ALGEBRAIC ANALYSIS AND RELATED TOPICS
BANACH CENTER PUBLICATIONS, VOLUME 53
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2000

AN ALGEBRAIC DERIVATIVE ASSOCIATED TO THE OPERATOR D^δ

V. M. ALMEIDA, N. CASTRO and J. RODRÍGUEZ

Departamento de Análisis Matemático, Universidad de La Laguna
38271 La Laguna (Tenerife), Canary Islands, Spain
E-mail: valmeida@ull.es

Abstract. In this paper we get an algebraic derivative relative to the convolution

$$(f * g)(t) = \int_0^t f(t-\psi)g(\psi)d\psi$$

associated to the operator D^δ, which is used, together with the corresponding operational calculus, to solve an integral-differential equation. Moreover we show a certain convolution property for the solution of that equation.

1. Introduction. W. Kierat and K. Skórnik [2], using the Mikusiński operational calculus, have solved the differential equation

$$t\frac{d^2x}{dt^2} + (c-t)\frac{dx}{dt} - ax = 0 \qquad (c, a \in \mathbb{C})$$

which for $c = 1$ reduces to the Laguerre differential equation and one of its solutions is

$$x_a(t) = \sum_{k=0}^{\infty} \binom{-a}{k} (-1)^k \frac{t^k}{\Gamma(k+1)}$$

satisfying the convolutional property

$$\frac{d}{dt}(x_a * x_b)(t) = x_{a+b}(t)$$

where $*$ represents the Mikusiński convolution.

We define the algebraic derivative

$$\mathcal{D}f(t) = \frac{-I^{\delta-1}}{\delta} tf(t), \qquad \text{for the convolution} \qquad (f*g)(t) = \int_0^t f(t-\psi)g(\psi)d\psi$$

where $I^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau$ represents the Riemann-Liouville fractional integral operator.

2000 *Mathematics Subject Classification*: 44A40, 26A33, 33A20.
The paper is in final form and no version of it will be published elsewhere.

The convolution $*$ is defined on the set

$$C_\delta = \left\{ f(t) = \sum_{k=1}^{\infty} a_k t^{k\delta-1} \text{ uniformly convergent on compact subsets of } [0,\infty) \right\}$$

introduced by Alamo and Rodríguez in [1].

Using a similar technique as in[2] and the appropriate operational calculus for $*$ we can get a solution of the following integral-diferential equation

$$-\mathcal{D}(D^\delta)^2 x + (1+\mathcal{D})D^\delta x - ax = 0 \quad (a \in \mathbb{C}) \quad (\delta > 1)$$

which we denote by $x_a(t)$, satisfying

$$D^\delta[x_a * x_b](t) = x_{a+b}(t).$$

2. An operational calculus for D^δ. The algebraic derivative $\mathcal{D}$. Let $\delta > 1$ be a fixed real number (when $\delta = 1$, it reduces to Kierat and Skórnik's case). As Alamo and Rodríguez [1] did, we define the set of positive real variable functions with complex values

$$C_\delta = \left\{ f(t) = \sum_{k=1}^{\infty} a_k t^{k\delta-1} \text{ uniformly convergent on compact subsets of } [0,\infty) \right\}.$$

They proved that $(C_\delta, +, \cdot_{\mathbb{C}})$ is a vector space.

Unlike these authors, we will consider in C_δ the Mikusiński convolution given by $(f*g)(t) = \int_0^t f(t-\psi)g(\psi)d\psi$.

From the definition of $*$ we get immediately the following propositions.

Proposition 1. *1.* $t^{k\delta-1} * t^{m\delta-1} = B(k\delta, m\delta)t^{(k+m)\delta-1}$.
2. $(f*g)(t) = \sum_{k=2}^{\infty} \{\sum_{j=1}^{k-1} a_j b_{k-j} B[j\delta,(k-j)\delta]\} t^{k\delta-1}$.

Here $B(u,v) = \int_0^1 (1-t)^{u-1} t^{v-1} dt$ *represents the beta function,* $f(t) = \sum_{k=1}^{\infty} a_k t^{k\delta-1}$ *and* $g(t) = \sum_{k=1}^{\infty} b_k t^{k\delta-1}$.

