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AUTOMORPHISMS OF C COMMUTING WITH PARTIAL INTEGRATION OPERATORS IN A RECTANGLE

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Abstract. Convolutional representations of the commutant of the partial integration operators in the space of continuous functions in a rectangle are found. Necessary and sufficient conditions are obtained for two types of representing functions, to be the operators in the commutant continuous automorphisms. It is shown that these conditions are equivalent to the requirement that the considered representing functions be joint cyclic elements of the partial integration operators.

1. Introduction. Let $\Delta_1 = [a_1, b_1]$ and $\Delta_2 = [a_2, b_2]$ be intervals containing zero and $\Delta = \Delta_1 \times \Delta_2$. Let $C(\Delta)$ be the space of continuous functions in the rectangle Δ. It is a Banach space with usual topology of uniform convergence on Δ.

We consider the partial integration operators l_1 and l_2 of Volterra type, defined by

$$l_1 f = \int_0^x f(\tau, y)\, d\tau \quad \text{and} \quad l_2 f = \int_0^y f(x, \sigma)\, d\sigma, \tag{1}$$

for $f, g \in C(\Delta)$ as right inverse of partial differentiation operators $\partial/\partial x$ and $\partial/\partial y$ in $C(\Delta)$. The operation

$$(f * g)(x, y) = \int_0^x \int_0^y f(x-\tau, y-\sigma) g(\tau, \sigma)\, d\tau\, d\sigma \tag{2}$$

for $f, g \in C(\Delta)$ is a separately continuous convolution of l_1 and l_2 without annihilators, according to N. Bozhinov [2]. This means that

$$l_i(f * g) = (l_i f) * g$$

for $f, g \in C(\Delta)$ and $i = 1, 2$. Moreover, the identity

$$l_1 l_2\{f(x, y)\} = \{1\} * f \tag{3}$$

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holds for all $f \in C(\Delta)$, where the symbol $\{1\}$ denotes the constant function equal to 1 in the rectangle Δ.

The usual Duhamel convolution may be considered in the space $C(\Delta)$ as a "coordinate" operation and one has to know on which variable it acts. In the case at least one of the functions does not depend on x or y, the operations $\overset{x}{*}$ and $\overset{y}{*}$ are introduced by the equalities

$$(f \overset{x}{*} g)(x,y) = \int_0^x f(x-\tau)g(\tau,y)\,d\tau, \tag{4}$$

$$(f \overset{y}{*} g)(x,y) = \int_0^y f(y-\sigma)g(x,\sigma)\,d\sigma. \tag{5}$$

They allow us to represent l_1 and l_2 as convolutional operators in $C(\Delta)$:

$$l_1\{f(x,y)\} = \{1\} \overset{x}{*} f(x,y) \quad \text{and} \quad l_2\{f(x,y)\} = \{1\} \overset{y}{*} f(x,y). \tag{6}$$

The convolution defined by (2) has an important property of "splitting". Namely, for all functions of the form

$$f(x,y) = f_1(x)\, f_2(y) \quad \text{with} \quad f_1(x) \in C(\Delta_1) \;\; \text{and} \;\; f_2(y) \in C(\Delta_2) \tag{7}$$

we have

$$(f * g)(x,y) = \big[f_1(x) \overset{x}{*} g_1(x)\big]\,\big[f_2(y) \overset{y}{*} g_2(y)\big]. \tag{8}$$

The linear combinations of the splittable functions, represented by (7), form a dense set in $C(\Delta)$. This follows from Weierstrass approximation theorem in $C(\Delta)$, since the polynomials of x and y are splittable functions.

