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ON LOCALLY BOUNDED ALGEBRAS

S. ROLEWICZ

Institute of Mathematics, Polish Academy of Sciences
Śniadeckich 8, 00-950 Warszawa, P.O. Box 137, Poland
E-mail: rolewicz@impan.gov.pl

Abstract. It is a survey talk concerning locally bounded algebras.

We shall start with the classical Wiener theorem.

WIENER THEOREM (Wiener (1933)). *Let $f(t)=\sum_{n=-\infty}^{\infty} a_n e^{int}$ be a periodic function. Suppose that*

$$\|f\|_1 = \sum_{n=-\infty}^{\infty} |a_n| < +\infty. \tag{1}$$

If $f(t) \neq 0$ for all t, then the function $1/f(t)$ can be developed in a trigonometric series $1/f(t) = \sum_{n=-\infty}^{\infty} b_n e^{int}$ such that

$$\left\| \frac{1}{f(t)} \right\|_1 = \sum_{n=-\infty}^{\infty} |b_n| < +\infty. \tag{2}$$

A little later the following generalization of the Wiener theorem was done by Lévy.

LÉVY THEOREM (Lévy (1933), (1934)). *Let $f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$ be a periodic function. Suppose that* (1) *holds. Let $\Phi(z)$ be an analytic function defined on an open set $U \supset \{z = f(t) : -\infty < t < \infty\}$. Then the function $\Phi(f(t))$ can be developed in a trigonometric series $\Phi(f(t)) = \sum_{n=-\infty}^{\infty} c_n e^{int}$ such that*

$$\|\Phi(f(t))\|_1 = \sum_{n=-\infty}^{\infty} |c_n| < +\infty. \tag{3}$$

Let $N(u) : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function such that $N(0) = 0$ for $u = 0$ only. Now a natural problem appears.

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PROBLEM 1. *Determine conditions on the function $N(\cdot)$ warranting that the following generalizations of the Wiener and Lévy theorems called their $N(\ell)$-versions hold.*

$N(\ell)$-VERSION OF WIENER THEOREM. *Let $f(t)=\sum_{n=-\infty}^{\infty} a_n e^{int}$ be a periodic function. Suppose that*

$$\|f\|_N = \sum_{n=-\infty}^{\infty} N(|a_n|) < +\infty. \tag{1_N}$$

If $f(t) \neq 0$ for all t, then the function $1/f(t)$ can be developed in a trigonometric series $1/f(t) = \sum_{n=-\infty}^{\infty} b_n e^{int}$ such that

$$\left\|\frac{1}{f(t)}\right\|_N = \sum_{n=-\infty}^{\infty} N(|b_n|) < +\infty. \tag{2_N}$$

$N(\ell)$-VERSION OF LÉVY THEOREM. *Let $f(t) = \sum_{n=-\infty}^{\infty} a_n e^{int}$ be a periodic function. Suppose that (1_N) holds. Let $\Phi(z)$ be an analytic function defined on an open set $U \supset \{z = f(t) : -\infty < t < \infty\}$. Then the function $\Phi(f(t))$ can be developed in a trigonometric series $\Phi(f(t)) = \sum_{n=-\infty}^{\infty} c_n e^{int}$ such that*

$$\|\Phi(f(t))\|_N = \sum_{n=-\infty}^{\infty} N(|c_n|) < +\infty. \tag{3_N}$$

Till now the answer is known in special cases only.

THEOREM 2 (Żelazko (1960), (1965)). *If $N(u) = u^p$, $0 < p \leq 1$, then the $N(\ell)$-versions of the Wiener and Lévy Theorems hold.*

THEOREM 3 (Rolewicz (1985)). *Let $N(u) : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function such that $N(0) = 0$ only for $u = 0$. Suppose that*

(a) $N(u + v) \leq N(u) + N(v)$ for sufficiently small u, v,
(b) there is $C > 0$ such that $N(uv) \leq CN(u)N(v)$ for sufficiently small u, v,
(c) there are $p > 0$ and a convex function $N_0(\cdot)$ such that $N(u) = N_0(u^p)$.

Then the $N(\ell)$-versions of the Wiener and Lévy theorems hold.

