ALGEBRAIC ANALYSIS AND RELATED TOPICS
BANACH CENTER PUBLICATIONS, VOLUME 53
INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
WARSZAWA 2000

SOME HIGHLIGHTS IN THE DEVELOPMENT OF ALGEBRAIC ANALYSIS

JOHN A. SYNOWIEC
Department of Mathematics, Indiana University
The Northwest Campus, 3400 Broadway
Gary, Indiana 46408-1197, U.S.A.
E-mail: jsynowie@indiana.edu

The term "algebraic analysis" has had, and continues to have, various meanings reflecting several different areas of study and approaches for over 200 years. The current definition, as given by Professor Przeworska-Rolewicz [1988], is: algebraic analysis is the theory of right invertible operators in linear spaces, in general without topology. As she points out in her encyclopedia article [1997], an essential distinction between algebraic analysis and operational calculus is that in the former, the notion of convolution is not necessary, there is no need for a field structure, and right inverses and initial operators are not commutative.

Here, the discussion will be very wide-ranging, in order to include many meanings of the term algebraic analysis. In addition to work which actually uses the term, we will briefly consider what it should, or might, mean. That is, algebraic analysis will be taken to mean the study of analysis using algebraic methods either exclusively, or at least, predominantly.

This paper is to be viewed as a work mainly of synthesis, rather than of new exploration. The work of many historians of mathematics will be cited. Of special note are the works of Bottazzini [1986], Deakin [1981, 1982], Grabiner [1981, 1990], Grattan-Guinness [1994], and Lützen [1979]; also, Davis [1936], although not a historical work, has much historical information.[1]

1. Algebraic analysis in the wide sense. The term algebraic analysis is sometimes used for studies which are primarily or wholly algebraic. For example, Mansion [1898] in

2000 *Mathematics Subject Classification*: 01-02, 01A50, 01A55, 01A60, 01A65.
Key words and phrases: algebraic analysis, operational calculus.
The paper is in final form and no version of it will be published elsewhere.

[1] At the Conference on Algebraic Analysis, Professor Przeworska-Rolewicz made available a preprint, *Two Centuries of Algebraic Analysis*, which contains further historical information (cf. the next paper).

a collection called *Mélanges mathématiques*, has sections titled algebraic analysis, which discuss purely algebraic matters.

In a recent work of Kashiwara, Kawai, & Kimura [1986], called *Foundations of Algebraic Analysis*, the authors state that "Algebraic Analysis" is not well-defined, but does have a common core, which is the essential use of algebraic methods such as cohomology theory. They go on to say that their title refers to a more special meaning, due to M. Sato: algebraic analysis is that analysis which holds onto its substance, and survives the shifts in fashion in Analysis, e.g., Euler's mathematics. In particular, this means the microlocal theory of linear partial differential equations (i.e., local analysis on the cotangent bundle). Sato is considered to be the founder of the Japanese school of algebraic analysis; see e.g., the collection edited by Kashiwara & Kawai [1988], called *Algebraic Analysis: Papers Dedicated to Professor Miko Sato on the Occasion of His Sixtieth Birthday*, which is a study of hyperfunction theory and microlocal analysis. A special part of this subject, involving D-modules and sheaf theory [e.g., Björk [1979], or Kashiwara and Schapiro 1990] may have more of a claim to the appellation algebraic analysis than most of microlocal analysis.

There are also many fields that could rightfully be called algebraic analysis, but are not. Examples are: the study of analysis over various algebraic systems such as Grassmann algebras (sometimes referred to as superanalysis in physics: Berezin [1987]), or over the quaternions and over Clifford algebras, Gurlebeck and Sprössig [1997], or Brackx, Delanghe, & Sommen [1982], or Ryan [1996]. There are even studies of analysis in abstract categories (Takahashi [1969]). (Perhaps analysis over p-adic numbers, or more generally, over local fields, should be considered algebraic analysis.) Abstract operator algebras also are candidates for this subject.