The divisors of zero of the convolution (2) in $C(\Delta)$ are described by J. Mikusiński and C. Ryll-Nardzewski in [10]:

Lemma 1.1 [10]. *Let $f(x,\,y)$ and $g(x,\,y)$ are two continuous functions in the rectangle*

$$\Delta_a \overset{\text{def}}{=} \{\,(x,\,y) :\, 0 \le x \le b_1, 0 \le y \le b_2; b_1 + b_2 < a\,\},$$

such that the convolution $f * g$ defined by (2) vanishes on the rectangle Δ_a. Then $f(x,\,y) = 0$ on Δ_b and $g(x,\,y) = 0$ on the rectangle Δ_c, where $b + c \ge a$.

Definition 1.1. The set of all operators $A\colon C(\Delta) \to C(\Delta)$ such that $Al_1 = l_1 A$ and $Al_2 = l_2 A$ is called the *commutant of* l_1 *and* l_2 in $C(\Delta)$.

Definition 1.2. A linear operator $M : C(\Delta) \to C(\Delta)$ is said to be a *multiplier* of the convolution algebra $(\,C(\Delta), *\,)$ if

$$M(\,f * g\,) = (Mf) * g \quad \text{for} \quad f, g \in C(\Delta).$$

Using identities $M\{1\} * \{1\} = \{1\} * M\{1\}$ and $l_1^p l_2^q\{1\} = \frac{x^p y^q}{p!q!}$ for $p, q = 0, 1, 2, \ldots$ and applying Weierstrass approximation theorem in $C(\Delta)$ it is easy to prove that the ring of multipliers of convolution algebra $(\,C(\Delta), *\,)$ coincides with the commutant of l_1 and l_2 in $C(\Delta)$.

Lemma 1.2. *A linear operator $M : C(\Delta) \to C(\Delta)$ commutes with the operators l_1 and l_2 in $C(\Delta)$ iff it is a multiplier of the convolution given in (2).*

2. Representation of the commutant of l_1 and l_2

THEOREM 2.1. *If the operator $M\colon C(\Delta) \to C(\Delta)$ commutes with the operators l_1 and l_2 in $C(\Delta)$, then it has a convolutional representation of the form*

$$M\,f = \frac{\partial^2}{\partial x \partial y}(m * f) \tag{9}$$

with $m \overset{\text{def}}{=} M\{1\} \in C(\Delta)$.

PROOF. Since M commutes with l_1 and l_2, it is a multiplier of convolution $*$. Thus we have the equality

$$M\{1\} * f = \{1\} * (\,Mf\,) = l_1 l_2 (\,Mf\,).$$

After differentiation on x and y and substituting $m \overset{\text{def}}{=} M\{1\}$ we obtain the representation (9). The multiplier M of convolution $*$ in $C(\Delta)$ is a continuous operator, due to R. Larsen [9], p. 13. Then m is a continuous function on Δ as a continuous image of the constant $\{1\}$.

The condition $m \in C(\Delta)$ does not ensure differentiability of the expression $m * f$ with respect to x and y. Therefore we consider two cases for the function m.

COROLLARY 2.1. *A linear operator $M : C(\Delta) \to C(\Delta)$, with $M\{1\} = m \in C^2(\Delta)$ commutes with l_1 and l_2 if and only if it has an integral representation*

$$\begin{aligned}(Mf)(x,y) = \left(\frac{\partial^2}{\partial x \partial y} m\right) * f + \left[\frac{\partial}{\partial x} m(x,0)\right] \overset{x}{*} f(x,y) \\ + \left[\frac{\partial}{\partial y} m(0,y)\right] \overset{y}{*} f(x,y) + m(0,0)\, f(x,y)\end{aligned} \tag{10}$$

for $f \in C(\Delta)$.

