TWO CENTURIES OF THE TERM
“ALGEBRAIC ANALYSIS”

DANUTA PRZEWORSKA-ROLEWICZ
Institute of Mathematics, Polish Academy of Sciences
Śniadeckich 8, 00-950 Warszawa, Poland
e-mail: rolewicz@impan.gov.pl

Abstract. The term “Algebraic Analysis” in the last two decades is used in two completely different senses. It seems that at least one is far away from its historical roots. Thus, in order to explain this misunderstanding, the history of this term from its origins is recalled.

The term “Analyse Algébrique” (“Algebraic Analysis”) in the last two decades is used in two completely different senses. It seems that at least one is far away from its historical roots. Thus, in order to explain this misunderstanding, I would like to recall the history of this term from its origins.

The term “Algebraic Analysis” was initially used by Lagrange two hundred years ago in the title of his book (cf. References, [1797-1813]) in order to point out that most of results have been obtained by algebraic operations on analytic quantities. As we shall see later, in that general and common sense this name was used in the 19th and 20th century.

To begin with, we should explain what is meant by Algebraic Analysis at present (cf. Encyclopaedia of Mathematics, [1997]).

The main idea of Algebraic Analysis in its present, more strict, sense derives from the fact that the differential operator $D = \frac{d}{dt}$ is right invertible in several function spaces.

Foundations of Algebraic Analysis are the following: Let $L(X)$ be the set of all linear operators with domains and ranges in a linear space $X$ (in general, without any topology) over a field $F$ of scalars with characteristic zero and let $L_0(X) = \{A \in L(X) : \text{dom } A = X\}$. Let $R(X)$ be the set of all right invertible operators in $L(X)$. Let $D \in R(X)$. Let $R_D \subset L_0(X)$ be the set of all right inverses for $D$, i.e. $DR = I$ (identity operator) if $R \in R_D$ (i.e. the Leibniz-Newton formula holds: $\frac{d}{dt} \int_a^t f(s)ds = f(t)$ for all functions $f$ from the space under consideration). Moreover, $\text{dom } D = RX \oplus \ker D$. For all $R, R' \in $
\[ \mathcal{R}_D, x \in X, Rx - R'x \in \ker D, \text{i.e. the difference of two primitives of } x \text{ is a constant. Let} \]
\[ \mathcal{F}_D = \{ F \in L_0(X) : F^2 = F; FX = \ker D \text{ and } \exists R_0 \in \mathcal{R}_D, FR = 0 \}. \]
Any \( F \in \mathcal{F}_D \) is said to be an \textit{initial} operator for \( D \) (corresponding to an \( R \)). One can prove that any projection \( F' \) onto \( \ker D \) is an initial operator for \( D \) corresponding to a right inverse \( R' = R - F'R \) independent of the choice of an \( R \in \mathcal{R}_D \). If two right inverses (resp. initial operators) commute with each other, then they are equal. Thus this theory is essentially \textbf{noncommutative}. An operator \( F \) is initial for \( D \) if and only if there is an \( R \in \mathcal{R}_D \) such that \( F = I - RD \) on \( \text{dom } D \). The last formula yields (by a two-lines induction) the \textit{Taylor Formula}:
\[ I = \sum_{k=0}^{n} R^n FD^n + R^n F^n \text{ on } \text{dom } D^n (n \in \mathbb{N}). \]

With these facts one can obtain Calculus and solutions to linear equations (under appropriate assumptions on resolving equations). If the field \( \mathbb{F} \) is algebraically closed then solutions of linear equations with scalar coefficients can be calculated by a decomposition of a rational function into vulgar fractions (as in Operational Calculus). If \( X \) is a commutative algebra with unit \( e \), \( \mathbb{F} = \mathbb{C} \) and \( D \) satisfies the \textit{Leibniz Condition}:
\[ D(xy) = xDy + yDx \text{ for } x, y \in \text{dom } D, \]
then the \textit{Trigonometric Identity} holds. Some results can be proved also for left invertible operators, even for operators having either finite nullity or finite deficiency. There is a rich theory of shifts and periodic problems. Recently, logarithms and antilogarithms have been introduced and studied (even in noncommutative algebras; cf. Przeworska-Rolewicz [1998]). It means that Algebraic Analysis is no more purely linear.

