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REDUCTION OF DIFFERENTIAL EQUATIONS

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Abstract. Let (F, D) be a differential field with the subfield of constants C ($c \in C$ iff $Dc = 0$). We consider linear differential equations

$$Ly = D^n y + a_{n-1} D^{n-1} y + \ldots + a_0 y = 0, \tag{1}$$

where $a_0, \ldots, a_n \in F$, and the solution y is in F or in some extension E of F ($E \supseteq F$).

There always exists a (minimal, unique) extension E of F, where $Ly = 0$ has a full system $y_1, \ldots, y_n$ of linearly independent (over C) solutions; it is called the *Picard-Vessiot extension* of F

$$E = PV(F, Ly = 0).$$

The Galois group $G(E|F)$ of an extension field $E \supseteq F$ consists of all differential automorphisms of E leaving the elements of F fixed. If $E = PV(F, Ly = 0)$ is a Picard-Vessiot extension, then the elements $g \in G(E|F)$ are $n \times n$ matrices, $n = ordL$, with entries from C, the field of constants.

Is it possible to solve an equation (1) by means of linear differential equations of lower order $\leq n - 1$? We answer this question by giving neccessary and sufficient conditions concerning the Galois group $G(E|F)$ and its Lie algebra $A(E|F)$.

I. Introduction. A derivation D of a ring A is an additive mapping $a \to Da$ of A into itself satisfying

$$D(a \cdot b) = Da \cdot b + a \cdot Db.$$

A differential field (F, D) is a commutative field F together with a derivation D. In any differential field (F, D) the elements c with derivative $Dc = 0$ form a subfield C, called

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the field of constants, see Kaplansky [1]. In this paper we assume that the characteristic of the field F is 0, and that the subfield of constants C is algebraically closed.

Let (F, D) be a differential field. We consider monic, linear, differential equations

$$Ly = D^n y + a_{n-1} D^{n-1} y + \ldots + a_0 y = 0, \tag{1}$$

and their inhomogeneous counterparts $Ly = b$, where $b, a_0, \ldots, a_{n-1} \in F$ and the solution y is in F or in some extension E of F $(E \supseteq F)$.

There always exists a (minimal, unique) extension E of F, where $Ly = 0$ has a full system $u_1, \ldots, u_n$ of linearly independent (over the constants) solutions; it is called the *Picard-Vessiot extension of* F and denoted by

$$E = PVF(Ly = 0) = PVF;$$

for its existence and uniqueness, see Magid [1].

We have

a) $PVF(Ly = 0) = F < u_1, \ldots, u_n >$, where $u_1, \ldots, u_n$ is a full system of linearly independent (over the constants) solutions of $Ly = 0$ and $< u_1, \ldots, u_n >$ means that we adjoin to F the variables u_j and $D^m u_j$ for all $j = 1, \ldots, n$ and $m \geq 1$, and form polynomials and rational functions in those variables with coefficients from F.

b) $PVF(Ly = 0)$ has the same field of constants as F.

By going—possibly—to a further extension, we can also find all solutions of the inhomogeneous equation $Ly = b$, and define analogously the *Picard-Vessiot extension* $PVF(Ly = b)$.

Looking closely at the existence proof of Magid [1], we see that we can prove a little more:

THEOREM 1. *Let* $(F, D) \overset{\pi}{\longleftrightarrow} (F', D')$ *be two isomorphic differential fields with subfields of constants* C *and* C' *respectively (it follows that* $C \overset{\pi}{\longleftrightarrow} C'$*). Let* L *be given by (1) and consider its isomorphic image*

$$\pi L y = L' y = {D'}^n y + {a'}_{n-1} {D'}^{n-1} y + \ldots + {a'}_0 y.$$

Then π *extends to a differential isomorphism* $\widetilde{\pi}$ *of the Picard-Vessiot extensions*

$$PVF(Ly = 0) \overset{\widetilde{\pi}}{\longleftrightarrow} PVF'(L'y = 0).$$

It follows that the Galois groups G *and* G' *of those extensions are isomorphic:*

$$G' = \widetilde{\pi} \circ G \circ \widetilde{\pi}^{-1}.$$

For the proof see Skórnik, Wloka [2].

Theorem 1 applies to various operator fields (Mikusiński operators, Bessel operators etc.)