PROOF. The following identity

$$m(x,y) = l_1 l_2\, m''_{xy} + m(0,y) + m(x,0) - m(0,0) \tag{11}$$

is evident for $m \in C^2(\Delta)$. If the operator M commutes with l_1 and l_2 in $C(\Delta)$, it has the representation given by (9). Then taking into account the equality (11) we get

$$\begin{aligned}M\,f = \frac{\partial^2}{\partial x \partial y}\big[(l_1 l_2\, m''_{xy}) * f\big] + \frac{\partial^2}{\partial x \partial y}\big[m(0,y) * f\big] \\ + \frac{\partial^2}{\partial x \partial y}\big[m(x,0) * f\big] - \frac{\partial^2}{\partial x \partial y}\big[m(0,0) * f\big].\end{aligned} \tag{12}$$

Since $*$ is a convolution of l_1 and l_2, right inverses of the partial differentiation operators $\partial/\partial x$ and $\partial/\partial y$ in $C(\Delta)$, we have

$$\frac{\partial^2}{\partial x \partial y}\, l_1 l_2 \left[m''_{xy} * f\right] = m''_{xy} * f. \tag{13}$$

According to identity (3), the equality

$$m(0,0) \frac{\partial^2}{\partial x \partial y}\big[\{1\} * f\big] = m(0,0)\, f(x,y) \tag{14}$$

is true. Since the set of all splittable functions is dense in $C(\Delta)$, we conclude from the theorem on differentiation under the integral sign (see e.g. [8], Th. 3, p. 665) and from Lemma 4 in [5], p. 14, that

$$\frac{\partial^2}{\partial x \partial y}\big[m(0,y) \,*\, f(x,y)\big] = m'_y(0,y) \overset{y}{*} f(x,y) + m(0,0)f(x,y) \tag{15}$$

and

$$\frac{\partial^2}{\partial x \partial y}\big[m(x,0) \,*\, f(x,y)\big] = m'_x(x,0) \overset{x}{*} f(x,y) + m(0,0)f(x,y). \tag{16}$$

Substituting (13), (14), (15) and (16) into (12) we get the desired representation (10).

Conversely, if M has the form (10) with $m \in C^2(\Delta)$, then the commutativity relations $Ml_1 = l_1 M$ and $Ml_2 = l_2 M$ can be verified directly.

Corollary 2.2. *If the function $m = M\{1\}$ is a splittable function*

$$m(x,y) = m_1(x)m_2(y), \qquad (x,y) \in \Delta \tag{17}$$

with components $m_1 \in BV \cap C(\Delta_1)$ and $m_2 \in BV \cap C(\Delta_2)$, then the representation in (9) is equivalent to the equality

$$\begin{aligned}(Mf)(x,y) = &\int_0^x \int_0^y f(x-\tau, y-\sigma)\, dm_1(\tau)\, dm_2(\sigma) + m_1(0)m_2(0)\, f(x,y)\\ &+ m_2(0)\int_0^x f(x-\tau, y)\, dm_1(\tau) + m_1(0)\int_0^y f(x, y-\sigma)\, dm_2(\sigma)\end{aligned} \tag{18}$$

for $f \in C(\Delta)$. Every operator M, given by (18) with splittable function $m \in C(\Delta)$ of the form (17), commutes with l_1 and l_2 in $C(\Delta)$.

The proof is an immediate consequence of Lemma 4 in [5], p. 14 and the density of the set of splittable functions in $C(\Delta)$. The commutativity relations of the operator (18) with l_1 and l_2 are clearly satisfied.

3. Automorphisms of $C(\Delta)$ commuting with l_1 and l_2

Definition 3.1. A linear operator $A\colon C(\Delta) \to C(\Delta)$ is a *topological automorphism* of $C(\Delta)$ onto itself if A is a one-to-one mapping and it is a continuous operator in $C(\Delta)$ together with its inverse A^{-1}.

Since $C(\Delta)$ is a Banach space and $*$ is a separately continuous annihilators-free convolution, the multipliers of the algebra $(C(\Delta), *)$ are continuous operators according to R. Larsen (see [9], p. 14). Then taking into account the inverse operator theorem in $C(\Delta)$ and Lemma 1.2, the problem of existence of continuous automorphisms in the commutant of l_1 and l_2, is reduced to the question of establishing a one-to-one mapping of $C(\Delta)$ onto itself by the operators of the forms (10) and (18).