Main advantages of Algebraic Analysis are:

- \textbf{simplifications of proofs} due to an algebraic description of problems under consideration;
- \textbf{algorithms} for solving “similar” problems, although these similarities could be rather far from each other and very formal;
- several \textbf{new results} even in the classical case of the operator \( \frac{d}{dt} \) (which was, indeed, unexpected).

There are several applications to ordinary and partial differential equations with scalar and variable coefficients, functional-differential equations and for discrete analogues of these equations, for instance, for difference equations. There are also some results for nonlinear equations.

It should be pointed out that in Algebraic Analysis a notion of \textit{convolution} is not necessary. Also there is no need to have a structure of a \textit{field}, like the \textit{Mikusiński field}. This, together with the noncommutativity of right inverses and initial operators, shows the essential distinction of Algebraic Analysis from Operational Calculus.

As we have mentioned at the beginning, the term “Algebraic Analysis” was first used in the title of a book by Joseph Louis de Lagrange in 1797: \textit{Théorie des Fonctions Analytiques contenant Les Principes du Calcul Différentiel, dégagés de toute considération}
 Nevertheless, it seems that this term had been used much earlier, for instance, by Euler and d'Alembert. In the time of Lagrange this term was in use also by Lacroix, Pfaff and others (cf. Jahnke [1993], Dhombre [1992]). A source for an algebraic treatment of analytic quantities was, no doubt, the Leibniz symbolic calculus. Some traces led back to algebraic investigations of Viète, hence a long time before the birth of analysis (cf. Bigaglia and Nastasi [1986]), some to Pascal, Fermat and Huygens (cf. Fenaroli and Penco [1979]).

The title could be explained by the fact that at that time the notions of limit, convergence, and so on, were not made precise. However, the main reason was to point out that most of the results were obtained by algebraic operations on analytic quantities.

The next book with the term “algebraic analysis” in its title was written in German by F. B. A. Lembert [1815], according to the library catalogue of the former Jacobson Schule in Seesen (Harz) (private communication of Professor Hans Lausch, Monash University, Clayton (Melbourne), December 1992).

The same term as a subtitle was used by Augustin Louis Cauchy [1821]. In his introduction he wrote that, as to methods, he had sought “to make them as rigorous as those of geometry, so as never to have recourse to justifications drawn from the generality of algebra”.

This may provoke the idea that the name “algebraic analysis” emphasized that the analyses under consideration were more or less “different” from other concepts of analysis at that time. And, indeed, it was so.

Unfortunately, Cauchy was forced by the authorities of L'École Polytechnique to change his way of teaching mathematical analysis so that, finally, Analyse algébrique completely disappeared after the academic year 1924/25 as an autonomous part of the course (cf. Gilain [1989]).

However, outside L'École Polytechnique, Lagrange's book was used as a handbook for several years without regard to these dramatic changes.

Hans Lausch wrote (again a private communication; April, 1989):

...On the theme “The occurrence of the term ‘algebraic analysis’ in history”: I ran across a biographical account by the historian Alfred Stern (1846-1936). Stern tells of his father Moritz Abraham Stern (1807-1894), who together with Riemann succeeded Dirichlet 1859 in Göttingen and was the first German Jew to hold a chair. His lectures covered a wide area, as his son reports: “...popular astronomy, algebraic analysis and elements of analytic geometry, ...”

Note that M. A. Stern was obliged to deliver lectures in algebraic analysis, since this was an essential part of the mathematical syllabus of the Prussian educational system according to the reforms of Wilhelm von Humboldt in 1809–10. This system was obligatory until the end of the 19th century (cf. Jahnke [1992], [1993]). Probably, the textbook
of Lembert was also prepared for that reason. C. G. Jacobi during his studies was under strong influence of that trend (cf. Knobloch, Pieper and Pulte [1995]).