Theorem 3 generalizes Theorem 2. Indeed, it is easy to see that the functions $N(u) = u^p$, $0 < p \leq 1$ satisfy conditions (a), (b), (c). However, there are also other functions satisfying conditions (a), (b), (c).

EXAMPLE 4 (Rolewicz (1985)). Let

$$N(u) = \begin{cases} 0 & \text{for } u = 0, \\ -u^p \log u & \text{for } 0 < u \leq e^{-\frac{2}{p}}, \\ \frac{2}{p} e^{-2} & \text{for } u \geq e^{-\frac{2}{p}}. \end{cases}$$

It is not difficult to show that $N(u)$ satisfies conditions (a), (b), (c).

Theorems 2 and 3 seem to be natural generalizations of the classical Wiener and Lévy theorems, however between the classical proofs and their $N(\ell)$-versions many years passed and a lot of modern mathematics is used. The subject of this paper is to describe how it was done.

The first essential step were new proofs of the Wiener and Lévy Theorems obtained by Banach algebras theory. The first important result in this theory was the result of Mazur (1938), who proved that if X is a normed field over the reals then it is either the field of real numbers or the field of complex numbers or the field of quaternions. Unfortunately the editor of Comptes Rendus required to abbreviate the paper, which was published without proof. The original proof of Mazur is published in the book of Żelazko (1973), p. 16–18. Another proof of this theorem, called now the Mazur-Gelfand theorem, for the complex case was given in the famous paper of Gelfand (1941). In that paper Gelfand proved fundamental results about Banach algebras and from that moment the fast development of the theory started. Here we recall only the basic facts about commutative Banach algebras with unit e, which will be useful in our further considerations.

Let X be a commutative algebra over complex numbers with unit e. Suppose that X is simultaneously a Banach space and that the multiplication is continuous. Then we call X a commutative Banach algebra over complex numbers with unit e. It is possible to introduce in X an equivalent submultiplicative norm $\|\cdot\|$, i.e. a norm $\|\cdot\|$ such that $\|xy\| \le \|x\|\|y\|$. A complex valued linear functional $\phi(\cdot)$ is called *multiplicative linear* if $\phi(xy) = \phi(x)\phi(y)$. If $\phi(x) \neq 0$ for all multiplicative linear functionals, then x is invertible.

Having these fundamental facts we are able to present a new proof of the Wiener theorem, given by Gelfand. Namely, we consider the algebra ℓ of absolutely summable sequences $x = \{x_n\},\ n \in \mathbb{Z}$, consisting of complex numbers with standard addition and multiplication by numbers. Clearly, ℓ is a Banach space with the norm

$$\|x\| = \sum_{n=-\infty}^{\infty} |x_n|.$$

Now we introduce multiplication as convolution of sequences $x = \{x_n\}$ and $y = \{y_n\}$,

$$z = x * y = \Big\{ z_n = \sum_{k=-\infty}^{\infty} x_{n-k} y_k,\ n \in \mathbb{Z} \Big\}.$$

Of course, with this multiplication, ℓ is a commutative Banach algebra over complex numbers with unit $e = \{e_n\}$, where $e_0 = 1$ and $e_n = 0$ otherwise. It is not difficult to show that every multiplicative linear functional on ℓ is of the form $\phi_t(x) = \sum_{n=-\infty}^{\infty} x_n e^{int}$, $t \in \mathbb{R}$. Observe that $f(t) = \phi_t(x)$ is a periodic function of period 2π. If $f(t) = \phi_t(x) \neq 0$ for all t, then x is invertible and $x^{-1} \in \ell$. Write $x^{-1} = \{x_n^{-1}\},\ n \in \mathbb{Z}$. Since ϕ_t are multiplicative, we get

$$\frac{1}{f(t)} = \phi_t(x^{-1}) = \sum_{n=-\infty}^{\infty} x_n^{-1} e^{int},$$

i.e. the Wiener theorem holds.