Differential algebra is a subject introduced by Joseph Fels Ritt (1893–1951). In its original version Ritt [1932] was concerned with systems of differential equations, ordinary or partial, which are algebraic in their unknowns and their derivatives. He says that it had been customary to assume canonical forms for systems, but this is an inadequate representation of general systems. The reasons for this inadequacy are: restrictions due to the use of implicit function theorems, lack of methods for coping with degeneracies which are likely to occur in the elimination process, and absence of techniques for preventing entrance of extraneous solutions. He goes on to say that these are merely symptoms of "the futility inherent in such methods of reduction." But there is a firm theory of algebraic elimination of the theory of systems of algebraic equations, which involves the theory of rings and ideals. The object of Ritt's study is to bring to the theory of systems of algebraic differential equations some of the completeness enjoyed by systems of algebraic equations.

In [1932], Ritt considers functions meromorphic on a given open connected set $\mathfrak{R}$ of the complex plane. Then a *field* (of such functions) is a set $\mathcal{F}$ satisfying the following.

(a) $\exists\, f \in \mathcal{F}$ such that $f \not\equiv 0$ on $\mathfrak{R}$.
(b) $f, g \in \mathcal{F} \Rightarrow f+g,\ f-g,\ fg \in \mathcal{F}$.
(c) $f, g \in \mathcal{F}$ and $g \not\equiv 0 \Rightarrow f/g \in \mathcal{F}$.
(d) $f \in \mathcal{F} \Rightarrow$ its derivative $f' \in \mathcal{F}$.

Ritt presents his results in this framework. In a revised version of this work, Ritt [1950] generalized this setup to an arbitrary field $\mathcal{F}$ of characteristic zero, together with an operation called differentiation, denoted by $a \mapsto a'$, where $a, a' \in \mathcal{F}$. This operation is assumed to satisfy

$$(a+b)' = a' + b', \qquad (ab)' = ba' + ab'.$$

This is Ritt's definition of a *differential field*, and he proceeds to develop his theory at this level of abstraction.

This certainly seems to be a subject which should be described as algebraic analysis. However, more recent studies of differential algebra, e.g., Kaplansky [1957], Kolchin [1973, 1985] are more concerned with algebra than analysis. (The entire subject has been described as being 99% due to Ritt and his students.) There is also a theory of difference algebra: see Cohn [1965].[2]

Many more examples could be given, but these will suffice for now; later we shall consider a few more possibilities in greater detail.

As this brief list of topics shows, we could easily stray very far from the current definition of algebraic analysis, so we need to limit the topics chosen for consideration. Emphasis will be on the theory of symbolic methods in the wide sense, and a few related topics will be surveyed.

2. Pre-history of algebraic analysis. Algebraic analysis, as a symbolic calculus, is a very old subject, perhaps going back to Leibniz. He was struck by the resemblance between the formula for the nth differential of a product and the binomial expansion:

$$d^n(xy) \text{ and } (a+b)^n.$$

(Letter of 1695 to John Bernoulli, published 1711.) Not only did Leibniz state his rule for the differential of a product, but he claimed that it is also valid for negative n if one takes d^{-1} to mean an antiderivative. In a letter of 1695 to L'Hospital, Leibniz also considered the possibility of non-integral values of n. However, he noted that this seems to give paradoxical results.

During the earliest years of the development of the calculus, and continuing well into the nineteenth century, power series were considered to be algebraic objects. Chrystal's *Algebra*, [1904] in editions in the early 20th century, considered the study of power series to be a part of algebra. Although there were occasional nods at convergence considerations, power series were generally handled as what are currently called formal power series. Newton's favorite tool in calculus was power series [1669]. When Euler [1748] wrote the first "pre-calculus" book, power series were very much in evidence, and were handled as a part of algebra.

