NORMED BARRELLED SPACES

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Introduction. In this paper we present a general “gliding hump” condition that implies the barrelledness of a normed vector space. Several examples of subspaces of $l^1$ are shown to be barrelled using the theorem. The barrelledness of the space of Pettis integrable functions is also implied by the theorem (this was first shown in [3]).

Results. The following theorem generalizes that given in [8].

Definition. Let $X$ be a normed space. $S \subset X$ is a bounding set if $S \subset \text{Sphere}(X)$ and if $(f_n) \subset X'$ is an unbounded sequence in the dual space of $X$, there exists $(x_n) \subset S$ such that $\sup_n |f_n(x_n)| = \infty$.

Theorem. Let $S$ be a bounding set in a normed space $X$. If for any sequence $(x_n) \subset S$, there exist a sequence $(d_k) \in \text{Ball}(l^1)$, $d_k \neq 0$, integers $N_k \geq 0$, $C > 0$, such that for every subsequence $(x_{n_k})$ of $(x_n)$ there is a further subsequence $(x_{n_{kl}})$ and $x \in X$ with $x = \sum_{j=1}^{\infty} t_j x_j$ and

$$
\|t_{n_{kl}} x_{n_{kl}}\| - \sum_{j=n_{kl}+N_{kl}+1}^{\infty} \|t_j x_j\| \geq C|d_{kl}|,
$$

then $X$ is barrelled. (The condition needs to hold only for the index $l$ greater than some integer).

Proof. Suppose $X$ is not barrelled. Let $(f_n) \subset X'$ be a pointwise bounded sequence that is unbounded in norm, and let $(x_n) \subset S$ satisfy $\sup_n |f_n(x_n)| = \infty$. We will use the notation

$$
\|f\|_* = \sup_{x \in S} \{|f(x)|\}.
$$

The paper is in final form and no version of it will be published elsewhere.
Note that \( \| f \|_* \leq \| f \| \). Choose \( n_1 \) and \( x_{n_1} \) such that \( \| f_{n_1} \|_* > \frac{1}{d_{k1}} \) and \( \| f_{n_1} \| - |f_{n_1}(x_{n_1})| < C \frac{|d_{k1}|}{2} \). Fix \( N_1 \). We will also use the notation
\[
M_k = \sup \{|f_n(x)| : n \in \mathbb{N}, 1 \leq i \leq n_k + N_k\} \tag{1}
\]
Note that \( M_k \) is finite by the pointwise boundedness of \( (f_n) \), and that \( M_k \) depends on \( n_k \) and \( N_k \). Choose \( n_2 > n_1 + N_1 \) and \( x_{n_2} \) such that \( \| f_{n_2} \|_* > M_1 \frac{2}{|d_{k2}|} \) and \( \| f_{n_2} \| - |f_{n_2}(x_{n_2})| < C \frac{|d_{k2}|}{2} \). Continue inductively to get
\[
\| f_{n_k} \|_* > M_{k-1} \frac{k}{|d_k|} \tag{2}
\]
and
\[
\| f_{n_k} \|_* - |f_{n_k}(x_{n_k})| < C \frac{|d_k|}{2}. \tag{3}
\]
Now choose \( x \in X \) that satisfies the hypotheses of the theorem. Then
\[
|f_{n_k}(x)| = \left| f_{n_k} \left( \sum_{j=1}^{n_{k_{i-1}} + N_{k_{i-1}}} t_j x_j \right) \right|
\]
\[
\geq |f_{n_k} (t_{n_{k_i}} x_{n_{k_i}})| - \left| f_{n_k} \left( \sum_{j=1}^{n_{k_{i-1}} + N_{k_{i-1}}} t_j x_j \right) \right|
\]
\[
\geq |f_{n_k} (t_{n_{k_i}} x_{n_{k_i}})| - M_{k_{i-1}} - \| f_{n_k} \|_* \left( \sum_{j=n_{k_{i-1}} + N_{k_{i-1}} + 1}^{\infty} \| t_j x_j \| \right)
\]
using (1) and the fact that \((t_j) \in \text{Ball}(l^1)\). Continuing,
\[
\geq \| t_{n_{k_i}} x_{n_{k_i}} \| \| f_{n_k} \|_* \left( 1 - C \frac{|d_{k_i}|}{2} \right) - M_{k_{i-1}} - \| f_{n_k} \|_* \left( \sum_{j=n_{k_{i-1}} + N_{k_{i-1}} + 1}^{\infty} \| t_j x_j \| \right)
\]
using (3). Then,
\[
\geq \| t_{n_{k_i}} x_{n_{k_i}} \| \| f_{n_k} \|_* \left( 1 - C \frac{|d_{k_i}|}{2} \right) - M_{k_{i-1}} - \| f_{n_k} \|_* (\| t_{n_{k_i}} x_{n_{k_i}} \| - C |d_{k_i}|)
\]
using (\( \ast \)) on the third term. Simplifying we get
\[
= \| f_{n_{k_i}} \|_* C |d_{k_i}| \left( 1 - \frac{\| t_{n_{k_i}} x_{n_{k_i}} \|}{2} \right) - M_{k_{i-1}}.
\]
Since \( \| t_{n_{k_i}} x_{n_{k_i}} \| \) will eventually be less than one we can write
\[
\geq \| f_{n_{k_i}} \|_* C \frac{|d_{k_i}|}{2} - M_{k_{i-1}} \geq \left( C \frac{k_i}{2} - 1 \right) M_{k_{i-1}}
\]
using (2). This goes to infinity as \( l \to \infty \), which contradicts the assumption of pointwise boundedness. \( \blacksquare \)
There are several examples of barrelledness in normed spaces that are implied by the theorem.