This proposition shows us that $*$ is a closed operation on C_δ, so we can conclude that $(C_\delta, +, *)$ is a subring of $(C, +, *)$. Here C represents the set of continuous complex functions of a positive real variable. Mikusiński [3] and Yosida [5] showed that the convolution $*$ has no zero divisors and there is no unit element on the set C, thus we can state the next proposition.

Proposition 2. $(C_\delta, +, *)$ *is a commutative non-unitary ring without zero divisors.*

Remark. It can be proved in a direct way that $(C_\delta, +, *)$ is a ring.

Therefore, C_δ can be extended to its field of fractions $M_\delta = C_\delta \times (C_\delta - \{0\})/\sim$, where the equivalence relation $\sim$ is defined, as usual, by $(f_1, g_1) \sim (f_2, g_2) \Leftrightarrow f_1 * g_2 = g_1 * f_2$; actually M_δ is a subfield of the Mikusiński field. The elements of M_δ will be called operators, and from now on we denote by $\frac{f}{g}$ the equivalence class of the pair (f, g).

The operations of sum, multiplication and product by a scalar can be defined on M_δ through

- $\dfrac{f_1}{g_1} + \dfrac{f_2}{g_2} = \dfrac{f_1 * g_2 + g_1 * f_2}{g_1 * g_2}$

- $\dfrac{f_1}{g_1} \cdot \dfrac{f_2}{g_2} = \dfrac{f_1 * f_2}{g_1 * g_2}$
- $\alpha \dfrac{f}{g} = \dfrac{\alpha f}{g}$

Alamo and Rodríguez [1] showed that the operator D^δ is an endomorphism on C_δ and proved that for all $f(t) = \sum_{k=1}^{\infty} a_k t^{k\delta-1}$ in C_δ

$$D^\delta I^\delta f(t) = f(t),$$

$$I^\delta D^\delta f(t) = f(t) - a_1 t^{\delta-1} = f(t) - [t^{1-\delta} f(t)]_{t=0} t^{\delta-1}, \tag{2.1}$$

$$(I^\delta)^m (D^\delta)^m f(t) = f(t) - \sum_{j=1}^{m} a_j t^{j\delta-1}. \tag{2.2}$$

These identities will be useful for our development.

The next proposition allows us to identify the operator I^δ and its positive integer powers with certain functions in C_δ.

Proposition 3. *Let $f(t) \in C_\delta$ and $k \in \mathbb{N}$, then we have*

1. $\frac{t^{\delta-1}}{\Gamma(\delta)} * f(t) = I^\delta f(t)$.

2. $\frac{t^{k\delta-1}}{\Gamma(k\delta)} * f(t) = (I^\delta)^k f(t) = I^{k\delta} f(t)$.

Proof. The first asertion is a consequence of the definition of the convolution $*$, and using induction method we can get the second one.

Following Mikusiński [3], we denote by $l_\delta = \frac{t^{\delta-1}}{\Gamma(\delta)} \equiv I^\delta$. So when we write $l_\delta f(t)$ we will understand $I^\delta f(t)$.

Now we remark that we can consider $C_\delta \subset M_\delta$ since C_δ is isomorphic to a subring of M_δ through the map $f \rightsquigarrow \frac{l_\delta f}{l_\delta}$. In a similar way the field $\mathbb{C}$ of complex numbers can be embedded into M_δ by associating with every $\alpha \in \mathbb{C}$ the so called numerical operator $[\alpha] = \frac{\alpha t^{\delta-1}}{t^{\delta-1}}$. The following basic properties of these numerical operators are immediate.

Proposition 4. *1.* $[\alpha] + [\beta] = [\alpha + \beta]$.
2. $[\alpha] \cdot [\beta] = [\alpha\beta]$.

From now on we denote the numerical operators $[\alpha]$ by α when it leads to no confusion.