Theorem 3.1. *A linear operator $M\colon C(\Delta) \to C(\Delta)$, commuting with the operators l_1 and l_2 in $C(\Delta)$ and having a representing function $m = M\{1\} \in C^2(\Delta)$, is a continuous automorphism of the space $C(\Delta)$ onto itself if and only if $m(0,0) \neq 0$.*

The theorem will be proved if we show that the equation

$$m''_{xy} * f + m'_x(x,0) \overset{x}{*} f + m'_y \overset{y}{*} f + m(0,0)\, f = g(x,y) \tag{19}$$

with given $m \in C^2(\Delta)$ has a unique solution f for every function $g \in C(\Delta)$, whenever $m(0,0) \neq 0$.

For that aim we consider the operator of the form

$$N f = m''_{xy} * f + m'_y(0,y) \overset{y}{*} f + m'_x(x,0) \overset{x}{*} f \tag{20}$$

with $m \in C^2(\Delta)$. The operators

$$N_2 f = m'_y(0,y) \overset{y}{*} f$$

and

$$N_3 f = m'_x(x,0) \overset{x}{*} f$$

are Volterra integral operators, due to representations (4) and (5) and therefore they are compact in $C(\Delta)$. Let us denote by N_p the first addend in (20), i.e. $N_p f = p * f$ with $p = m''_{xy} \in C(\Delta)$. The following lemma is true:

LEMMA 3.1. *Every operator of the form $N_p f = p * f$ with $p \in C(\Delta)$ is a compact operator in the space $C(\Delta)$.*

PROOF. Let B be the unit ball in $C(\Delta)$. To prove that N_p is compact is suffices to show that the image $N_p(B)$ of B is a precompact set in $C(\Delta)$. This is fulfilled, when $N_p(B)$ is a uniformly bounded and equicontinuous set, according to Ascoli's theorem ([7], Th. 0.4.11).

Denote the area of the rectangle Δ by $S(\Delta)$, the lengths of the intervals $\Delta_i, i = 1,2$, by $d(\Delta_i),\ i = 1,2$ and let $K_p = \max_{(x,y)\in\Delta} |p(x,y)|$. Then the inequality

$$\max_{(x,y)\in\Delta} |(N_p f)(x,y)| \leq K_p\, S(\Delta), \qquad f \in B$$

shows that N_p is uniformly bounded. For the sake of definiteness, assume that $s > 0$, $t > 0$ and $(x+s,\, y+t) \in \Delta$. Denoting

$$F^{s,t} f(x,y) = (N_p f)(x+s,\, y+t) - (N_p f)(x,y)$$

we have the estimation

$$\begin{aligned} |\, F^{s,t} f(x,y)\, | \leq & \int_0^x \int_0^y |p(x+s-\tau, y+t-\sigma) - p(x-\tau, y-\sigma)|\, d\tau\, d\sigma \\ & + K_p\, |s|\, d(\Delta_2) + K_p\, |t|\, d(\Delta_1) + K_p\, |s|\, |t|, \end{aligned}$$

for $f \in B$.

Fix $\varepsilon > 0$. By the uniform continuity of p on Δ, there exists a number $\delta > 0$ such that

$$\max_{(x,y)\in\Delta} |\, F^{s,t} f(x,y)\, | < \varepsilon \quad \text{for} \quad f \in B,$$

provided that $\sqrt{s^2+t^2} < \delta$.

This proves the lemma.

Proof of Theorem 3.1. Let the operator M given in the form (10) be a one-to-one linear mapping of $C(\Delta)$ onto itself and assume $m(0,0) = 0$. Using formulas (2), (4) and (5) we obtain $(Mf)(0,0) = 0$ for all $f \in C(\Delta)$, which contradicts the surjectivity of the operator M.

