space, the Fourier inversion formula gives the formal representation

$$a(x, \xi) = \int_{\mathbb{R}^{2n}} e^{-2\pi i(q\bullet x+p\bullet\xi)}\widehat{a}(p, q)dpdq$$

from which $a(X, D)$ can be defined formally as

$$a(X, D) = \int_{\mathbb{R}^{2n}} e^{-2\pi i(q\bullet X+p\bullet D)}\widehat{a}(p, q)dpdq. \tag{1}$$

This is Weyl's quantization rule, usually called Weyl correspondence. The right hand side of (1) is in fact well defined as a Bochner integral with values in $L^2(\mathbb{R}^{2n})$, provided that a and $\widehat{a}$ are both integrable functions. In fact, it can be shown that the operator $e^{-2\pi i(q\bullet X+p\bullet D)}$ is a unitary operator. For the proof, we refer to the book by G. B. Folland [10], where many other insights and connections of the Weyl correspondence are discussed. We also refer to [19] for a presentation of the mathematical aspects of the Weyl correspondence.

An abstract realization of the Weyl correspondence means to replace the n-tuples X and D in (1) with other operators. It seems fair to say that E. Nelson [17] was the first to considered an abstract realization of (1), followed by M. E. Taylor [22] and R. F. V. Anderson [7]. A. McIntosh and A. Pryde [15] used the abstract realization of (1) in the context of Clifford algebras.

Following these authors, we will now present an abstract realization of the Weyl correspondence that is relevant to the application we have in mind.

Given a complex Banach algebra X and given elements $A_1, \ldots, A_k \in X$, the exponential $e^{2\pi i\xi\bullet A}$ makes sense in X for any complex value of the parameters $\xi_1, \ldots, \xi_k$, and defines a continuous function of ξ with values in X. So, the Bochner integral

$$\int_{\mathbb{R}^k} e^{-2\pi i\xi\bullet A}\widehat{f}(\xi)d\xi \tag{2}$$

will exist provided that $\|e^{-2\pi i\xi\bullet A}\widehat{f}(\xi)\|_X$ is a Lebesgue integrable function of $\xi \in \mathbb{R}^k$. Formally, (2) is obtained by making the substitution $x \to A$ in the right hand side of the Fourier inversion formula

$$f(x) = \int_{\mathbb{R}^k} e^{-2\pi i\xi\bullet x}\widehat{f}(\xi)d\xi. \tag{3}$$

In some cases, this substitution becomes a way of defining the action of the function f on the k-tuple A. To check this assertion, let us look at the following case. Suppose that X is the Banach algebra of linear bounded operators on a complex Hilbert space H. In other words, $X = L(H)$. Assume also that A is a k-tuple of self-adjoint operators in $L(H)$. Then, one can observe that (2) is, in the sense of vector valued tempered distributions, the Fourier transform of the function of exponential type $\xi \to e^{-2\pi i\xi\bullet A}$. Paley-Wiener theorem implies that this Fourier transform is a distribution of compact support. Thus, we can apply the distribution$(e^{-2\pi i\xi\bullet A})^\wedge$ to any polynomial P in k variables. After some calculations, one can conclude ([22], p. 94) that this action is $P_s(A)$, where P_s indicates the symmetrization of P. When $k = 1$, this action is simply $P(A)$.

Going back to our initial setting, it becomes clear that to know for which functions f is $f(A)$ defined, one needs to obtain the best possible estimate for $\|e^{-2\pi i\xi\bullet A}\|_X$ in terms of $\xi \in \mathbb{R}^k$. For example, when $X = L(H)$ and A is a k-tuple of self-adjoint operators, the exponential $e^{-2\pi i\xi\bullet A}$ is a unitary operator. So, it suffices to consider integrable functions f whose Fourier transform is also integrable. We will see in Section 3 that this is not always the case.

To keep our presentation on target, we will not allude to any of the very interesting ideas connected with the notion of functional calculi in several variables. A good reference for further study is the paper by Z. Słodkowski and W. Żelazko [21]. We will not mention either the algebraic formalization of the notion of functional calculus [1].