The proof of the Lévy theorem in this way is little more complicated. Namely, we need to define analytic functions on algebras. We recall the definition of spectrum. The *spectrum* $\sigma(x)$ of a given $x \in X$ is the set of those complex numbers λ such that $x - \lambda e$ is not invertible. It can be shown that spectra are always compact sets. Let $F(\cdot)$ be an analytic functions defined on an open set $U \supset \sigma(x)$. By $f(x)$ we denote an element $y \in X$

such that for an arbitrary multiplicative linear functional $\phi(\cdot)$ we have

$$\phi(y) = F(\phi(x)). \tag{4}$$

Such an element y always exists and it can be obtained in the following way. We consider the integral

$$y = \int_\gamma F(\lambda)(x - \lambda e)^{-1} d\lambda, \tag{5}$$

where γ is the union of a finite number of smooth curves contained in U and such that $\sigma(x)$ is surrounded by γ. It looks formally as the classical Cauchy formula, we only have to remember that we integrate a function of a complex variable with values in X. The theory of integration of functions of a real and complex variable with values in Banach spaces is well developed. We can consider in this case the Riemann integral as well as a generalization of the Lebesgue integral, called in this case the Bochner integral.

In a similar way as in the case of the Wiener theorem, we consider the algebra ℓ. We take y defined by (5). Of course, $y \in \ell$. On the other hand by (4) we have for all t

$$\phi_t(y) = F(\phi_t(x)) \tag{4_t}$$

for all t, i.e.

$$F(f(t)) = \sum_{n=-\infty}^{\infty} y_n e^{int},$$

where $y = \{y_n\},\ n \in \mathbb{Z}$ and $y \in \ell$, i.e. the Lévy theorem holds.

We have obtained only new proofs of the classical Wiener and Lévy theorems.

The crucial point of this talk is that this new approach permits us to obtain the $N(\ell)$-versions of the Wiener and Lévy theorems given by Theorems 2 and 3. In order to do it, it is necessary to introduce locally bounded algebras.

We shall return to the classical characterization of normed spaces done by Kolmogorov. Let X be a linear topological space, i.e. a linear and topological space such that the operations of addition and multiplication by scalars are continuous. A set $K \subset X$ is called *bounded* if for each neighbourhood U of zero there is a scalar $\lambda(K, U)$ such that

$$\lambda(K, U)K \subset U.$$

Kolmogorov Theorem (Kolmogorov (1935)). *A linear topological space X is isomorphic to a normed space if and only if there is a convex bounded neighbourhood of zero $U \in X$.*

Thus the generalization of the theory of normed spaces can go in two directions.

- To consider linear topological spaces X in which there is a basis of convex neighbourhoods U_α of zero. Such spaces are called *locally convex spaces.*
- To consider linear topological spaces X in which there is a basis of bounded neighbourhoods U_α of zero. Such spaces are called *locally bounded spaces.*

The investigations of locally convex spaces started in the 30-ties. In particular, the fast development of this theory was later stimulated by the theory of distributions. The big advantage is that in those spaces the Hahn-Banach theorem holds.

The investigations of locally bounded spaces started later. The reason was that in those spaces the Hahn-Banach theorem does not hold, thus they can be considered as a mathematical pathology.

As far as I know, the first result in this direction was done by Aoki (1942), who proved that in such spaces the topology can be given by a p-homogeneous norm $\|\cdot\|$, i.e. such that

(1) $\|x\| = 0$ if and only if $x = 0$,
(2) $\|x + y\| \leq \|x\| + \|y\|$,
(3) there is p, $0 < p \leq 1$ such that $\|tx\| = |t|^p\|x\|$.

Unfortunately, since the war, this result was unknown among mathematicians and this theorem in a slightly better form was rediscovered by myself in 1957 (Rolewicz (1957)).

Having locally bounded spaces, we can start with the theory of locally bounded algebras. Namely, we say that X is a *locally bounded algebra* if it is an algebra and a locally bounded space and the multiplication is continuous. By the Aoki-Rolewicz theorem we know that the topology in X can be given by a p-homogeneous norm $\|\cdot\|'$. We introduce a new norm $\|\cdot\|$ by the formula

$$\|x\| = \sup_{y \neq 0} \frac{\|xy\|'}{\|y\|'}.$$

It is easy to see that $\|\cdot\|$ is a p-homogeneous submultiplicative norm equivalent to the norm $\|\cdot\|'$ (Żelazko (1960)).