Proposition 5. *Let $v_\delta \in M_\delta$ be the algebraic inverse of l_δ. For any function $f(t) = \sum_{k=1}^{\infty} a_k t^{k\delta-1}$,*

$$v_\delta f(t) = D^\delta f(t) + \Gamma(\delta) a_1 \tag{2.3}$$

$$v_\delta^m f(t) = (D^\delta)^m f(t) + \sum_{j=1}^{m} a_j \Gamma(j\delta) v_\delta^{m-j} \tag{2.4}$$

Proof. To see (2.3), having the identity (2.1) we act on both sides by the operator v_δ and take into account that $a_1 t^{\delta-1}$ is identified with $\frac{l_\delta a_1 t^{\delta-1}}{l_\delta} \in M_\delta$, so $v_\delta a_1 t^{\delta-1} = \frac{a_1 t^{\delta-1}}{l_\delta} = [\Gamma(\delta) a_1] = \Gamma(\delta) a_1$. For (2.4) it is analogous, acting on both sides of (2.2) by v_δ^m.

The next step is to define an operator over C_δ which will be an algebraic derivative.

DEFINITION 1. Let $f \in C_\delta$. We define the operator $\mathcal{D}$ as follows:

$$\mathcal{D}f(t) = -\frac{I^{\delta-1}}{\delta} t f(t).$$

We need to know how $\mathcal{D}$ acts on any member of C_δ.

PROPOSITION 6. *$\mathcal{D}f(t) \in C_\delta$ for all $f(t) \in C_\delta$.*

PROOF. It is not dificult to show that if $f(t) = \sum_{k=1}^{\infty} a_k t^{k\delta-1}$, then

$$-\frac{I^{\delta-1}}{\delta} t f(t) = \sum_{k=1}^{\infty} b_k t^{(k+1)\delta-1}$$

where $b_k = -a_k \frac{k\Gamma(k\delta)}{\Gamma[(k+1)\delta]}$. An equivalent and more manageable expression is

$$-\frac{I^{\delta-1}}{\delta} t f(t) = (-t^{\delta-1}) * \Big[\sum_{k=1}^{\infty} c_k t^{k\delta-1}\Big]$$

where $c_k = \frac{k a_k}{\Gamma(\delta)}$.

Now we establish a proposition which shows that $\mathcal{D}$ is an algebraic derivative on C_δ.

PROPOSITION 7. *For any functions f and g in C_δ, we have:*

1. $\mathcal{D}[f(t) + g(t)] = \mathcal{D}f(t) + \mathcal{D}g(t)$.
2. $\mathcal{D}(f * g)(t) = ([\mathcal{D}f] * g)(t) + (f * [\mathcal{D}g])(t)$.

PROOF. 1. It immediately follows by taking into account that $\frac{-I^{\delta-1}}{\delta} t$ is a linear operator.

2. Let $f(t) = \sum_{k=1}^{\infty} a_k t^{k\delta-1}$ and $g(t) = \sum_{k=1}^{\infty} b_k t^{k\delta-1}$, then we have:

$$\mathcal{D}(f * g)(t) = \mathcal{D} \sum_{k=2}^{\infty} \Big\{ \sum_{j=1}^{k-1} a_j b_{k-j} B[j\delta, (k-j)\delta] \Big\} t^{k\delta-1}.$$

If we denote $c_k = \sum_{j=1}^{k-1} a_j b_{k-j} B[j\delta, (k-j)\delta]$, by using the result obtained in the proof of proposition 6 and the second identity of proposition 1, we can get

$$\mathcal{D}(f * g)(t) = (-t^{\delta-1}) * \sum_{k=2}^{\infty} \frac{k}{\Gamma(\delta)} c_k t^{k\delta-1}.$$

In a similar way, it can be proved that

$$([\mathcal{D}f] * g)(t) = (-t^{\delta-1}) * \Big[\sum_{k=2}^{\infty} \Big\{ \sum_{j=1}^{k-2} \frac{j}{\Gamma(\delta)} a_j b_{k-j} B[j\delta, (k-j)\delta] \Big\} t^{k\delta-1} \Big]$$

and

$$(f * [\mathcal{D}g])(t) = (-t^{\delta-1}) * \Big[\sum_{k=2}^{\infty} \Big\{ \sum_{j=1}^{k-2} \frac{(k-j)}{\Gamma(\delta)} a_j b_{k-j} B[j\delta, (k-j)\delta] \Big\} t^{k\delta-1} \Big]$$

and using the last three identities the proof is concluded.