We remark that (2) is an extension to the non-commutative case of classical rules used to define functions of a single operator, for instance the holomorphic calculus discussed in [20]. However, unlikely the one variable case, the functional calculus given by (2) is not multiplicative in the variable f. That is to say, the product $f_1 f_2$ is not mapped in general to $f_1(A)\circ f_2(A)$. For instance, the polynomial T_1T_2 is mapped to the operator $(A_1A_2 + A_2A_1)/2$.

A remarkable feature of (2) is that one only needs to establish its validity for single operators. In fact, one can write

$$\int_{\mathbb{R}^k} e^{-2\pi i\xi\bullet A}\widehat{f}(\xi)d\xi = \int_{|\xi|\le 1} e^{-2\pi i\xi\bullet A}\widehat{f}(\xi)d\xi + \int_1^\infty \int_{S^{k-1}} e^{-2\pi i r(\varpi\bullet A)}\widehat{f}(r\varpi)r^{n-1}d\varpi dr. \tag{4}$$

It only remains to observe that $\|\varpi \bullet A\|_X$ is uniformly bounded for $\varpi \in S^{k-1}$.

It is interesting to point out that (2) can be studied also in connection with operants, a term coined by E. Nelson to indicate the new mathematical objects he introduced to study the functional calculus of non-commuting operators. The abstract realization of the Weyl correspondence is also related to the work of M. L. Lapidus [14] and V. Nazaikinskii, V. Shatalov and B. Sternin [16].

Pseudo-differential operators appear in more than one way in connection with the Weyl correspondence (1). In fact, pseudo-differential operators were introduced, (see for instance [13]), to provide a refinement of the theory of singular integrals developed by Calderón and Zygmund. The idea was to associate a so called symbol to an operator, in such a way that operations among symbols would carry considerable information about corresponding operations among operators. This idea was a refinement of ideas already present in some of the work by Calderón and Zygmund. The Weyl correspondence works in the opposite direction, associating operators to functions. Furthermore, the action of the Weyl correspondence on smooth functions f with sufficient decay at infinity is given by the following oscillatory integral ([10], p. 79)

$$a(X,D)(f) = \int\int e^{-2\pi i(x-y)\bullet\xi} a\left(\frac{x+y}{2},\xi\right) f(y)dyd\xi.$$

This is precisely the representation of pseudo-differential operators that L. Hörmander [11] called the Weyl calculus of pseudo-differential operators.

The use of (2) to define functions of pseudo-differential operators goes back to the work of J. Alvarez, A. P. Calderón, and J. Hounie ([2], [4], [5], [6]). In [2], [4], [5] we construct Banach algebras of pseudo-differential operators and we define the action of functions with finitely many derivatives. We consider in [2] L^2-continuous operators in the framework of Euclidean space. In [4] we consider L^2-continuous operators on compact manifolds without border. In [5] we extend these results to the case of operators that are bounded on L^p. The purpose of [6] is to define the action of C^∞ functions on a single pseudo-differential operator of non-positive order in the Hörmander class [12].

The operators we will consider here form a self-adjoint Banach algebra. This algebra was introduced in [2], p. 24. Our goal is to define a functional calculus without resorting to the notion of characteristic used in [2], [4], [5]. As we will see later, the fact that we do not use characteristics, implies that we lose the star representation of the operator Banach algebra into a space of differential operators. However, from the functional calculus point of view, our approach will render a more direct way of defining functions of the operators.

Although we can define in a similar way a Banach algebra of L^p-bounded operators, our functional calculus do not extend to this case. The obstruction is that it remains unknown how to obtain an optimal estimate for the norm of the exponential operator $e^{-2\pi i\xi\bullet A}$ in terms of $\xi\in\mathbb{R}^k$. This problem is solved in [5] in the case of operators acting on functions defined on a compact manifold without border. In [6] the problem is solved in the case of one operator, not a Banach algebra structure. The method of proof requires to consider symbols with infinitely many derivatives.

Our work is organized as follows. In Section 2 we introduce our Banach algebra of pseudo-differential operators. Actually, in the notation of [12], p. 139, we will consider the case $\rho=1,\delta=0$. With no extra work but with some notational complications, we could consider the case $0\le\delta<\rho\le 1$.

In Section 3 we use the abstract realization of the Weyl correspondence to construct a functional calculus with respect to functions with finitely many derivatives. To keep the technicalities to a minimum, we will explain in detail the case of translation invariant operators and we will discuss afterwards what modifications are needed in the general case. The treatment of the translation invariant case will also allow us to justify what type of estimate for the exponential operator could be considered optimal.