Next the title “algebraic analysis” was used in the following books:

- Oskar Schlömilch [1845]; 5th ed. 1873;
- J. Dienger [1851];
- W. Gallenkamp [1860];
- M. A. Stern [1860];
- G. Novi [1863];
- Johann Lieblein (Professor of Technical University in Prag), [1867], a collection of exercises for the book of O. Schlömilch;
- Karl Hattendorf [1877];
- A. Capelli and G. Garbieri [1886];
- Salvatore Pincherle [1893];
- A. Capelli [1894], whose book concerned algebraic curves,
- Ernesto Cesàro [1894], who wrote in his Prefazione:

\[\ldots Forse un giorno mi deciderò a pubblicare un libro di “istituzioni analitiche” fonda tre cattedre diverse sotto i nomi di Algebra, Geometria analitica e Calcolo. Per ora, pure stando a disagio in un programma necessariamente eterogeneo e pieno di addentellamenti fittizi con altre materie, io mi propongo di guidare il lettore, con mosse rapide a sicura, aa far larga messe di fatti analitici, ponendo a base (non a fine) un’esposizione rigoros dei principi dell’Analisi algebrica.\]

In his book second book (in German, [1904]) Cesàro wrote (p. 683):

\[\ldots so kann man sagen, dass die Integration die inverse Operation der Differentiation ist.\]

This means that Cesàro not only made an attempt at a common treatment of Algebra, Linear Algebra, Calculus and Differential Equations, but also followed the ideas of Leibniz and Lagrange.

- Heinrich Burkhardt [1903];
- E. Cesàro [1904], a German translation;
- Salvatore Pincherle [1906];
- D. O. Grave (Дмитр Александрович Граве, 1863–1939, in Russian 1911 and in Russian and Ukrainian 1938–1939); Grave devoted his books to an analysis of algebraic problems which appeared in connection with systems of differential equations describing movements of three bodies.

We should mentioned here a large survey of Algebraic Analysis itself by Alfred Pringsheim and Georg Faber given in Encyclopädie der Mathematischen Wissenschaften [1909-1921].

D. Laugwitz in his book [1996], in Section 0.4.2 entitled Algebraische Analysis, gives an overview of Riemann’s contributions in this direction.

In the academic years 1973/74 and 1974/75 I was delivering lectures for the first and second year students at the Cybernetics Faculty of the Military Engineering Academy

At the International Conference on Generalized Functions and Operational Calculi held in Varna, September 29–October 6, 1975, I had a talk, in which I described the differences between Operational Calculi and the newly born “modern” Algebraic Analysis in the following way:

*By Operational Calculus in a common sense is meant: 1. a method of integration which uses algebraic properties of the derivation operator; 2. applications of this method for solving differential equations, mainly ordinary differential equations with scalar coefficients.*

The first algebraic connection between the derivation and the integration is as old as the Calculus itself. Namely, G. W. Leibniz observed in a non-published paper in 1675 that the symbol used by him as a symbol of derivation can be treated as an “inverse” of the symbol of integration. He applied many times this fact and he wrote about it in his “Historia et origo calculi differentialis”.

The further history of Operational Calculus, in particular, in the last fifty years, is well known.

*Algebraic Analysis appears when (…) for a right invertible operator acting in a linear space one is interested not only in one right inverse, but simultaneously, in the family of all right inverses and the family of initial operators, which are, in general, non-commutative.*

We point out that nothing like a convolution is used in Algebraic Analysis. (cf. Przeworska-Rolewicz [1979])

This distinction between Operational Calculi and Algebraic Analysis was immediately adopted by the mathematical community working in this field and related topics.

Note that my first my papers concerning the theory of right invertible operators and induced families of initial operators and right inverses appeared in 1972 (cf. Przeworska-Rolewicz, Studia Math. 48 (1973), 129–144).

The next use of the term “Algebraic Analysis” in a book was (not counting several collections concerning microlocal analysis; cf. References) in the book of Masaki Kashiwara, Takahiro Kawai, Tatsuo Kimura [1986], also concerned with microlocal analysis.