**Corollary 1.** Let $S$ be a bounding set in a normed space $X$. Suppose that for any sequence $(x_n) \subset S$ and any null sequence $(t_n)$ of real numbers, we have for every subsequence $(t_{n_k}x_{n_k})$ of $(t_n x_n)$ there is a further subsequence $(t_{n_k}x_{n_k})$ such that $\sum t_{n_k}x_{n_k}$ converges in $X$. Then $X$ is barrelled.

**Proof.** Choose $t_n = \frac{1}{n}$, $d_k = \frac{1}{k!}$ and $N_k = 0$ in the theorem. Let $(n_k) = (k)$, and $x = \sum t_k x_k$. Then we can show that

$$\|t_{n_k}x_{n_k}\| - \sum_{j=n_k-1+1}^{\infty} \|t_j x_j\| \geq \frac{1}{2} \frac{1}{k_l!}.$$  

We have $\|t_{n_k}x_{n_k}\| = \frac{1}{k_l!}$ and noting that $t_j = 0$ for $j \neq k_l$, it is easy to check that

$$\sum_{j=n_k-1+1}^{\infty} \|t_j x_j\| = \sum_{j=1}^{\infty} \frac{1}{(k_l+j)!} \leq \frac{1}{2} \frac{1}{k_l!}$$

for $k_l \geq 3$, so the result follows from the theorem.

A topological vector space $X$ is a $K$-space if for every null sequence $(x_n)$ in $X$, every subsequence of $(x_n)$ has a further subsequence $(x_{n_k})$ such that $\sum_{k=1}^{\infty} x_{n_k}$ converges in $X$. $X$ is an $A$-space if for every bounded sequence $(x_n)$ in $X$ and every null sequence of real (or complex) numbers $(t_n)$, every subsequence of $(t_n x_n)$ has a subsequence $(t_{n_k}x_{n_k})$ such that $\sum_{k=1}^{\infty} t_{n_k}x_{n_k}$ converges in $X$. The corollary above implies that normed $A$-spaces, and thus normed $K$-spaces, are barrelled. (See [9] for more information on these spaces).

It is shown in the paper [3] that the space of Pettis integrable functions defined on an atomless measure space satisfies property $K$ with respect to a bounding set and so is barrelled. We refer the interested reader to the paper for details.

The following corollary is a general condition for a dense subspace of $l^1$ to be barrelled. $\Phi$ denotes the span of $\{e^i : i \in N\}$, the canonical unit vectors.

**Corollary 2.** Let $\Phi \subset E \subset l^1$. $E$ is barrelled if there exist a sequence $(d_k) \in \text{Ball}(l^1)$ with $d_k \neq 0$ for all $k$, $C > 0$, and integers $N_k \geq 0$ such that for every increasing sequence of integers $(n_k)$, there is a subsequence $(n_{k_l})$, and $x \in E$ such that

$$\|x_{n_k}e^{n_k}\| - \sum_{j=n_{k_l-1}+N_{k_l-1}+1}^{\infty} \|x_{j}e^j\| \geq C|d_{k_l}|.$$  

(Again, the condition need only hold for the index $l$ greater than some integer).