Let now $m(0,0) \neq 0$ and let us consider the equation (19) in $C(\Delta)$. Since the operator N of the form (20) is compact and $m(0,0) \neq 0$, this is a Fredholm integral equation of the second kind. Its corresponding homogeneous equation has the form

$$\frac{\partial^2}{\partial x \partial y}(m * f) = 0. \tag{21}$$

Then we get immediately $\frac{\partial}{\partial x}(m * f) = \varphi(x)$ and $\frac{\partial}{\partial y}(m * f) = \phi(y)$ and also

$$(m * f)(x,y) = \int_0^x \varphi(\tau)\, d\tau + \int_0^y \phi(\sigma)\, d\sigma. \tag{22}$$

Since

$$(m * f)(0,y) = 0 \qquad \forall y \in \Delta_2$$

and

$$(m * f)(x,0) = 0 \qquad \forall x \in \Delta_1,$$

due to definition (2), using the equality (22) we conclude $m * f = 0$ for $(x,y) \in \Delta$. The function m is not a divisor of zero of the convolution $*$ provided that $m(0,0) \neq 0$. Then the homogeneous equation (21) has only the trivial solution $f = 0$. Therefore by Fredholm alternative, the equation (19) has a unique solution for every function $g \in C(\Delta)$.

When the operator M has the representation given in (18), for our aims we consider the following equation

$$\begin{aligned} m_1(0)m_2(0)\, f(x,y) + m_2(0)\int_0^x f(x-\tau,y)\, dm_1(\tau) \\ +m_1(0)\int_0^y f(x,y-\sigma)\, dm_2(\sigma) \\ +\int_0^x\int_0^y f(x-\tau,y-\sigma)\, dm_1(\tau)\, dm_2(\sigma) = g(x,y), \end{aligned} \tag{23}$$

with given functions $m,\ g \in C(\Delta)$. Here m has the form (17) and $m_1 \in BV \cap C(\Delta_1)$, $m_2 \in BV \cap C(\Delta_2)$.

Let $m_1(0)m_2(0) \neq 0$ and denote $\lambda_1 = -1/m_1(0)$, respectively $\lambda_2 = -1/m_2(0)$. Then the equation (23) is equivalent to the equation

$$\begin{aligned} f(x,y) = \lambda_1\lambda_2 g(x,y) + \lambda_1\int_0^x f(x-\tau,y)\, dm_1(\tau) + \lambda_2\int_0^y f(x,y-\sigma)\, dm_2(\sigma) \\ -\lambda_1\lambda_2\int_0^x\int_0^y f(x-\tau,y-\sigma)\, dm_1(\tau)\, dm_2(\sigma). \end{aligned} \tag{24}$$

For the sake of simplicity, let us restrict the considerations to the rectangle $\Delta = [0,\, b_1] \times [0,\, b_2]$, with $\Delta_1 = [0,\, b_1]$ and $\Delta_2 = [0,\, b_2]$.

LEMMA 3.2. *Let $g \in C(\Delta)$ and $m = m_1(x)m_2(y)$ be a splittable continuous function on Δ, such that $m_1 \in BV \cap C(\Delta_1)$ and $m_2 \in BV \cap C(\Delta_2)$. Then the equation (24) has a unique solution, which is a continuous function in Δ.*

PROOF. Let $0 < s < b_1$ and $0 < t < b_2$ be arbitrarily chosen and denote by $V_0^s m_1$, respectively by $V_0^t m_2$ the total variations of the functions $m_i,\ i = 1,2$ in the intervals

