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NORMED BARRELLED SPACES

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Introduction. In this paper we present a general "gliding hump" condition that implies the barrelledness of a normed vector space. Several examples of subspaces of l^1 are shown to be barrelled using the theorem. The barrelledness of the space of Pettis integrable functions is also implied by the theorem (this was first shown in [3]).

Results. The following theorem generalizes that given in [8].

Definition. Let X be a normed space. $S \subset X$ is a *bounding set* if $S \subset Sphere(X)$ and if $(f_n) \subset X'$ is an unbounded sequence in the dual space of X, there exists $(x_n) \subset S$ such that $\sup_n |f_n(x_n)| = \infty$.

Theorem. *Let S be a bounding set in a normed space X. If for any sequence $(x_n) \subset S$, there exist a sequence $(d_k) \in Ball(l^1)$, $d_k \neq 0$, integers $N_k \geq 0$, $C > 0$, such that for every subsequence (x_{n_k}) of (x_n) there is a further subsequence $(x_{n_{k_l}})$ and $x \in X$ with $x = \sum_{j=1}^{\infty} t_j x_j$ and*

$$\|t_{n_{k_l}} x_{n_{k_l}}\| - \sum_{\substack{j=n_{k_{l-1}}+N_{k_{l-1}}+1 \\ j \neq n_{k_l}}}^{\infty} \|t_j x_j\| \geq C|d_{k_l}|, \tag{$\star$}$$

then X is barrelled. (The condition needs to hold only for the index l greater than some integer).

Proof. Suppose X is not barrelled. Let $(f_n) \subset X'$ be a pointwise bounded sequence that is unbounded in norm, and let $(x_n) \subset S$ satisfy $\sup_n |f_n(x_n)| = \infty$. We will use the notation

$$\|f\|_* = \sup_{x \in S}\{|f(x)|\}. \tag{1}$$

The paper is in final form and no version of it will be published elsewhere.

Note that $\|f\|_* \leq \|f\|$. Choose n_1 and x_{n_1} such that $\|f_{n_1}\|_* > \frac{1}{|d_1|}$ and $\|f_{n_1}\| - |f_{n_1}(x_{n_1})| < C\frac{|d_1|}{2}$. Fix N_1. We will also use the notation

$$M_k = \sup\{|f_n(x_i)| : n \in \mathbf{N},\ 1 \leq i \leq n_k + N_k\} \quad (1)$$

Note that M_k is finite by the pointwise boundedness of (f_n), and that M_k depends on n_k and N_k. Choose $n_2 > n_1 + N_1$ and x_{n_2} such that $\|f_{n_2}\|_* > M_1\frac{2}{|d_2|}$ and $\|f_{n_2}\|_* - |f_{n_2}(x_{n_2})| < C\frac{|d_2|}{2}$. Continue inductively to get

$$\|f_{n_k}\|_* > M_{k-1}\frac{k}{|d_k|} \tag{2}$$

and

$$\|f_{n_k}\|_* - |f_{n_k}(x_{n_k})| < C\frac{|d_k|}{2}. \tag{3}$$

Now choose $x \in X$ that satisfies the hypotheses of the theorem. Then

$$\begin{aligned}
|f_{n_{k_l}}(x)| &= \Big|f_{n_{k_l}}\Big(\sum_{j=1}^{n_{k_{l-1}}+N_{k_{l-1}}} t_j x_j\Big) + f_{n_{k_l}}(t_{n_{k_l}} x_{n_{k_l}}) + f_{n_{k_l}}\Big(\sum_{\substack{j=n_{k_{l-1}}+N_{k_{l-1}}+1 \\ j\neq n_{k_l}}}^{\infty} t_j x_j\Big)\Big| \\
&\geq |f_{n_{k_l}}(t_{n_{k_l}} x_{n_{k_l}})| - \Big|f_{n_{k_l}}\Big(\sum_{j=1}^{n_{k_{l-1}}+N_{k_{l-1}}} t_j x_j\Big)\Big| - \Big|f_{n_{k_l}}\Big(\sum_{\substack{j=n_{k_{l-1}}+N_{k_{l-1}}+1 \\ j\neq n_{k_l}}}^{\infty} t_j x_j\Big)\Big| \\
&\geq |f_{n_{k_l}}(t_{n_{k_l}} x_{n_{k_l}})| - M_{k_{l-1}} - \|f_{n_{k_l}}\|_*\Big(\sum_{\substack{j=n_{k_{l-1}}+N_{k_{l-1}}+1 \\ j\neq n_{k_l}}}^{\infty} \|t_j x_j\|\Big)
\end{aligned}$$