Żelazko in his papers (1960), (1965) developed the theory of locally bounded algebras. In particular, he proved ℓ^p-versions ($0 < p \leq 1$) of the Wiener and Lévy theorems (see Theorem 2). The theory of locally bounded algebras is similar to the theory of Banach algebras. However, Żelazko's proofs were different, due to the problem of existence of integrals of functions of real and complex variable with values in locally bounded spaces (recall that in those spaces the Hahn-Banach theorem may not hold). Even more, the existence of the Riemann integral of continuous functions is strictly related with the local convexity. We recall now some fundamental facts about linear metric spaces.

Let X be a linear metric space. In such spaces topology can be given by an F-norm $\|\cdot\|$, i.e. a function which satisfies the following conditions:

(1) $\|x\| = 0$ if and only if $x = 0$,
(2) $\|x + y\| \leq \|x\| + \|y\|$,
(3) $\|tx\|$ is continuous on the product $\mathbb{R} \times X$.

The space X equipped with an F-norm $\|\cdot\|$ is called an F^*-space. If X is complete then it is called an F-space (cf. Banach (1932)). Having the notions of F^*-spaces and F-norm we can formulate

Mazur-Orlicz Theorem (Mazur-Orlicz (1948)). *Let $(X, \|\cdot\|)$ be an F^*-space. If every continuous function $f(t)$, $0 \leq t \leq 1$, with values in X is Riemann integrable, then the space X is locally convex and complete.*

As a consequence of the Mazur-Orlicz theorem, we get

THEOREM 5. *Let $(X, \|\cdot\|)$ be an F^*-space with F-norm $\|\cdot\|$. If there is a constant $C > 0$ such that for all polynomials $w(t) : [0,1] \to X$*

$$\Big\| \int_0^1 w(t)dt \Big\| \le C \int_0^1 \|w(t)\|dt,$$

then the space X is locally convex.

PROOF.* By (1) the functional $F(w) = \int_0^1 w(t)dt$ is uniformly continuous on the set of all polynomials. Thus it can be extended to all continuous functions admitting values in the completion X^c of the space X. Therefore, by the Mazur-Orlicz theorem, the space X is locally convex. ■

The Mazur-Orlicz theorem stopped the investigation of integrals in non-locally convex spaces till the middle 60-ties, when independently Gramsch (1965) and D. Przeworska-Rolewicz and S. Rolewicz (1966) showed that in locally bounded spaces the Riemann integral exists for every analytic function having values in such spaces. This result permits to use the technique of Cauchy integrals in locally bounded algebras. Till now, we have considered analytic functions of one variable. Replacing Cauchy integrals by Weyl integrals, Shilov (1951) (for finitely generated algebras) and Arens and Calderón (1955) (in general) obtained the following

SHILOV-ARENS-CALDERÓN THEOREM. *Let X be a Banach algebra. Let Φ be the set of all multiplicative linear functionals defined on X. Let $x_1, \ldots, x_n \in X$. Let $\sigma(x_1, \ldots, x_n) = \{(z_1, \ldots, z_n) : z_i = \phi(x_i), \phi \in \Phi, i = 1, \ldots, n,\}$ be the joint spectrum of the elements $x_1, \ldots, x_n$. Let $\Phi(z_1, \ldots, z_n)$ be an analytic function defined on an open set $U \subset \mathbb{C}^n$ such that $\sigma(x) \subset U$. Then there is $y \in X$ such that*

$$\phi(y) = \Phi(\phi(x_1), \ldots, \phi(x_n)) \tag{6}$$

for every multiplicative linear functional ϕ.

D. Przeworska-Rolewicz and S. Rolewicz (1966) and Gramsch (1967) showed that the Shilov-Arens-Calderón Theorem is valid if we replace Banach algebras by locally bounded algebras. As a consequence, we have

THEOREM 6 (Rolewicz (1985)). *Let $N(u) : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function such that $N(0) = 0$ only for $u = 0$. Suppose that*

(a) $N(u + v) \le N(u) + N(v)$ for sufficiently small u, v,
(b) there is $C > 0$ such that $N(uv) \le CN(u)N(v)$ for sufficiently small u, v,
(c) there are $p > 0$ and a convex function $N_0(\cdot)$ such that $N(u) = N_0(u^p)$.