$[0,\, s]$ and $[0,\, t]$. Introduce the operator

$$T\,h(x,y) = \lambda_1\lambda_2 g(x,y) + \lambda_1 \int_0^x h(x-\tau,y)\,dm_1(\tau) + \lambda_2 \int_0^y h(x,y-\sigma)\,dm_2(\sigma)$$

$$-\lambda_1\lambda_2 \int_0^x \int_0^y h(x-\tau,y-\sigma)\,dm_1(\tau)\,dm_2(\sigma) \tag{25}$$

in the space $C(\Delta)$. The following estimations are true

$$|Th_1 - Th_2| \le |\lambda_1| \max_{\substack{0\,\le\,\tau\,\le\,x\,\le\,s\\ 0\,\le\,y\,\le\,t}} |h_1(x-\tau,y) - h_2(x-\tau,y)|\, V_0^s m_1$$
$$+|\lambda_2| \max_{\substack{0\,\le\,x\,\le\,s\\ 0\,\le\,\sigma\,\le\,y\,\le\,t}} |h_1(x,y-\sigma) - h_2(x,y-\sigma)|\, V_0^t m_2$$
$$+|\lambda_1|\,|\lambda_2| \max_{\substack{0\,\le\,\tau\,\le\,x\,\le\,s\\ 0\,\le\,\sigma\,\le\,y\,\le\,t}} |h_1(x-\tau,y-\sigma) - h_2(x-\tau,y-\sigma)|\, V_0^s m_1\, V_0^t m_2$$

and

$$\max_{\substack{x\,\in\,[0,\,s]\\ y\,\in\,[0,\,t]}} |Th_1 - Th_2| \le \big(\,|\lambda_1| V_0^s m_1 + |\lambda_2| V_0^t m_2 + |\lambda_1\lambda_2|\, V_0^s m_1 V_0^t m_2\,\big)$$
$$.\max_{\substack{x\,\in\,[0,\,s]\\ y\,\in\,[0,\,t]}} |\,h_1 - h_2\,|.$$

Therefore T is a contracting mapping in $C([0,s]\times[0,t])$ iff the inequality

$$|\lambda_1|\, V_0^s m_1 + |\lambda_2|\, V_0^t m_2 + |\lambda_1|\,|\lambda_2|\, V_0^s m_1 V_0^t m_2 < 1 \tag{26}$$

holds. Since each of the functions m_i is uniformly continuous in $\Delta_i,\ i=1,2$, there exists a natural number n_0 such that the condition (26) with $s = \frac{b_1}{n_0}$ and $t = \frac{b_2}{n_0}$ is fulfilled.

Hence the continuous solution of equation (24) may be found, using successive approximations of four types in every "subrectangle" of Δ with lengths of the sides being the chosen s and t. Thus, after n_0^2 steps, we obtain a continuous solution of the equation (24) in Δ.

THEOREM 3.2. *Suppose a linear operator $M\colon C(\Delta)\to C(\Delta)$ commutes with l_1 and l_2 and has a splittable representing function $m = M\{1\}\in C(\Delta)$ with components $m_1\in BV\cap C(\Delta_1)$ and $m_2\in BV\cap C(\Delta_2)$. A necessary and sufficient condition for such an operator to be a continuous automorphism of the space $C(\Delta)$ onto itself is $m_1(0)m_2(0)\neq 0$.*

PROOF. It is enough to show that an operator given in the form (18) establishes a one-to-one mapping in the space $C(\Delta)$, whenever $m_1(0)m_2(0)\neq 0$.

One may prove that the condition $m_1(0)m_2(0)\neq 0$ is necessary for the bijectivity of the operator M, in the same way as in the proof of Theorem 3.1. On the other hand, Lemma 3.2 shows that this inequality is sufficient for the operator M to fulfil a one-to-one correspondence of $C(\Delta)$ onto itself.

4. Joint cyclic elements of l_1 and l_2 in $C(\Delta)$

DEFINITION 4.1. A function $k\in C(\Delta)$ is a *joint cyclic element* of the operators l_1 and l_2 in $C(\Delta)$, if the set of linear combinations of the expressions $l_1^p l_2^q\, k$ with $p,q\in \mathbf{N}_0$ is everywhere dense in $C(\Delta)$.

Using the convolution defined in (2) we describe two kinds of joint cyclic elements of l_1 and l_2 in $C(\Delta)$.