The reviewer of this last book, J. L. Brylinsky (Bulletin AMS, 18 (1988), 104-105) began his review with the following statement:

“Algebraic analysis” is a term coined by Mikio Sato …

A few months before, on the turn of 1987 and 1988 there was published a monograph of the present author [1988]. In its review (Zbl 696.47002) M. Z. Nashed writes:

*What is “Algebraic Analysis”? The name “Algebraic Analysis” was used by Lagrange in a subtitle to the second revised and enlarged edition of his “Théorie des fonctions analytiques” (1813). The same subtitle was used by Cauchy in 1821 in his “Cours d’analyse de l’École Royale Polytechnique, 1re partie, Analyse algébrique”. In his introduction he
D. PRZEWORSKA-ROLEWICZ wrote “As to methods, I have sought to make them as rigorous as those of geometry, so as never to have recourse to justifications drawn from the generality of algebra”.

The term “algebraic analysis” appears in the title of over a dozen books without a clear delineation of what it describes; often it is used in contexts where the common thread is tenuous or doesn’t exist. Of the older books we mention “Istituzioni di Analisi Algebrica” by A. Capelli (Napoli, 1894); “Corso di Analisi Algebrica con Introduzione al Calcolo Infinitesimale” by E. Cesàro (Torino, 1894); “Elementares Lehrbuch der Algebraischen Analysis und der Infinitesimal Rechnung”, also by E. Cesàro (Leipzig, 1904), “Course of Algebraic Analysis” (in Russian, Kiev, 1911) and “Treatise on Algebraic Analysis” (in Russian and Ukrainian, Kiev, 1938-1939; Zbl. 20, 197) by D. O. Grave. Capelli’s book concerns algebraic curves, the two books by Grave are devoted to algebraic problems, while Cesàro’s book is an attempt at a common treatment of Algebra, Linear Algebra, Calculus and Differential Equations, close to what is often called nowadays “linear analysis”.

In 1988 two volumes entitled “Algebraic Analysis” (Vol I: Zbl. 665.00008) were published. Edited by M. Kashiwara and T. Kawai, the two volumes consist of papers dedicated to Professor Mikio Sato, “the initiator of algebraic analysis in the twentieth century”, whose research seems to aim at the renaissance of “Algebraic Analysis” of Euler, and deals with the theory of hyperfunctions (which Sato invented in 1957) and with other topics not related to the classical books mentioned earlier. Finally, we mention “Foundations of Algebraic Analysis”, Princeton (1986; Zbl. 605.35001) by M. Kashiwara, T. Kawai and T. Kimura which is concerned with microlocal analysis.

The author of the book under review has her own very interesting explanation of what led to the type of “Algebraic Analysis” considered in her book. But it is clear from above that “Algebraic Analysis” means markedly different things to different authors; one has to examine the meaning from the context in which it is used. For the present book, this is best highlighted by quoting titles of the main chapters and key phrases: Calculus of right invertible operators, general solution of equations with right invertible operators, initial and boundary value problems, well-posed and ill-posed boundary value problems, periodic operators and elements, shift operators and shift invariant subspaces, D-algebras, perturbations and nonlinear problems, metric properties in algebraic analysis.

The common thread and concepts throughout the book (9 chapters) are the proper definition of initial operators for right invertible operators and their fundamental properties, and “Calculus in Algebraic Analysis” by which the author means the theory of right invertible operators in linear spaces (without any topology, in general) - think of indefinite integrals!

The first edition of the preprint Short story of term “Algebraic Analysis” was prepared by the present author (cf. Przeworska-Rolewicz [1996]) in the following way.

Items until 1940 were found more or less at random. Items from 1940 on were found in Mathematical Reviews by means of MathSciNet asking for the term “algebraic analysis”. Until 1994 there were 318 items. Not all of them are of the same “kind”. I made a selection in the following way. First, I cancelled a few which did not contain this term in any form. Next, there are chosen collections which contain this term in their titles. All individual papers in these collections are also cancelled. The remaining papers and books contain...
the term “algebraic analysis” either in an explicit form in their titles or in their review (which is denoted by “aa in review”) or in the author’s summaries (denoted by “aa in summary”) or, in a few cases, in the cited introductions (again denoted similarly). This means that, by assumption, this is not a full bibliography, excerpts only. However, these excerpts (since 1940) can be easily completed by means of MathSciNet.