2. The classes $\mathcal{M}$ and $\mathcal{R}$. Given $m=1,2,\ldots$ we consider operators A acting on the Schwartz class $S(\mathbb{R}^n)$ as

$$A(f)=\sum_{j=0}^{m-1}\int_{\mathbb{R}^n} e^{-2\pi i x\bullet\xi}a_j(x,\xi)\widehat{f}(\xi)d\xi+R(f) \tag{5}$$

where:

1. The function $a_j(x,\xi)$ has continuous derivatives in x,ξ up to the orders $M_j=2([\frac{n}{2}]+m+1)-j$ and $N_j=n+m+1-j$, respectively. Moreover, the quantity $\|a_j\|_j$ defined as

$$\|a_j\|_j=\sup_{x,\xi\in\mathbb{R}^n}\sum_{0\le|\alpha|\le M_j,0\le|\beta|\le N_j}(1+|\xi|)^{j+|\beta|}|\partial_x^\alpha\partial_\xi^\beta a_j(x,\xi)| \tag{6}$$

is finite.

2. The operator R is linear and continuous on $L^2(\mathbb{R}^n)$. Furthermore, for each n-tuple α with $|\alpha| = m$, the operators $\partial^\alpha R$ and $R\partial^\alpha$ are also continuous on $L^2(\mathbb{R}^n)$. Because of these continuity properties, operators in the class $\mathcal{R}$ are usually called regularizing operators of order m.

We indicate with $\mathcal{M}$ and $\mathcal{R}$ the classes of operators A and R, respectively.

On $\mathcal{R}$ we define the norm

$$\|R\|_{\mathcal{R}} = \|R\|_{L(L^2)} + \sup_{|\alpha|=m} (\|\partial^\alpha R\|_{L(L^2)} + \|R\partial^\alpha\|_{L(L^2)}). \tag{7}$$

In the language of [12], the function a_j belongs to the class $S^{-j}_{1,0}$ defined in terms of a finite number of derivatives. We define in $\mathcal{M}$ the norm

$$\|A\|_{\mathcal{M}} = \inf\Big\{ \sum_{j=0}^{m-1} \|a_j\|_j + \|R\|_{\mathcal{R}} \Big\}.$$

where the infimum is taken over all possible representations of A as in (5). With this norm, the space $\mathcal{M}$ becomes a normed space.

Sometimes we will indicate

$$Op(a_j) = \int_{\mathbb{R}^n} e^{-2\pi i x \bullet \xi} a_j(x,\xi) \widehat{f}(\xi) d\xi.$$

When $R = 0$ in (5), the operator A defines uniquely the function $\sum_{j=0}^{m-1} a_j(x,\xi)$, which then can be considered the symbol of the operator. However, each function a_j in the sum is not uniquely determined by A. So, if one wants to have a fine tuned notion of symbol attached to each operator, it is necessary to consider the notion of characteristic (see for instance [2], p. 45). Roughly speaking, a characteristic is a differential operator with vector valued coefficients. The operations among these differential operators are defined in such a way that there is a continuous homomorphism of algebras between operators and differential operators. Moreover, this homomorphism is a $*$-homomorphism ([9], p.88) of the operator algebra onto the space of characteristics.

For future reference we state the following continuity result, adapted from [2], p. 4.

THEOREM 1. *Assume that the function $a(x,\xi)$ has continuous derivatives in x, ξ up to the orders $[\frac{n}{2}] + 1$ and $n + 1$ respectively, satisfying*

$$\sup_{x,\xi\in\mathbb{R}^n} \sum_{0\le|\alpha|\le[\frac{n}{2}]+1, 0\le|\beta|\le n+1} (1+|\xi|)^{j+|\beta|} |\partial_x^\alpha \partial_\xi^\beta a(x,\xi)| = M < \infty.$$

Then, the pseudo-differential operator

$$A(f) = \int_{\mathbb{R}^n} e^{-2\pi i x \bullet \xi} a(x,\xi) \widehat{f}(\xi) d\xi$$

initially defined on $S(\mathbb{R}^n)$, extends to a continuous operator on $L^2(\mathbb{R}^n)$. Moreover, there exists $C = C(n) > 0$ such that

$$\|A\|_{L(L^2)} \le CM.$$

REMARK 2. Theorem 1 implies that each term $Op(a_j)$ in (5) is continuous on $L^2(\mathbb{R}^n)$.