Now we can extend the definition of $\mathcal{D}$ to the field M_δ, as usual, by:

$$\mathcal{D}\frac{f}{g} = \frac{[\mathcal{D}f] * g - f * [\mathcal{D}g]}{g * g} \qquad (f \in C_\delta,\ g \in (C_\delta - \{0\})),$$

$$\mathcal{D}\frac{p}{q} = \frac{[\mathcal{D}p] \cdot q - p \cdot [\mathcal{D}q]}{q^2} \qquad (p \in M_\delta,\ q \in (M_\delta - \{0\})).$$

The next proposition shows the behavior of the algebraic derivative over some particular members of M_δ and will be used to solve an integral-differential equation.

PROPOSITION 8. *Let* $1 = \frac{t^{\delta-1}}{t^{\delta-1}}$ *the unit of* M_δ, $0 = \frac{0}{t^{\delta-1}}$, $v_\delta = \frac{1}{l_\delta}$ *the algebraic inverse of* l_δ *in* M_δ *and* $n \in \mathbb{N}$. *Then:*

1. $\mathcal{D}1 = 0$.
2. $\mathcal{D}\alpha = 0$ *(α being a numerical operator).*
3. $\mathcal{D}(\alpha p) = \alpha \mathcal{D}p$ *(for any $p \in M_\delta$).*
4. $\mathcal{D}l_\delta^n = -nl_\delta^{n+1}$.
5. $\mathcal{D}v_\delta^n = nv_\delta^{n-1}$.
6. $\mathcal{D}(1-\alpha l_\delta)^n = n\alpha l_\delta^2(1-\alpha l_\delta)^{n-1}$.
7. $\mathcal{D}(v_\delta - \alpha)^n = n(v_\delta - \alpha)^{n-1}$.

PROOF. (1) and (2) follow by a simple calculation. (3) is a direct consequence of (2). In (4) we will use induction. Since in our case

$$\mathcal{D}l_\delta = \mathcal{D}\frac{t^{\delta-1}}{\Gamma(\delta)} = -\frac{t^{2\delta-1}}{\Gamma(2\delta)} = -l_\delta^2,$$

if we suppose that (4) is true for $n = k$, then

$$\mathcal{D}l_\delta^{k+1} = \mathcal{D}(l_\delta \cdot l_\delta^k) = [\mathcal{D}l_\delta] \cdot l_\delta^k + l_\delta \cdot [\mathcal{D}l_\delta^k] = -(k+1)l_\delta^{k+2}.$$

For (5) we consider the fact that $v_\delta = \frac{1}{l_\delta}$, so it is not difficult to see that $\mathcal{D}v_\delta = 1$ using (1) and (4), afterwards we can use induction again. Finally, to get (6) and (7),

$$\begin{aligned}\mathcal{D}(1-\alpha l_\delta)^n &= \mathcal{D}\left[\sum_{k=1}^{n}\binom{n}{k}(-\alpha l_\delta)^{n-k}\right] = -\sum_{k=1}^{n}\binom{n}{k}(-\alpha)^{n-k}(n-k)l_\delta^{n-k+1} \\ &= -\sum_{k=1}^{n} n\binom{n-1}{k}(-\alpha)l_\delta^2(-\alpha l_\delta)^{n-k-1} = n\alpha l_\delta^2(1-\alpha l_\delta)^{n-1}\end{aligned}$$

however $(v_\delta - \alpha)^n = \frac{(1-\alpha l_\delta)^n}{l_\delta^n}$, using (6), (4) and the definition of $\mathcal{D}$ on M_δ the proof can be concluded.

REMARK. The last proposition holds for $n \in \mathbb{Z}$ since $p^{-n} = \frac{1}{p^n}$ for any $p \in M_\delta$.

The second identity of the last proposition tell us that the algebraic derivative of the numerical operators is zero, but furthermore we can establish the inverse result.