using (1) and the fact that $(t_j) \in Ball(l^1)$. Continuing,

$$\geq \|t_{n_{k_l}} x_{n_{k_l}}\|\|f_{n_{k_l}}\|_*\Big(1 - C\frac{|d_{k_l}|}{2}\Big) - M_{k_{l-1}} - \|f_{n_{k_l}}\|_*\Big(\sum_{\substack{j=n_{k_{l-1}}+N_{k_{l-1}}+1 \\ j\neq n_{k_l}}}^{\infty} \|t_j x_j\|\Big)$$

using (3). Then,

$$\geq \|t_{n_{k_l}} x_{n_{k_l}}\|\|f_{n_{k_l}}\|_*\Big(1 - C\frac{|d_{k_l}|}{2}\Big) - M_{k_{l-1}} - \|f_{n_{k_l}}\|_*(\|t_{n_{k_l}} x_{n_{k_l}}\| - C|d_{k_l}|)$$

using $(\star)$ on the third term. Simplifying we get

$$= \|f_{n_{k_l}}\|_* C|d_{k_l}|\Big(1 - \frac{\|t_{n_{k_l}} x_{n_{k_l}}\|}{2}\Big) - M_{k_{l-1}}.$$

Since $\|t_{n_{k_l}} x_{n_{k_l}}\|$ will eventually be less than one we can write

$$\geq \|f_{n_{k_l}}\|_* C\frac{|d_{k_l}|}{2} - M_{k_{l-1}} \geq \Big(C\frac{k_l}{2} - 1\Big)M_{k_{l-1}}$$

using (2). This goes to infinity as $l \to \infty$, which contradicts the assumption of pointwise boundedness. ∎

There are several examples of barrelledness in normed spaces that are implied by the theorem.

COROLLARY 1. *Let S be a bounding set in a normed space X. Suppose that for any sequence $(x_n) \subset S$ and any null sequence (t_n) of real numbers, we have for every subsequence $(t_{n_k} x_{n_k})$ of $(t_n x_n)$ there is a further subsequence $(t_{n_{k_l}} x_{n_{k_l}})$ such that $\sum t_{n_{k_l}} x_{n_{k_l}}$ converges in X. Then X is barrelled.*

PROOF. Choose $t_n = \frac{1}{n!}$, $d_k = \frac{1}{k!}$ and $N_k = 0$ in the theorem. Let $(n_k) = (k)$, and $x = \sum t_{k_l} x_{k_l}$. Then we can show that

$$\|t_{n_{k_l}} x_{n_{k_l}}\| - \sum_{\substack{j=n_{k_{l-1}}+1 \\ j \neq n_{k_l}}}^{\infty} \|t_j x_j\| \geq \frac{1}{2}\frac{1}{k_l!}.$$

We have $\|t_{n_{k_l}} x_{n_{k_l}}\| = \frac{1}{k_l!}$ and noting that $t_j = 0$ for $j \neq k_l$, it is easy to check that

$$\sum_{\substack{j=n_{k_{l-1}}+1 \\ j \neq n_{k_l}}}^{\infty} \|t_j x_j\| = \sum_{j=1}^{\infty} \frac{1}{(k_{l+j})!} < \frac{1}{2}\frac{1}{k_l!}$$

for $k_l \geq 3$, so the result follows from the theorem. ∎

A topological vector space X is a *K-space* if for every null sequence (x_n) in X, every subsequence of (x_n) has a further subsequence (x_{n_k}) such that $\sum_{k=1}^{\infty} x_{n_k}$ converges in X. X is an *A-space* if for every bounded sequence (x_n) in X and every null sequence of real (or complex) numbers (t_n), every subsequence of $(t_n x_n)$ has a subsequence $(t_{n_k} x_{n_k})$ such that $\sum_{k=1}^{\infty} t_{n_k} x_{n_k}$ converges in X. The corollary above implies that normed A-spaces, and thus normed K-spaces, are barrelled. (See [9] for more information on these spaces).

It is shown in the paper [3] that the space of Pettis integrable functions defined on an atomless measure space satisfies property K with respect to a bounding set and so is barrelled. We refer the interested reader to the paper for details.

The following corollary is a general condition for a dense subspace of l^1 to be barrelled. Φ denotes the span of $\{e^i : i \in \mathbf{N}\}$, the canonical unit vectors.