THEOREM 3. *$(\mathcal{R}, \|\ \|_{\mathcal{R}})$ and $(\mathcal{M}, \|\ \|_{\mathcal{M}})$ are self-adjoint Banach algebras.*

PROOF. For the proof of this result we refer to Theorem 3.1 in [2], p. 25. ∎

REMARK 4. As part of the proof of Theorem 3 one is able to conclude that $\mathcal{R}$ is an ideal of $\mathcal{M}$.

3. Functions of self-adjoint operators in the class $\mathcal{M}$. We start by observing that in the previous section we used the letter ξ to indicate the momentum variable in the representation (5) of the operator A and we used the letter f to indicate the functions on which the operator A acts. To avoid any confusion with the notation used in the introduction, we will now write the abstract realization of the Weyl correspondence (2) as

$$\int_{\mathbb{R}^k} e^{-2\pi it \bullet A}\widehat{\varphi}(t)dt \tag{8}$$

for appropriate functions φ. As we observed in the introduction, the bulk of the work of defining functions of operators using (8) is in the treatment of the case $k = 1$.

We will consider translation invariant operators, that is to say operators of the form

$$A(f) = \int_{\mathbb{R}^n} e^{-2\pi ix \bullet \xi} a(\xi)\widehat{f}(\xi)d\xi \tag{9}$$

where $a(\xi) = \sum_{j=0}^{m-1} a_j(\xi)$. While avoiding excessive technical details, this case exemplifies all the important features of the general case, which we will discuss afterwards.

THEOREM 5. *Given an operator A as in (9), assume that A is self-adjoint as an operator acting on $L^2(\mathbb{R}^n)$. Then, given a function φ that is continuous and has continuous derivatives up to the order $3m+n-1$, the abstract realization of the Weyl correspondence*

$$\int_{-\infty}^{\infty} e^{-2\pi itA}\widehat{\varphi}(t)dt$$

defines an operator in $\mathcal{M}$.

PROOF. As we mentioned in the introduction, it suffices to show that

$$\|e^{-2\pi itA}\|_{\mathcal{M}} \leq C(1 + |t|)^{3m+n-1}. \tag{10}$$

When $m = 1$ we can write $e^{-2\pi itA} = Op(e^{-2\pi ita_0})$. Since A is self-adjoint, we can assume that a_0 is a real function. So, we obtain

$$\|e^{-2\pi itA}\|_{\mathcal{M}} = \|Op(e^{-2\pi ita_0})\|_{\mathcal{M}} \leq C(1 + |t|)^{m+n-1}. \tag{11}$$

We now assume that $m \geq 2$. According to Theorem 3, the operator $e^{-2\pi itA}$ is well defined, by the power series $\sum_{l=0}^{\infty} \frac{(-2\pi i)^l}{l!} A^l$, as an operator in $\mathcal{M}$. Moreover, we can write

$$e^{-2\pi itA} = Op(e^{-2\pi ita}) = Op(e^{-2\pi ita_o} \dots e^{-2\pi ita_{m-1}}). \tag{12}$$

To obtain a representation of $e^{-2\pi itA}$ as in (5), we observe that

$$e^{-2\pi ita_o} \dots e^{-2\pi ita_{m-1}} =$$
$$\sum_{j=0}^{m-1} e^{-2\pi ita_o} \sum_{\alpha_1+2\alpha_2+\dots+(m-1)\alpha_{m-1}=j} \frac{(-2\pi it)^{|\alpha|}}{\alpha!} a_1^{\alpha_1} \dots a_{m-1}^{\alpha_{m-1}} + b(t)$$

where $b(t)$ includes all the terms of order $\leq -m$. Because the operator $R(t) = Op(b(t))$ belongs to $\mathcal{R}$, we have the representation

$$e^{-2\pi itA} = Op\Big(\sum_{j=0}^{m-1} e^{-2\pi ita_o} \sum_{\alpha_1+2\alpha_2+\ldots+(m-1)\alpha_{m-1}=j} \frac{(-2\pi it)^{|\alpha|}}{\alpha!} a_1^{\alpha_1}\ldots a_{m-1}^{\alpha_{m-1}}\Big) + R(t). \quad (13)$$