According to Grabiner [1997], Colin Maclaurin's *Treatise of Fluxions* [1742] was an important link between the calculus of Newton and the Continental analysts. The *Treatise* is Maclaurin's major work on analysis. It influenced John Landen's work on series, and

[2]The paper of Professor Wloka presented at this Conference is a new contribution to the analytical side of differential algebra.

was praised by Lagrange, Woodhouse, and Waring. Lagrange [1797] cited the French edition of Maclaurin's *Treatise*.

The earliest manifestation of algebraic analysis is generally considered to be the well-known work of Joseph-Louis Lagrange (1736–1813). This began with a memoir [1772] and culminated in his celebrated *Théorie des fonctions analytiques* [1797, 2nd ed. 1813], written for students of the École Polytechnique in Paris. In [1772] Lagrange referred to Leibniz and the Bernoullis as his source of ideas. Putting

$$\Delta u = u(x+\xi,\ y+\Psi,\ z+\zeta,\ t+\theta,\ldots) - u(x,y,z,t,\ldots) \tag{2.1}$$

he obtained various results, among them a celebrated formula

$$\Delta^\lambda u = (e^{\frac{du}{dx}\xi + \frac{du}{dy}\Psi + \frac{du}{dz}\zeta + \cdots} - 1)^\lambda, \tag{2.2}$$

where for λ negative, $d^{-1} = \int$, and $\Delta^{-1} = \sum$. Lagrange did not consider his arguments to be proofs, but this use of analogy led him to results which would otherwise have been very difficult to obtain. This led him to the summation of series. The first published proof of Lagrange's theorems is due to Laplace [1776], who also discussed formula (2.2) at length.

In his book [1797], Lagrange mentions John Landen (1719–1790) as having ideas similar to his own for overcoming the lack of rigor in the foundations of calculus. Lagrange cites a public lecture of Landen in 1758, and his book [1764]. But Landen's work on "residual analysis" had little influence outside of England. Lagrange also wrote *Leçons sur le calcul des fonctions*. [1799, 2nd ed. 1806] which is mostly another version of the first part of the *Théorie des fonctions analytiques*, with some expanded coverage, but omitting the second and third parts, applications to geometry and to mechanics.

Lagrange was a great admirer of Descartes, Leibniz, and especially Euler. In particular, Lagrange was inspired by Euler in his use of power series. According to Euler, analysis is a method of applying algebra to solve problems. He considered power series to be entirely algebraic. (See Fraser [1987], Grabiner [1981, 1990].) Lagrange wanted to provide a rigorous foundation for calculus by basing it on algebra, which he assumed to have a secure foundation. In particular, he wanted to eliminate any appeal to geometric intuition, or to ideas of motion (mechanics), and to use algebra alone. In fact, there are no diagrams in this book. Lagrange accepted Euler's ideas, and believed that algebraic operations apply to infinite processes such as series. He tried to prove that all functions have power series expansions. Unfortunately, the concept of function was not at all clear at that time.

Strictly speaking, Lagrange's "algebraic analysis" is not really a symbolic calculus, but an attempt at an algebraic foundation of calculus.

Mathematicians soon began to doubt the rigor of Lagrange's algebraic calculus; among the first to do so were Abel Burja (1752–1816), Józef Maria Hoene-Wroński (1776–1853), Jan B. Śniadecki (1756–1830), and Bernard Bolzano (1781–1848).

Since Śniadecki and his mathematical work are not well known, we shall briefly describe these. He studied in France, where he met Cousin, Laplace, and d'Alembert. He wrote a book called *The Theory of the Algebraic Calculus Applied to Curved Lines* [1783]. This work, written in Polish, seems to be one of the earliest books to use the term "algebraic calculus" in its title.

Only two of four proposed volumes were published, due to the great personal expense he had to suffer. The first two volumes cover material similar to that in Euler [1748]. The topics are: algebra, solution of algebraic equations, expansions of elementary functions in power series, the basics of coordinate geometry, conics, some special surfaces, and certain problems of differential geometry. The unpublished volumes III and IV were to cover differential and integral calculus. It must be noted that Śniadecki wrote a history, which was critical of Lagrange, especially his version of calculus. See Dianni [1977] for details of Śniadecki's calculus.