**Proof.** The canonical unit vectors form a bounding set in $l^1$, so the result follows directly from the theorem.

There are many examples of dense, barrelled subspaces of $l^1$. See [6] for an application of these spaces. We next show that some of these examples are implied by the corollary.
Corollary 3. Let $\Phi \subset E \subset l^1$. Suppose that $E$ is monotone (that is, $\chi_A \cdot x \in E$ for all $A \subset N$ and $x \in E$, where $\cdot$ stands for coordinatewise multiplication) and there is a fixed sequence $(b_k) \in l^1 \setminus \Phi$ such that for any increasing sequence of integers $(i_k)$ there is a subsequence $(i_{k_l})$ and $x \in E$ for which $x_{i_{k_l}} = b_{k_l}$. Then $E$ is barrelled.

Proof. We need to find a subsequence of $(b_k)$ that satisfies the hypotheses of the Corollary 2. We can find $A \subset N$ such that $\chi_A \cdot (b_k) = (g_k e_i)$, where $(|g_k|)$ is a non-zero, decreasing sequence that satisfies $|g_k| > \sum_{j>k} |g_j|$. Now let $d_k = |g_k|$. Then, since $E$ is monotone, we can find an infinite subset $B$ and $x \in E$ such that $\chi_B \cdot (|x_{i_{k_l}}|) = d_{k_l}$. Let $N_k = 0$. Note that $x_j = 0$ for $j \neq i_{k_l}$. Then

$$|x_{i_{k_l}}| \geq \sum_{j=i_{k_l-1}+1}^{\infty} |x_j| \geq d_{k_l}$$

So the result follows from Corollary 2.

The following result is due to Bennett [2].

Corollary 4. $l^0 = \cap_{0<p<1} l^p$ is a barrelled subspace of $l^1$.

Proof. $l^0$ is monotone and satisfies the hypotheses of Corollary 3 (say with $b_k = 2^{-k}$) so this result follows.

Bennett actually showed that scarce copies of $l^0$ (copies that satisfy a sparseness condition) are also barrelled. See [1] for details. This follows easily from the above result.

The following example is due to Ruckle [4]. A sequence space is symmetric if $x_{\pi(n)} \in E$ for all $(x_n) \in E$ and for any permutation $\pi$ of $N$.

Corollary 5. Let $\Phi \subset E \subset l^1$, $E \neq \Phi$, and $E$ symmetric. Then $E$ is a dense, barrelled subspace of $l^1$.

Proof. Actually, Ruckle’s proof contains ideas similar to those used in the main Theorem. We will need to define $N_k$ to be something other than 0 for the first time. We essentially follow his construction and notation.

Let $x \in E \setminus \Phi$, and $(h_j)$ an increasing sequence in $N$ such that $h_j - h_{j-1} > 1$ and $|x_{h_j}| < |x_{h_{j-1}}|$.

Let $\pi$ be the permutation of the integers which interchanges $h_{2n-1}$ and $h_{2n}$ and leaves the other integers the same.

If $v = x - x_\pi$ then $v \in E$ and

$$v_j = 0 \text{ for } j \notin \{h_1, h_2, \ldots\}$$

$$v_{h_{2n-1}} = -v_{h_{2n}} \neq 0 \text{ for all } n \in N.$$ 

Let $\{n_1, n_2, \ldots\}$ be an increasing sequence of integers for which

$$\sum_{j>m} |v_{h_{2n-1}}| + |v_{h_{2n}}| < \frac{1}{2} |v_{h_{2n-1}}|.$$ (4)
Denote by $\theta$ the permutation which interchanges $h_{2n_j-1}$ and $h_{2n_j}$, and leaves the remaining integers unchanged. Let $u = \frac{1}{2}(v - v_0)$. Then $u \in E$, $u_j = 0$ for $j \not\in \{h_{2n_j-1}, h_{2n_j}, \ j = 1, 2, 3, \ldots\}$

$$uh_{2n_j-1} = -uh_{2n_j} \neq 0.$$ 

Note that if $d_k = |uh_{2n_k-1}|$, then

$$\sum_{k>m} d_k < \frac{1}{4}d_m.$$  \hfill (5)

We will define a sequence $y$ that is a permutation of $u$ and that satisfies the hypotheses of Corollary 2 with $N_k = 1$ for all $k$.