Since A is a self-adjoint operator, we can assume that $a_o(\xi)$ is a real valued function. Thus, we obtain the following estimate for $\|e^{-2\pi itA}\|_{\mathcal{M}}$.

$$\|e^{-2\pi itA}\|_{\mathcal{M}} \leq C(1+|t|)^{m+n+1} + \|R(t)\|_{\mathcal{R}}. \quad (14)$$

It remains to estimate the norm $\|R(t)\|_{\mathcal{R}}$ as a function of t. To do this we write

$$\mathcal{E}(t) = \sum_{j=0}^{m-1} e^{-2\pi ita_o} \sum_{\alpha_1+2\alpha_2+\ldots+(m-1)\alpha_{m-1}=j} \frac{(-2\pi it)^{|\alpha|}}{\alpha!} a_1^{\alpha_1}\ldots a_{m-1}^{\alpha_{m-1}}$$

and

$$R(t) = e^{-2\pi itA} - Op(\mathcal{E}(t)). \quad (15)$$

We observe that $e^{-2\pi itA}$ and $Op(\mathcal{E}(t))$ are smooth functions of t with values in $\mathcal{M}$ and that $R(t)$ is a smooth function of t with values in $\mathcal{R}$. So, if we differentiate with respect to t both sides of (15) we obtain

$$R'(t) = e^{-2\pi itA}(-2\pi iA) - Op(\mathcal{E}'(t)). \quad (16)$$

We can write (16) as

$$R'(t) = R(t)(-2\pi iA) + Op(\mathcal{E}(t))(-2\pi iA) - Op(\mathcal{E}'(t)).$$

We claim that $Op(\mathcal{E}(t))(-2\pi iA) - Op(\mathcal{E}'(t))$ is a regularizing operator. In fact, after some straightforward computations we have

$$Op(\mathcal{E}(t))(-2\pi iA) - Op(\mathcal{E}'(t)) =$$
$$Op\Big(e^{-2\pi ita_o}\sum_{j=0}^{m-1}\sum_{l=1}^{m-1} \sum_{\alpha_1+2\alpha_2+\ldots+(m-1)\alpha_{m-1}=j} (-2\pi i)\frac{(-2\pi it)^{|\alpha|}}{\alpha!} a_1^{\alpha_1}\ldots a_l^{\alpha_l+1}\ldots a_{m-1}^{\alpha_{m-1}}\Big) -$$
$$Op\Big(e^{-2\pi ita_o}\sum_{j=1}^{m-1} \sum_{\alpha_1+2\alpha_2+\ldots+(m-1)\alpha_{m-1}=j} (-2\pi i)|\alpha|\frac{(-2\pi it)^{|\alpha|-1}}{\alpha!} a_1^{\alpha_1}\ldots a_{m-1}^{\alpha_{m-1}}\Big).$$

With the change of variable $(\beta_1, \ldots \beta_l, \ldots, \beta_{m-1}) = (\alpha_1, \ldots, \alpha_l + 1, \ldots, \alpha_{m-1})$ we can write

$$\sum_{j=0}^{m-1}\sum_{l=1}^{m-1} \sum_{\alpha_1+2\alpha_2+\ldots+(m-1)\alpha_{m-1}=j} \frac{(-2\pi it)^{|\alpha|}}{\alpha!} a_1^{\alpha_1}\ldots a_l^{\alpha_l+1}\ldots a_{m-1}^{\alpha_{m-1}} =$$
$$\sum_{j=0}^{m-1}\sum_{l=1}^{m-1} \sum_{\beta_1+2\beta_2+\ldots+(m-1)\beta_{m-1}=j+l} \beta_l\frac{(-2\pi it)^{|\beta|-1}}{\beta!} a_1^{\beta_1}\ldots a_{m-1}^{\beta_{m-1}} =$$
$$\sum_{s=1}^{2m-2}\sum_{l=1}^{m-1} \sum_{\beta_1+2\beta_2+\ldots+(m-1)\beta_{m-1}=s} \beta_l\frac{(-2\pi it)^{|\beta|-1}}{\beta!} a_1^{\beta_1}\ldots a_{m-1}^{\beta_{m-1}} =$$

$$\sum_{s=1}^{2m-2} \sum_{\beta_1+2\beta_2+\ldots+(m-1)\beta_{m-1}=s} |\beta| \frac{(-2\pi it)^{|\beta|-1}}{\beta!} a_1^{\beta_1} \ldots a_{m-1}^{\beta_{m-1}}.$$