THEOREM 4.1. *A function $k \in C^2(\Delta)$ is a joint cyclic element of l_1 and l_2 in $C(\Delta)$ if and only if* $k(0,0) \neq 0$.

PROOF. Analogously to the proof of Corollary 2.1, we may write the representation

$$k = k''_{xy} * \{1\} + k'_x(x,0) \overset{x}{*} \{1\} + k'_y(0,y) \overset{y}{*} \{1\} + k(0,0),$$

for the function $k \in C^2(\Delta)$. Applying several times the operators l_1 and l_2 to the last equality we get

$$\begin{aligned} l_1^p l_2^q\, k = k''_{xy} * \left(\frac{x^p y^q}{p!q!}\right) + k'_x(x,0) \overset{x}{*} \left(\frac{x^p y^q}{p!q!}\right) \\ + k'_y(0,y) \overset{y}{*} \left(\frac{x^p y^q}{p!q!}\right) + k(0,0)\left(\frac{x^p y^q}{p!q!}\right). \end{aligned} \tag{27}$$

Let k be a cyclic element of l_1 and l_2. Then for every $f \in C(\Delta)$ there exists a sequence $\{ f_n \}_{n=1}^{\infty}$ of elements, linear combinations of the form

$$f_n(x,y) = \sum_{p=0}^{p_n} \sum_{q=0}^{q_n} \alpha_{pq}^{(n)}\, (l_1^p l_2^q\, k) \tag{28}$$

with constants $\alpha_{pq}^{(n)}$. This sequence tends to f uniformly on Δ. Denote by

$$P_n(x,y) = \sum_{p=0}^{p_n} \sum_{q=0}^{q_n} \alpha_{pq}^{(n)} \left(\frac{x^p y^q}{p!q!}\right)$$

the polynomials of x and y similar to the representation (28) of f_n. From the linearity of operations $*$, $\overset{x}{*}$ and $\overset{y}{*}$ and according to the equation (27), we have

$$f_n(x,y) = k''_{xy} * P_n + k'_x(x,0) \overset{x}{*} P_n(x,y) + k'_y(0,y) \overset{y}{*} P_n(x,y) + k(0,0)P_n(x,y). \tag{29}$$

If $k(0,0) = 0$, then we obtain $f_n(0,0) = 0$ for $n \in \mathbf{N}$, due to definitions in (2), (4) and (5). Therefore, only the functions with property $f(0,0) = 0$ can be approximated by means of linear combinations of $l_1^p l_2^q\, k$ with $p, q \in \mathbf{N}_0$. This proves the necessity of the condition $k(0,0) \neq 0$.

Let now $k(0,0) \neq 0$ and let us fix an arbitrary function $f \in C(\Delta)$. According to the proof of Theorem 3.1, the equation

$$f = k''_{xy} * g + k'_x(x,0) \overset{x}{*} g + k'_y(0,y) \overset{y}{*} g + k(0,0)g \tag{30}$$

is of the form (19) and has a unique solution g for every $f \in C(\Delta)$. Then we may choose a sequence of polynomials of two variables

$$P_n(x,y) = \sum_{p=0}^{p_n} \sum_{q=0}^{q_n} \alpha_{pq}^{(n)} \left(\frac{x^p y^q}{p!q!}\right), \tag{31}$$

which tends to g in the topology of $C(\Delta)$, due to the respective approximation theorem in $C(\Delta)$. We form a new sequence

$$f_n(x,y) = \sum_{p=0}^{p_n}\sum_{q=0}^{q_n} \alpha_{pq}^{(n)}\,(l_1^p l_2^q\,k)$$
$$= k''_{xy} \ast P_n + k'_x(x,0) \overset{x}{\ast} P_n + k'_y(0,y) \overset{y}{\ast} P_n + k(0,0)P_n.$$

Operations $\ast$, $\overset{x}{\ast}$ and $\overset{y}{\ast}$ are continuous in $C(\Delta)$, then we obtain $\lim_{n\to\infty} f_n = f$ uniformly in Δ, according to equation (30) and the choice of the sequence (31). This means the function $k \in C(\Delta)$ is a joint cyclic element of l_1 and l_2 in $C(\Delta)$.