COROLLARY 2. *Let $\Phi \subset E \subset l^1$. E is barrelled if there exist a sequence $(d_k) \in Ball(l^1)$ with $d_k \neq 0$ for all k, $C > 0$, and integers $N_k \geq 0$ such that for every increasing sequence of integers (n_k), there is a subsequence (n_{k_l}), and $x \in E$ such that*

$$\|x_{n_{k_l}} e^{n_{k_l}}\| - \sum_{\substack{j=n_{k_{l-1}}+N_{k_{l-1}}+1 \\ j \neq n_{k_l}}}^{\infty} \|x_j e^j\| \geq C|d_{k_l}|.$$

(Again, the condition need only hold for the index l greater than some integer).

PROOF. The canonical unit vectors form a bounding set in l^1, so the result follows directly from the theorem. ∎

There are many examples of dense, barrelled subspaces of l^1. See [6] for an application of these spaces. We next show that some of these examples are implied by the corollary.

Corollary 3. *Let $\Phi \subset E \subset l^1$. Suppose that E is monotone (that is, $\chi_A \cdot x \in E$ for all $A \subset \mathbf{N}$ and $x \in E$, where $\cdot$ stands for coordinatewise multiplication) and there is a fixed sequence $(b_k) \in l^1 \backslash \Phi$ such that for any increasing sequence of integers (i_k) there is a subsequence (i_{k_l}) and $x \in E$ for which $x_{i_{k_l}} = b_{k_l}$. Then E is barrelled.*

Proof. We need to find a subsequence of (b_k) that satisfies the hypotheses of the Corollary 2. We can find $A \subset \mathbf{N}$ such that $\chi_A \cdot (b_k) = (g_k e^{i_k})$, where $(|g_k|)$ is a non-zero, decreasing sequence that satisfies $|g_k| > \sum_{j>k} |g_j|$. Now let $d_k = |g_k|$. Then, since E is monotone, we can find an infinite subset B and $x \in E$ such that $\chi_B \cdot (|x_{i_{k_l}}|) = d_{k_l}$. Let $N_k = 0$. Note that $x_j = 0$ for $j \neq i_{k_l}$. Then

$$|x_{i_{k_l}}| - \sum_{\substack{j=i_{k_{l-1}}+1 \\ j \neq i_{k_l}}}^{\infty} |x_j| \geq d_{k_l}$$

So the result follows from Corollary 2. ∎

The following result is due to Bennett [2].

Corollary 4. *$l^0 = \cap_{0<p<1} l^p$ is a barrelled subspace of l^1.*

Proof. l^0 is monotone and satisfies the hypotheses of Corollary 3 (say with $b_k = 2^{-k}$) so this result follows. ∎

Bennett actually showed that scarce copies of l^0 (copies that satisfy a sparseness condition) are also barrelled. See [1] for details. This follows easily from the above result.

The following example is due to Ruckle [4]. A sequence space is *symmetric* if $x_{\pi(n)} \in E$ for all $(x_n) \in E$ and for any permutation π of $\mathbf{N}$.

Corollary 5. *Let $\Phi \subset E \subset l^1$, $E \neq \Phi$, and E symmetric. Then E is a dense, barrelled subspace of l^1.*

Proof. Actually, Ruckle's proof contains ideas similar to those used in the main Theorem. We will need to define N_k to be something other than 0 for the first time. We essentially follow his construction and notation.

Let $x \in E \backslash \Phi$, and (h_j) an increasing sequence in $\mathbf{N}$ such that

$$h_j - h_{j-1} > 1 \text{ and } |x_{h_j}| < |x_{h_{j-1}}|.$$

Let π be the permutation of the integers which interchanges h_{2n-1} and h_{2n} and leaves the other integers the same.

If $v = x - x_\pi$ then $v \in E$ and

$$v_j = 0 \text{ for } j \notin \{h_1, h_2, \ldots\}$$
$$v_{h_{2n-1}} = -v_{h_{2n}} \neq 0 \text{ for all } n \in \mathbf{N}.$$

Let $\{n_1, n_2, \ldots\}$ be an increasing sequence of integers for which

$$\sum_{j>m} |v_{h_{2n_j-1}}| + |v_{h_{2n_j}}| < \frac{1}{2} |v_{h_{n_m}}|. \tag{4}$$

Denote by θ the permutation which interchanges h_{2n_j-1} and h_{2n_j}, and leaves the remaining integers unchanged. Let $u = \frac{1}{2}(v - v_\theta)$. Then $u \in E$,

$$u_j = 0 \text{ for } j \notin \{h_{2n_j-1}, h_{2n_j},\ j = 1,2,3,\dots\}$$
$$u_{h_{2n_j-1}} = -u_{h_{2n_j}} \neq 0.$$

Note that if $d_k = |u_{h_{2n_k-1}}|$, then

$$\sum_{k>m} d_k < \frac{1}{4} d_m. \tag{5}$$

We will define a sequence y that is a permutation of u and that satisfies the hypotheses of Corollary 2 with $N_k = 1$ for all k.