As far as the completed version of Śniadecki's book goes, it seems to follow in the Eulerian tradition of calculus and is an early version of numerous books using the term algebraic analysis in their titles.

Wroński and his work are better known than Śniadecki; in [1811] he criticized Lagrange's algebraic calculus, and proposed his own "universal series" as a substitute. His supreme law was that every function f can be given in the form

$$f(x) = \sum_{0}^{\infty} A_n \Omega_n(x),$$

where the Ω_n are any functions of x, and the numerical coefficients A_n involve determinants which are known today as Wronskians. In [1812], Wroński repeated his claim, and gave a more severe criticism of Lagrange. Although Wroński's claim appears outrageous, it seems to have been taken seriously by Banach [1939], who proved that under very general assumptions, Wroński's method is possible for a large class of functions, depending on the functions Ω_n. He also gave a functional analytic interpretation of the supreme law.

None of these mathematicians had the influence or the stature to sway opinion against the work of such an eminent mathematician as Lagrange. This was left to Augustin-Louis Cauchy (1789–1857), who was heavily indebted to Lagrange for many of his own ideas on calculus. However, Cauchy rejected algebra as a basis for calculus. Instead, he gave his own version, mainly in his *Cours d'analyse. 1re Partie. Analyse algébrique* [1821]. This was only the first part of a planned course of analysis, which was to have been offered in three parts:

I. Analyse algébrique
II. Calcul différentiel et intégral (including ordinary differential equations)
III. Application du calcul différentiel et intégral à la géométrie.

As Cauchy describes this work, it treats various kinds of real and imaginary functions, convergent and divergent series solutions of algebraic and trigonometric equations, and decomposition of rational fractions. Although many of the topics he discusses are the same as those described by Euler and by Lagrange, Cauchy makes the study of convergence a key issue in this work. Thus Cauchy continued the tradition of viewing algebraic analysis as the study of infinite series, but with a rigorous study of convergence as a key point.

This use of the term algebraic analysis for the study of elementary functions via infinite series persisted throughout the nineteenth century. Among numerous works of this type, we mention only a few: Capelli & Garbieri [1894], Capelli [1909], Schlömilch [1845] (this work presented much on the convergence of continued fractions), and Cesàro

[1904]. Cesàro's book begins with infinite series, and includes discussions of complex numbers and quaternions, the theory of algebraic equations, in addition to the basics of calculus. It also covers a large amount of material on geometry, even more than on algebra. (He also mentions the course of algebraic analysis of Capelli and Garbieri [1894].) For work on algebraic analysis in Germany up to 1840, see Jahnke [1993].

The sense of the term algebraic analysis used by Lagrange, Cauchy, Schlömilch, Cesàro and many others is not a precursor of the contemporary meaning. The true precursors were many lesser-known mathematicians, beginning with some of the immediate followers of Lagrange, such as Arbogast, Brisson, the brothers Français, and Servois, but also including Cauchy. They were followed by the English algebraic symbolists. From them, the line of ascendance reached the best-known symbolists, Boole and Heaviside.

3. The early French symbolists and the English algebraic symbolists. It must be remembered that at the beginning of the 18^{th} century, not only were the concepts of abstract algebra lacking, but even the idea of convergence was unclear. Also, the definition of a function was not yet standard. As a result, mathematicians had to struggle with these ideas, as well as the particular problems that they addressed.