PROPOSITION 9. *Given* $p \in M_\delta$, *if* $\mathcal{D}p = 0$ *then* p *is a numerical operator.*

PROOF. Let $p = \frac{f}{g}$ and $\mathcal{D}p = 0$. Since

$$\mathcal{D}p = \frac{([\mathcal{D}f] * g)(t) - (f * [\mathcal{D}g])(t)}{(g * g)(t)}$$

it follows that:

$$([\mathcal{D}f] * g)(t) - (f * [\mathcal{D}g])(t) = 0. \tag{2.5}$$

If we denote $f(t) = \sum_{k=1}^{\infty} a_k t^{k\delta-1}$ and $g(t) = \sum_{k=1}^{\infty} b_k t^{k\delta-1}$, then we have

$$([\mathcal{D}f] * g)(t) = (-t^{\delta-1}) * \left[\sum_{k=2}^{\infty}\left\{\sum_{j=1}^{k-1}\frac{j}{\Gamma(\delta)}a_j b_{k-j} B(j\delta, (k-j)\delta)\right\}t^{k\delta-1}\right]$$

and

$$(f * [\mathcal{D}g])(t) = (-t^{\delta-1}) * \left[\sum_{k=2}^{\infty} \left\{ \sum_{j=1}^{k-1} \frac{(k-j)}{\Gamma(\delta)} a_j b_{k-j} B(j\delta, (k-j)\delta) \right\} t^{k\delta-1} \right]$$

so, (2.5) implies that

$$\sum_{j=1}^{k-1} (2j-k) a_j b_{k-j} B[j\delta, (k-j)\delta] = 0 \quad (\forall k \geq 2). \tag{2.6}$$

Now let us suppose $b_1 \neq 0$. If we take in (2.6) $k = 3$ and $k = 4$ we can get respectively

$$a_1 b_2 = a_2 b_1 \quad \text{and} \quad a_1 b_3 = a_3 b_1;$$

next it is easy to prove that $a_m b_n = a_n b_m$ when $a_1 b_n = a_n b_1$ and $a_1 b_m = a_m b_1$.

Finally, in order to get that $a_1 b_k = a_k b_1$ for any $k \geq 2$ we take into account the following identities

$$\sum_{j=1}^{k-1} (2j-k) a_j b_{k-j} B[j\delta, (k-j)\delta]$$

$$= \sum_{j=1}^{r-1} (2j-2r)(a_j b_{2r-j} - a_{2r-j} b_j) B[j\delta, (2r-j)\delta] \quad (k = 2r),$$

$$\sum_{j=1}^{k-1} (2j-k) a_j b_{k-j} B[j\delta, (k-j)\delta]$$

$$= \sum_{j=1}^{r} [2j - (2r+1)](a_j b_{2r+1-j} - a_{2r+1-j} b_j) B[j\delta, (2r+1-j)\delta] \quad (k = 2r+1).$$

Therefore if $b_1 \neq 0$ we can establish that $a_k = \frac{a_1}{b_1} b_k$ for any $k \geq 1$, in other words

$$\frac{f}{g} = \frac{\alpha g}{g} = \frac{\alpha t^{\delta-1}}{t^{\delta-1}} = [\alpha] \quad \left(\alpha = \frac{a_1}{b_1}\right).$$

To conclude the proof we remark that, however $b_1 = 0$ and $a_1 \neq 0$ allow us to prove that $b_k = 0$ for any k in opposition to the fact that $g(t) \in C_\delta - \{0\}$, $b_1 = 0$ implies $a_1 = 0$ so we can start with $b_2 \neq 0$ and so on.

3. The use of $\mathcal{D}$ to solve an integral-differential equation. As an application of the results obtained in the preceding section, we will solve the integral-differential equation

$$\begin{aligned} &-\mathcal{D}(D^\delta)^2 x(t) + (1+\mathcal{D}) D^\delta x(t) - a x(t) = 0 \quad (x(t) \in C_\delta) \quad (a \in \mathbb{C}) \\ &[t^{1-\delta} x(t)]_{t=0} = 0. \end{aligned} \tag{3.1}$$

Making use of (2.1), (2.2), (2.3), (2.4) and proposition 8, the equation (3.1) becomes

$$\frac{\mathcal{D}x(t)}{x(t)} = \frac{a-1}{v_\delta} - \frac{a}{v_\delta - 1} = \frac{l_\delta [l_\delta (1-a) - 1]}{1 - l_\delta}. \tag{3.2}$$

Several facts are immediately deduced from this expression.