Let (i_k) be any sequence of integers with $i_k > i_{k-1} + 1$ and $i_1 > 2$. Let

$$y_1 = -d_1,\ y_{i_1} = d_1,\ y_{i_k} = d_k,\ y_{i_k+1} = -d_{k+1},\ \text{and } y_j = 0 \text{ otherwise.}$$

Then y is a permutation of u since it exhausts $\pm a_k$ and has infinitely many zeros. We will show that y satisfies the hypotheses of Corollary 2.

We need to show that

$$\|y_{i_k} e^{i_k}\| - \sum_{\substack{j=i_{k-1}+2 \\ j \neq n_k}}^{\infty} \|y_j e^j\| \geq \frac{1}{2} |d_k|.$$

This follows easily from conditions (4) and (5) above. Note that we need $N_k = 1$ for the result to follow. ∎

The following corollary uses the idea of a *modulus*. A modulus is a non-negative, subadditive function q on $[0, \infty)$ which is continuous and 0 at 0.

Corollary 6. *Subspaces of l^1 determined by a modulus q are barrelled.*

Proof. Ruckle shows in [5] that the space of all sequences s in l^1 that satisfy $\sum_j q(s_j) < \infty$ is symmetric and properly contains Φ, so this result follows from the previous corollary. ∎

The following result is due to Saxon [7]. If b is any fixed sequence in l^1 with infinite support, then the *dilation space* E_b is the span of Φ and the vectors $\sum_i b_i e^{n_i}$ as (n_i) ranges through all increasing subsequences of $\mathbf{N}$.

Corollary 7. *Dilation subspaces of l^1 that properly contain Φ are barrelled.*

Proof. Let b be a fixed sequence in l^1 with infinite support, and let $b_{i_j} = d_j$ be a subsequence of b that is non-zero, $|d_j|$ decreasing, and $|d_j| > 2 \sum_{l>j} |d_l|$. We can construct a sequence in E_b that satisfies the hypotheses of Corollary 2 by a cancellation process similar to that used in the corollary above on symmetric spaces. In what follows, the subsequence (b_{i_j}) is shown in brackets. We can define dilations of (b_i), denoted (c_i) and (f_i), as follows:

$$\begin{aligned}
(b_i) &= b_1\ \langle b_2 \rangle\ b_3\ b_4\ \langle b_5 \rangle\ b_6\ \langle b_7 \rangle\ b_8\ \dots \\
(c_i) &= b_1\ 0\ \langle b_2 \rangle\ b_3\ b_4\ 0\ \langle b_5 \rangle\ b_6\ 0\ \langle b_7 \rangle\ b_8\ \dots \\
(f_i) &= b_1\ \langle b_2 \rangle\ 0\ b_3\ b_4\ \langle b_5 \rangle\ 0\ b_6\ \langle b_7 \rangle\ 0\ b_8\ \dots \\
(c_i) - (f_i) &= 0\ \ {-b_2}\ b_2\ 0\ 0\ \ {-b_5}\ b_5\ 0\ \ {-b_7}\ b_7\ \dots
\end{aligned}$$

Let $(x_i) = (c_i) - (f_i)$ and $N_j = i_{j+1} - i_j$. Note that the sequence N_j is fixed. Given any increasing sequence of integers (i_k) we can find a subsequence (i_{k_l}) so that we can dilate the sequence x to define a sequence that satisfies $y_{i_{k_l}} = d_l$, $y_{i_{k_{l-1}}+N_{k_{l-1}}} = -d_l$, and $y_j = 0$ otherwise. This can be accomplished by adding zeros between the $-d_l$ and d_l terms. We can check that the hypotheses of the theorem are satisfied:

$$|y_{i_{k_l}}| - \sum_{\substack{j=i_{k_{l-1}}+N_{k_{l-1}}+1 \\ j\neq i_{k_l}}}^{\infty} |y_j| > \frac{1}{2}|d_l| > \frac{1}{2}|d_{k_l}|.$$

The last inequality follows from the definition of d_k and the fact that $|d_k|$ is decreasing. ■

In fact, we do not know of a dense, barrelled subspace of l^1 for which the barrelledness is not implied by Corollary 2. It would be very interesting to have an example of such a space or, even better, a gliding hump characterization of the dense, barrelled subspaces of l^1.

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