Let $(i_k)$ be any sequence of integers with $i_k > i_{k-1} + 1$ and $i_1 > 2$. Let $y_1 = -d_1$, $y_{i_k} = d_k$, $y_{i_k+1} = -d_{k+1}$, and $y_j = 0$ otherwise. 

Then $y$ is a permutation of $u$ since it exhausts $\pm a_k$ and has infinitely many zeros. We will show that $y$ satisfies the hypotheses of Corollary 2.

We need to show that

$$\|y_{i_k}e_{i_k}\| - \sum_{j=1}^{i_k+1} \|y_{j}e_{j}\| \geq \frac{1}{2}|d_k|.$$ 

This follows easily from conditions (4) and (5) above. Note that we need $N_k = 1$ for the result to follow. \hfill ■

The following corollary uses the idea of a *modulus*. A modulus is a non-negative, subadditive function $q$ on $[0, \infty)$ which is continuous and 0 at 0.

**Corollary 6.** Subspaces of $l^1$ determined by a modulus $q$ are barrelled.

**Proof.** Ruckle shows in [5] that the space of all sequences $s$ in $l^1$ that satisfy $\sum_j q(s_j) < \infty$ is symmetric and properly contains $\Phi$, so this result follows from the previous corollary. \hfill ■

The following result is due to Saxon [7]. If $b$ is any fixed sequence in $l^1$ with infinite support, then the dilation space $E_b$ is the span of $\Phi$ and the vectors $\sum_i b_i e_{n_i}$ as $(n_i)$ ranges through all increasing subsequences of $\mathbb{N}$.

**Corollary 7.** Dilation subspaces of $l^1$ that properly contain $\Phi$ are barrelled.

**Proof.** Let $b$ be a fixed sequence in $l^1$ with infinite support, and let $b_{i_j} = d_j$ be a subsequence of $b$ that is non-zero, $|d_j|$ decreasing, and $|d_j| > 2 \sum_{i>j} |d_i|$. We can construct a sequence in $E_b$ that satisfies the hypotheses of Corollary 2 by a cancellation process similar to that used in the corollary above on symmetric spaces. In what follows, the subsequence $(b_{i_j})$ is shown in brackets. We can define dilations of $(b_i)$, denoted $(c_i)$ and $(f_i)$, as follows:

$$(b_i) \quad = b_1 \langle b_2 \rangle b_3 \langle b_5 \rangle b_6 \langle b_7 \rangle b_8 \ldots$$

$$(c_i) \quad = b_1 0 \langle b_2 \rangle b_3 0 \langle b_5 \rangle b_6 0 \langle b_7 \rangle b_8 \ldots$$

$$(f_i) \quad = b_1 \langle b_2 \rangle 0 \langle b_3 \rangle b_4 \langle b_5 \rangle 0 b_6 \langle b_7 \rangle 0 b_8 \ldots$$

$$(c_i) - (f_i) = 0 - b_2 b_2 0 0 - b_5 b_5 0 - b_7 b_7 \ldots$$
Let \( (x_i) = (c_i) - (f_i) \) and \( N_j = i_{j+1} - i_j \). Note that the sequence \( N_j \) is fixed. Given any increasing sequence of integers \( (i_k) \) we can find a subsequence \( (i_{k_l}) \) so that we can dilate the sequence \( x \) to define a sequence that satisfies \( y_{i_{k_l}} = d_l \), \( y_{i_{k_l-1} + N_{k_l-1}} = -d_l \), and \( y_j = 0 \) otherwise. This can be accomplished by adding zeros between the \(-d_l\) and \(d_l\) terms. We can check that the hypotheses of the theorem are satisfied:

\[
|y_{i_{k_l}}| - \sum_{j=i_{k_l-1} + N_{k_l-1} + 1}^{\infty} |y_j| > \frac{1}{2} |d_l| > \frac{1}{2} |d_{k_l}|.
\]

The last inequality follows from the definition of \( d_k \) and the fact that \( |d_k| \) is decreasing.

In fact, we do not know of a dense, barrelled subspace of \( l^1 \) for which the barrelledness is not implied by Corollary 2. It would be very interesting to have an example of such a space or, even better, a gliding hump characterization of the dense, barrelled subspaces of \( l^1 \).

**References**