Proposition 10. *1.* $x_a = l_\delta(1-l_\delta)^{-a} \in M_\delta$ *is a solution of (3.2).*

2. $x_a(t) = \dfrac{t^{\delta-1}}{\Gamma(a)}\,{}_1\Psi_1\left[\begin{matrix}(a,1);\\ & t^\delta\\ (\delta,\delta);\end{matrix}\right]$ *is a solution of (3.1).*

3. $$D^\delta \int_0^t \frac{(t-\tau)^{\delta-1}}{\Gamma(a)}\,{}_1\Psi_1\left[\begin{matrix}(a,1);\\ & (t-\tau)^\delta\\ (\delta,\delta);\end{matrix}\right]\frac{\tau^{\delta-1}}{\Gamma(b)}\,{}_1\Psi_1\left[\begin{matrix}(b,1);\\ & \tau^\delta\\ (\delta,\delta);\end{matrix}\right]d\tau$$
$$= \frac{t^{\delta-1}}{\Gamma(a+b)}\,{}_1\Psi_1\left[\begin{matrix}(a+b,1);\\ & t^\delta\\ (\delta,\delta);\end{matrix}\right]$$
where ${}_1\Psi_1$ *represents the Wright generalized hypergeometric functions (cf. [4]).*

Proof. 1. We have
$$\mathcal{D}(1-l_\delta)^{-a} = \mathcal{D}\left[1+\sum_{k=1}^{\infty}\binom{-a}{k}(-1)^k\frac{t^{k\delta-1}}{\Gamma(k\delta)}\right]$$
$$= \sum_{k=1}^{\infty}(-a)\binom{-a-1}{k-1}(-1)^{k-1}\frac{t^{(k+1)\delta-1}}{\Gamma[(k+1)\delta]} = (-a)l_\delta^2(1-l_\delta)^{-a-1}$$
thus
$$\frac{\mathcal{D}[l_\delta(1-l_\delta)^{-a}]}{l_\delta(1-l_\delta)^{-a}} = \frac{l_\delta[l_\delta(1-a)-1]}{1-l_\delta}.$$

2. The solution $x_a = l_\delta(1-l_\delta)^{-a}$ admits a representation of the form (cf. [3, p. 171])
$$x_a = l_\delta(1-l_\delta)^{-a} = \sum_{k=0}^{\infty}\binom{-a}{k}(-1)^k l_\delta^{k+1} = t^{\delta-1}\sum_{k=0}^{\infty}\frac{(a)_k}{\Gamma(k+1)}\frac{t^{k\delta}}{\Gamma[(k+1)\delta]}$$
$$= \frac{t^{\delta-1}}{\Gamma(a)}\sum_{k=0}^{\infty}\frac{\Gamma(k+a)}{\Gamma(k\delta+\delta)}\frac{t^{k\delta}}{\Gamma(k+1)}$$
thus (cf. [4, p. 50]),
$$x_a(t) = \frac{t^{\delta-1}}{\Gamma(a)}\,{}_1\Psi_1\left[\begin{matrix}(a,1);\\ & t^\delta\\ (\delta,\delta);\end{matrix}\right].$$

3. It is consequence of the preceding items.

Remark. If $-a \in \mathbb{N}$ the series which appears in the proof of the last proposition becomes a polynomial of fractional degree.

References

[1] J. A. Alamo and J. Rodríguez, *Cálculo operacional de Mikusiński para el operador de Riemann-Liouville y su generalizado*, Rev. Acad. Canar. Cienc. 1 (1993), 31–40.

[2] W. Kierat and K. Skórnik, *A remark on solutions of the Laguerre differential equation*, Integral Transforms and Special Functions 1 (1993), 315–316.

[3] J. Mikusiński, *Operational Calculus*, Pergamon, Oxford, 1959.
[4] H. M. Srivastava and H. L. Manocha, *A Treatise on Generating Functions*, Ellis Horwood, 1984.
[5] K. Yosida, *Operational Calculus. A Theory of Hyperfunctions*, Springer-Verlag, New York, 1984.