EXAMPLE 1. Let $\mathbb{C}(z)$ be the field of rational functions in the complex variable $z \in \mathbb{C}$. Then $(\mathbb{C}(z), \frac{d}{dz})$ is a differential field with $C = \mathbb{C}$. Defining $D\{f(t)\} = \{-tf(t)\}$ for functions $\{f(t)\}$, and extending D by the quotient rule, we find that the field of Mikusiński operators $(\mathfrak{M}, D)$ is a differential field with $C_{\mathfrak{M}} = \mathbb{C}$, see Mikusiński [1]. Let $\mathbb{C}(s)$ denote the field of rational functions in the (Mikusiński) operator $s = \frac{1}{\{1\}}$. We have (Wloka [1])

$$D = \frac{d}{ds}, \quad \text{and} \quad (\mathbb{C}(z), \frac{d}{dz}) \overset{\pi}{\longleftrightarrow} (\mathbb{C}(s), D),$$

where the isomorphism π is given by $\pi = id$ on $\mathbb{C}$ and $\pi(z) = s$.

The Galois group $G(E|F)$ of an extension field $E \supseteq F$ consists of all differential automorphisms of E leaving the elements of F fixed. If $E = PVF(Ly = 0)$ is a Picard-Vessiot extension, then the elements $g \in G(E|F)$ are $n \times n$ matrices, $n = ordL$, with elements from C, the field of constants. G is an algebraic matrix group, closed in the Zariski topology:

$$G(PVF(Ly = 0)|F) \subseteq GL(n, C).$$

II. m-Reduction. We define an m-reduction chain, $m = 0, 1, 2, \ldots$, as a chain of intermediate differential fields:

$$F = F_1 \subseteq F_2 \subseteq \ldots \subseteq F_l, \tag{2}$$

such that for $i = 2, \ldots, l$, either

a) F_i is a finite algebraic extension of F_{i-1} (we put $m = 0$ in this case), or

b) F_i is a Picard-Vessiot extension of F_{i-1} associated with an (inhomogeneous) differential equation $L_{i-1}y = b_{i-1}$ with coefficients (and b_{i-1}) in F_{i-1} and with $ordL_{i-1} \leq m$.

For $m \geq 2$ we may take in the definition above only homogeneous equations $L_{i-1}y = 0$, because a solution of $Dy = a$ (integrals!) is a solution of the homogeneous equation

$$D^2y - \frac{Da}{a}Dy = 0.$$

An equation (1) is called *m-reducible* if there exists an m-reduction chain (2) such that the Picard-Vessiot extension $PVF(Ly = 0)$ of F, associated with $Ly = 0$, lies in F_l:

$$PVF(Ly = 0) \subseteq F_l.$$

The case $m = 1$ is well known, it is Liouville reduction, see Kaplansky [1], Magid [1]. The case $m = 2$ was studied in Singer [1], it is called Euler reduction, or reduction by Mathematical Physics.

III. Conditions on groups and Lie algebras. Now we bring into play the Galois group $G(E|F)$ of an equation $Ly = 0$, here we put $E = PVF(Ly = 0)$. We use the language of linear algebraic groups and their Lie algebras, see Humphreys [1, 2].

An *algebraic group* G is *simple* if it has no proper infinite normal subgroups. A *Lie algebra* g is *simple* if it has no proper ideals. An algebraic group G has a Lie algebra g, it is also the Lie algebra of the component of unity G_I, and we have: G_I is simple if and only if g is simple. We call an *equation* (1) *simple* if $G_I(E|F)$ is simple or if $g(E|F)$ is simple, here we denoted $E = PVF(Ly = 0)$.

Example 2 (Lang [1]). The group

$$SL(n, C) \underset{\text{def}}{=} \{u \in GL(n, C) |\, det\, u = 1\}$$

is simple and we have

$$dim\, SL(n, C) = dim\, sl(n, C) = n^2 - 1, \quad n \geq 1,$$

where

$$sl(n, C) = \{v \in M(n, C) |\, \mathrm{trace}\, v = 0\}$$

is the Lie algebra of $SL(n, C)$.

Using the Fundamental Theorem of differential Galois Theory, see Magid [1], we get a lower bound for m-reducibility, see Skórnik, Wloka [2].

THEOREM 2. *Let $Ly = 0$ be a simple equation with Galois group $G(E|F)$, where $E = PVF(Ly = 0)$. This equation is not m-reducible for*

$$m < \sqrt{dimG(E|F)}. \tag{3}$$

Using commutative algebra and representation theory M.F. Singer [2,3] got for $m = n - 1$ a final result:

THEOREM 3. *Let $Ly = 0$ be a homogeneous differential equation of order n, $n \geq 3$. Let $E = PVF(Ly = 0)$ be the associated Picard-Vessiot extension of F. We assume that*

$$G(E|F) \subseteq SL(n, C),$$

or that the coefficient a_{n-1} in (1) is zero ($a_{n-1} = 0$).