The present paper is a revised and extended version of Short Story, since it contains, like foundations of Algebraic Analysis and items from Zentralblatt für Mathematik und ihre Grenzgebiete since 1943 (collected in the same manner, by CompactMATH) due to the kind help of Professor Bernd Wegner from Technische Universität Berlin. This paper is also essentially enriched thanks to Professor Ernst Albrecht from Universität des Saarlandes in Saarbrücken, who kindly sent me a xerox copy of the survey article Algebraische Analysis by Alfred Pringsheim and Georg Faber from Encyclopädie der Mathematischen Wissenschaften [1909–1921].

There are also added items from MathSciNet up to date and a few others found again at random.

Note that one book (in Spanish) had in 1960 the fifth edition (!). Another one (in Serbo-Kroatische) had in 1970 the third edition (cf. References). I am not able to find earlier references, because these books have not been reviewed in Zentralblatt.

A conclusion follows if you look through References. The term “algebraic analysis” was used through centuries and is still used whenever authors wish to point out their algebraic approach to analytic problems (or, possibly, to their far generalizations). For that reason, one can find in References papers in Theoretical Physics, Logics, Graph Theory, System Theory, and so on.

Acknowledgements. The author would like to express her appreciation and gratitude to Professors Hans Lausch, Monash University, Clayton (Melbourne), Ernst Albrecht, Universität des Saarlandes, Saarbrücken, and Bernd Wegner, Technische Universität Berlin, for their support in collecting material used in this paper.

References

COLLECTIONS:

1975

1976

1978
1979

1981

1984

1986

1988


1989


1992

1994


1995
Algebraic analysis of solvable lattice models. Dedicated to Mikio Sato and Ludwig D. Fadeev. 
Providence, RI, 1995.

1996
Bibun hoteishiki no kansu kaisekiteki oyobi daisu kaisekiteki kenkyu. Research on functional 

1997
Chokyokusho kaiseki ni okeru daisu kaisekiteki hoho (in Japanese) [Algebraic analysis methods 
Tokui setsudo no daisu kaisekigaku (in Japanese) [Algebraic analysis of singular perturbations]. 

PAPERS AND BOOKS:

1797–1813
LAGRANGE J. P., Théorie des Fonctions Analytiques contenant Les Principes du Calcul Diffé-
rentiel, dégagés de toute considération d’infiniment petits, d’évanouissans, de limites et de 
fluxions, et réduit à l’analyse algébrique de quantités finies. 2nd revised and enlarged ed., 
Mme Ve Courcier, Imprimeur-Libraire pour les Mathématiques, Paris, 1813 (1st ed. - 1797).

1815

1821
Paris, 1821.

1845

1851
DIENER J., Grundzüge der Algebraischen Analysis. Karlsruhe, 1851.

1860
GALLENKAMP W., Die Elemente der Mathematik. III. Teil. Die algebraische Analysis und die 

1863

1867
Satov, Prag, 1867.

1877
HATTENDORF K., Algebraische Analysis. Hannover, 1877.
1886

1893
Pincherle S., Analisi Algebrica. Milano, 1893

1894

1903

1904
Cesàro E., Elementares Lehrbuch der Algebraischen Analysis und der Infinitesimal Rechnung mit zahlreichen Übungsbeispielen. B. G. Teubner, Leipzig, 1904; translated from the Italian manuscript by Doctor Gerhard Kowalewski, Universität Greisswald.

1906

1909

1911

1938–39

1943

1955
Elston F. G., The last theorem of Fermat not only a problem of algebraic analysis but also a probability problem? Math. Mag. 28 (1955), 150–152.

1958

1960
1969

1970

1972

1973

1974

1975

1976

1977

1978

1979


1980


1981


1982


1983


1984


Georgescu G., *Algebraic analysis of the logic with the quantifier “there exist uncountably many”.* Algebra Universalis, 1, 19 (1984), 99–105. (aa in review)


1985

1986

1987
TWO CENTURIES OF "ALGEBRAIC ANALYSIS"  61


1988


1989


1990


1991


1992


1993


1994


1995


**1996**


1997


1998


1999


Przeworska-Rolewicz D., True shifts induced by right invertible operators are hypercyclic.