THEOREM 4.2. *Let $k \in C(\Delta)$ be a splittable function $k(x,y) = k_1(x)k_2(y)$ with components $k_1 \in BV \cap C(\Delta_1)$ and $k_2 \in BV \cap C(\Delta_2)$. A necessary and sufficient condition for a function k to be a joint cyclic element of l_1 and l_2 in $C(\Delta)$ is $k_1(0)k_2(0) \neq 0$.*

PROOF. The operators l_1 and l_2 are right inverses of the partial differentiation operators $\partial/\partial x$ and $\partial/\partial y$ in $C(\Delta)$. Thus, using the identity $l_1 l_2\, k = \{1\} \ast k$ and the splitting property $k = k_1 k_2$, the function k can be represented in the form

$$k(x,y) = \frac{\partial^2}{\partial x \partial y}\,[\,\{1\} \ast k\,] = k_1(0)k_2(0) +$$
$$+\int_0^x\int_0^y dk_1(\tau)\,dk_2(\sigma) + k_2(0)\int_0^x dk_1(\tau) + k_1(0)\int_0^y dk_2(\sigma).$$

Then we have

$$l_1^p l_2^q k = \int_0^x\int_0^y \frac{\tau^p\sigma^q}{p!q!}\,dk_1(x-\tau)\,dk_2(y-\sigma) + k_1(0)k_2(0)\frac{x^p y^q}{p!q!}$$
$$+k_2(0)\int_0^x \frac{\tau^p y^q}{p!q!}\,dk_1(x-\tau) + k_1(0)\int_0^y \frac{x^p\sigma^q}{p!q!}\,dk_2(y-\sigma) \tag{32}$$

for $p, q \in \mathbf{N}_0$.

Let k be a joint cyclic element of l_1 and l_2 in $C(\Delta)$. Hence, for every function $f \in C(\Delta)$ there exists a sequence $\{\, f_n \,\}_{n=1}^{\infty}$ of the form (28), which converges to f uniformly in Δ. The elements of this sequence are represented by polynomials (31), due to (32):

$$f_n(x,\, y) = \int_0^x\int_0^y P_n(\tau,\sigma)\,dk_1(x-\tau)\,dk_2(y-\sigma) + k_1(0)k_2(0)P_n(x,y)$$
$$+k_2(0)\int_0^x P_n(\tau,y)\,dk_1(x-\tau) + k_1(0)\int_0^y P_n(x,\sigma)\,dk_2(y-\sigma). \tag{33}$$

If $k_1(0)k_2(0) = 0$, it follows from (33) that only the functions f with $f(0,0) = 0$ can be approximated by linear combinations of the expressions (32). This contradiction shows that the condition $k_1(0)k_2(0) \neq 0$ is necessary.

Let now $k_1(0)k_2(0) \neq 0$ and let $f \in C(\Delta)$ be an arbitrary fixed function. The equation

$$k_1(0)k_2(0)g(x,y) + k_2(0)\int_0^x g(x-\tau,y)\,dk_1(\tau)$$
$$+k_1(0)\int_0^y g(x,y-\sigma)\,dk_2(\sigma)$$
$$\int_0^x\int_0^y g(x-\tau,y-\sigma)\,dk_1(\tau)\,dk_2(\sigma) = f(x,y)$$

has a unique solution $g \in C(\Delta)$ for every $f \in C(\Delta)$, due to Lemma 3.2. Then in the same way as in the proof of Theorem 4.1 we may form a sequence $\{f_n\}_{n=1}^{\infty}$ of linear combinations of $l_1^p l_2^q k$, which converges to f uniformly in Δ.

This proves the theorem.

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