$Ly = 0$ cannot be solved in terms of linear differential equations of lower order $\leq n-1$ (it is <u>not</u> $(n-1)$-reducible) if and only if

a) *$g(E|F)$ is simple, and*

b) *$g(E|F) \subseteq sl(n, C)$ has <u>no</u> (nonzero) representation of degree less than n.*

Since $g(E|F)$ is simple, each (nonzero) representation is faithful, and we may reformulate the second condition b):

$$g(E|F) \not\subseteq gl(n - 1, C).$$

We get the easy corollary (from Theorem 2 or from Theorem 3):

COROLLARY 1. *If the equation $Ly = 0$ is simple and we have*

$$\dim G(E|F) = \dim g(E|F) > (n - 1)^2, \quad n \geq 3,$$

then this equation is <u>not</u> $(n-1)$-reducible.

Generic equations behave as they should, see Skórnik, Wloka [2].

THEOREM 4. *Let $b_0, \ldots, b_{n-1}4$ be indeterminates over a differential field F. The generic equation ("general equation" in Magid [1])*

$$L_{GL}y = D^n y + b_{n-1}D^{n-1}y + \ldots + b_0 y = 0, \quad n \geq 2,$$

is not $(n-1)$-reducible, hence it cannot be reduced to equations of lower order.

All simple Lie algebras and their representations are known, see Humphreys [1,2] and also all linear algebraic (matrix) groups, which belong to simple algebras, see Zalesskij [1].

Using this information we get theorems and corollaries just by checking cases. This is the way to prove the following

COROLLARY 2. *The same assumptions as in Theorem 3. Let* $n = 3, 5$ *and* $a_{n-1} = 0$ *in* (1).

1) *The condition*

$$g(E|F) \text{ is simple and } \dim g(E|F) > (n-1)^2$$

is necessary and sufficient for the <u>*non*</u> $(n-1)$*-reducibility of the equation* (1).

2) *Only equations* (1) *with Lie algebra* $g(E|F) = sl(n, C)$ are <u>*not*</u> $(n-1)$*-reducible.*

IV. Simple Fuchsian equations. C. Tretkoff and M. Tretkoff [1] solved the inverse Galois problem for the differential field $(\mathbb{C}(z), \frac{d}{dz})$; by isomorphy (Theorem 1) it is also solved for the field

$$(\mathbb{C}(s), \frac{d}{ds} = D) \quad \text{of Mikusiński operators;}$$

i.e. for every closed algebraic matrix group $G \subseteq GL(n, \mathbb{C})$, there exists an ordinary, linear, Fuchsian differential equation $L_G y = 0$ of <u>*order n*</u> (with polynomial coefficients in z or s)

$$L_G y = p_n(z) y^{(n)} + p_{n-1}(z) y^{(n-1)} + \ldots + p_0(z) y = 0, \tag{4}$$

such that the Picard-Vessiot extension $E = PVF(L_G y = 0)$ over $F = \mathbb{C}(z)$ (or $\mathbb{C}(s)$) has as its Galois group G:

$$G(E|F) = G.$$

The inverse Galois theorem and Theorem 2 imply an existence theorem.

THEOREM 5. *Let F be* $(\mathbb{C}(z), \frac{d}{dz})$ *or* $(\mathbb{C}(s), \frac{d}{ds})$. *For each simple group* $G \subset GL(n, \mathbb{C})$ *there exists a Fuchsian equation* (4) *of* $order L_G = rank G = n$, *which is not* m*-reducible for any*

$$m < \sqrt{dim G}.$$

Combining Theorem 3 with the inverse Galois theorem we get a more general existence theorem.

Using Example 2 we get from Theorem 5 the following corollary.

COROLLARY 3. *For each group* $SL(n, \mathbb{C})$, $n \geq 2$, *there exists a Fuchsian equation*

$$L_{SL} y = 0, \qquad ord\, L_{SL} = n$$

which is not $(n-1)$*-reducible.*

REMARK. For Fuchsian equations $L_f y = 0$ over $F = (\mathbb{C}(s), \frac{d}{ds})$ we have

$$PVF(L_f y = 0) \subseteq \mathfrak{M} \qquad \text{(Mikusiński operators)}$$

see Wloka [1], thus we need not go outside $\mathfrak{M}$ with our PV-extensions.

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