Let $x_1(t), \ldots, x_n(t)$ be periodic functions such that their coefficients belong to $N(\ell)$. Let $\Phi(z_1, \ldots, z_n)$ be an analytic function defined on an open set U such that $\sigma(x) = \{(x_1(t), \ldots, x_n(t)) : t \in \mathbb{R}\} \subset U$. Then $\Phi(x_1(t), \ldots, x_n(t)) = \sum_{n=-\infty}^{\infty} c_n e^{int}$ where $c = (c_1, c_2, \ldots) \in N(\ell)$.

Theorem 6 for the space $N(\ell)$, where $N(u) = u^p$, $0 < p \le 1$, was proved by Gramsch (1967) and D. Przeworska-Rolewicz and S. Rolewicz (1966).

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* The presented proof was given by S. Kwapień. The original proof was much longer.

Let $N^+(\ell)$ denote the space of those sequences $x = (x_0, x_1, \ldots)$ of complex numbers, for which we have

$$\|x\|^{(N)} = \sum_{n=0}^{\infty} N(|x_n|) < +\infty. \tag{$1^{(N)}$}$$

It is easy to see that $N^+(\ell)$ is a locally bounded space with norm $\|x\|^{(N)}$. By simple calculation, we get that, if conditions (a), (b), (c) hold then $N^+(\ell)$ is an algebra where multiplication is convolution of sequences $x = \{x_n\}$ and $y = \{y_n\}$:

$$z = x * y = \Big\{z_n = \sum_{k=0}^{\infty} x_{n-k} y_k,\ n = 0, 1, 2, \ldots\Big\}.$$

It is not difficult to show that every multiplicative linear functional $\phi(\cdot)$ defined on $N^+(\ell)$ is of the form $\phi(x) = \sum_{n=0}^{\infty} x_n z^n,\ |z| \leq 1$. Knowing this fact, we can prove the $N^+(\ell)$-version of the Lévy Theorem.

$N^+(\ell)$-Version of Lévy Theorem. *Let $N(u) : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function such that $N(0) = 0$ only for $u = 0$. Suppose that*

(a) $N(u + v) \leq N(u) + N(v)$ for sufficiently small u, v,
(b) there is $C > 0$ such that $N(uv) \leq CN(u)N(v)$ for sufficiently small u, v,
(c) there are $p > 0$ and a convex function $N_0(\cdot)$ such that $N(u) = N_0(u^p)$.

Let $f(t) = \sum_{n=0}^{\infty} a_n e^{int}$ be a periodic function. Suppose that

$$\|f\|^{(N)} = \sum_{n=0}^{\infty} N(|a_n|) < +\infty. \tag{$1^{(N)}$}$$

Let $\Phi(z)$ be an analytic function defined on an open set $U \supset \{z = f(t) : -\infty < t < \infty\}$. Then the function $\Phi(f(t))$ can be developed in a trigonometric series $\Phi(f(t)) = \sum_{n=0}^{\infty} c_n e^{int}$ such that

$$\|\Phi(f(t))\|^{(N)} = \sum_{n=0}^{\infty} N(|c_n|) < +\infty. \tag{$3^{(N)}$}$$

Using the Shilov-Arens-Calderón Theorem for locally bounded algebras we can obtain for $N^+(\ell)$ spaces a theorem similar to Theorem 6.

Theorem 7 (Rolewicz (1985)). *Let $N(u) : [0, +\infty) \to [0, +\infty)$ be a non-decreasing function such that $N(0) = 0$ only for $u = 0$. Suppose that*

(a) $N(u + v) \leq N(u) + N(v)$ for sufficiently small u, v,
(b) there is $C > 0$ such that $N(uv) \leq CN(u)N(v)$ for sufficiently small u, v,
(c) there are $p > 0$ and a convex function $N_0(\cdot)$ such that $N(u) = N_0(u^p)$.

Let $x_1(z), \ldots, x_n(z)$ be functions defined on a closed unit disc, which are analytic in the interior. Suppose that their coefficients belong to $N^+(\ell)$. Let $\Phi(z_1, \ldots, z_n)$ be an analytic function defined on an open set U such that $\sigma(x) = \{(x_1(z), \ldots, x_n(z)) : |z| \leq 1\} \subset U$. Then for $z,\ |z| \leq 1\ \Phi(x_1(z), \ldots, x_n(z)) = \sum_{n=0}^{\infty} c_n z^n$, where $c = (c_1, c_2, \ldots) \in N^+(\ell)$.

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