Thus, we obtain that

$$Op(\mathcal{E}(t))(-2\pi iA) - Op(\mathcal{E}'(t)) =$$

$$Op\left(e^{-2\pi ita_o} \sum_{s=m}^{2m-2} \sum_{\beta_1+2\beta_2+\ldots+(m-1)\beta_{m-1}=s} (-2\pi i)|\beta| \frac{(-2\pi it)^{|\beta|-1}}{\beta!} a_1^{\beta_1} \ldots a_{m-1}^{\beta_{m-1}}\right).$$

This is a regularizing operator, $R_1(t)$, as claimed. Moreover

$$\|R_1(t)\|_{\mathcal{R}} \leq C(1+|t|)^{3m+n-2}.$$

Since $R(0) = 0$, from the formula

$$R'(t) = R(t)(-2\pi iA) + R_1(t)$$

we can obtain by integration

$$R(t) = \int_0^t e^{-2\pi i(t-s)A} R_1(s)ds.$$

Because the operator A is self-adjoint as an operator acting on $L^2(\mathbb{R}^n)$, we have the estimate

$$\|e^{-2\pi i(t-s)A}\|_{L(L^2)} = 1. \tag{17}$$

So, we obtain

$$\|R(t)\|_{\mathcal{R}} \leq C(1+|t|)^{3m+n-1}.$$

Thus, according to (14),

$$\|e^{-2\pi itA}\|_{\mathcal{M}} \leq C(1+|t|)^{3m+n-1}.$$

This concludes the proof of Theorem 5. ∎

Remark 6. Theorem 5 shows that in the case of pseudo-differential operators one cannot expect to obtain anything better than a polynomial estimate for the exponential operator. As expected in the case of translation invariant operators, the estimate (10) provides a significant improvement over the analogous estimate proved in [2].

When the operator A is not translation invariant, the exponential $e^{-2\pi itA}$ no longer coincides with $Op(e^{-2\pi ita})$, as shown by the calculus of pseudo-differential operators ([12], p. 147). Moreover, one has to take into account in general the regularizing term R. These are the technical complications added to the method by which we proved Theorem 5. However, once we obtain the correct representation of $e^{-2\pi itA}$ to use instead of (13), the idea of the proof applies quite directly. The price to pay is the need to increase the power of $(1+|t|)$ in (10), because there are already derivatives used in obtaining a representation of the operator $e^{-2\pi itA}$. The form of this symbol can be obtained in a way similar to the work done in [1], p. 66.

Theorem 5 does not apply to the case of a Banach algebra of L^p-bounded operators because the problem of finding an alternative to estimate (17) remains open.

We observed in the introduction that Paley-Wiener's theorem implies that $\varphi(A)$ is zero when $supp(\varphi) \subset \{|z| > 2\pi\|A\|_{\mathcal{M}}\}$. Furthermore, if we consider the set ([1], p. 39)

$$\Gamma = \bigcap_{R\geq 0} \overline{\{a_0(x,\xi) : x \in \mathbb{R}^n, |\xi| \geq R\}}$$

it can be shown that Γ is contained in the spectrum of A. Moreover, the operator $\varphi(A)$ is a regularizing operator if the function φ is zero in a neighborhood of Γ.

The k-dimensional version of Theorem 5 follows from (4) and the fact that the operator $\varpi \bullet A$ belongs to $\mathcal{M}$ uniformly with respect to $\varpi \in S^{k-1}$. We obtain the estimate

$$\|e^{-2\pi ir(\varpi\bullet A)} f(r\varpi)r^{k-1}\|_{\mathcal{M}} \leq C(1+r)^{3m+n+k-2}.$$

In the case of k commuting operators, we need to consider the joint spectrum [21] in order to obtain a result analogous to the one mentioned above ([2], p. 58).

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