Louis François Antoine Arbogast (1759–1803) was a follower of Lagrange. In the spring of 1789, he presented to the Academy of Sciences in Paris a paper [1789] on the new principles of differential and integral calculus independent of the theory of infinitesimals and that of limits. This was never published. It was, like Lagrange's work, an attempt at basing calculus on algebra, i.e., on power series. (Lagrange's paper of 1772 is not mentioned explicitly by Arbogast.) However, it did lead Arbogast to further studies, and in fact, to his best known work, *Du calcul des dérivations* [1800]. The principal aim of this book was to give simple and precise rules for finding power series expansions, and he gives a vast number of examples of this process. Arbogast mentions only one person as having similar ideas: Edward Waring (1734–1798). Waring's book [1776; Arbogast gives the date as 1785] discusses his direct and inverse method of deduction, which Arbogast describes as parallel to the calculus of derivations. However, Waring gave no details of his method, and he was noted for his obscure style, so the method seems to have had little influence.

Arbogast thought of his derivations as a generalization of the derivative, and he considered differential calculus as a special case of his theory. His derivation, D, is a linear operator, which he separated from the function being acted on. Thus (using modern notation), from

$$\Delta f(x) = (e^{hD} - 1)f(x), \tag{3.1}$$

he obtained

$$\Delta = (e^{hD} - 1). \tag{3.2}$$

Arbogast's most important contributions to symbolic methods were the concept of an operation (derivation), and the idea that one can calculate with operators, which was made possible by his separation of the symbol of derivation from the object acted on by the operation, as in (3.2).

Barnabé Brisson (1777–1828) in [1808] introduced the idea of representing a differential equation as a differential operator acting on a function, and used an algebra of operators to find solutions of partial differential equations. Thus, to solve the equation

$$y + Ly = f(x),$$

where L is a linear (differential) operator, he used

$$y = (1 + L)^{-1} f(x) = (1 - L + L^2 - L^3 + \cdots) f(x),$$

and calculated the right side term-by-term. Several of his later memoirs were unpublished, but formed the starting point for Cauchy's work on symbolic methods.

François (Joseph) Français (1768–1810) assisted Arbogast with his *Calcul des dérivations* and collected his papers. He admired Brisson's work, but criticized its lack of rigor. His methods were very similar to those popular in England in the late 1830s. His brother, Jacques Fréderic Français (1775–1833) was attracted to the calculus of derivations by his brother's work. In [1812–13] Jacques used a form of factorization into linear operators to solve a differential-difference equation.

François-Joseph Servois (1767–1847) was one of the chief precursors of the English school of symbolic algebraists. His essay on a new mode of exposition of the principles of differential calculus [1814] presented a calculus of operations which was motivated by the search for a rigorous foundation for calculus. Unlike his predecessors, he did not always distinguish between functions and operations. He proved properties of linear commutative operators in this work, which we describe currently as ring properties. This explained why they could be manipulated like algebraic magnitudes. He introduced the terms "commutative" law and "distributive" law. Servois' memoir inspired the studies of Robert Murphy and George Boole on symbolic operations.

Such algebras are important in the history of algebra as well as of analysis, as they are the first case where the objects of study are neither numbers nor geometric magnitudes. Applications of the kind mentioned above led to a limited acceptance of the use of differential operators. In the 1820s Cauchy used his Fourier integral method (with complex integrands) as a basis for Brisson's procedures for solving differential equations. (See below.) This was an early instance of indirect methods for symbolic operators, in which the given differential equations are first transformed and symbolic methods are used on the transformed equations. (The Laplace transform is another example of such indirect methods.)

A different use of operations is due to J. B. Fourier [1822] who used operations not to solve equations, but to express concisely and to verify solutions of partial differential equations. For example, he got the solution of

$$\frac{dv}{dt} = \frac{d^2 v}{dx^2} \tag{3.3}$$

as

$$v = e^{tD^2} \varphi(x), \text{ where } \varphi \text{ is an arbitrary function.} \tag{3.4}$$

Then to verify that this is a solution, "differentiate" both sides of (3.4) with respect to t:

$$\frac{dv}{dt} = D^2 e^{tD^2} \varphi(x) = D^2 v = \frac{d^2 v}{dx^2}.$$

Fourier also used a single symbol, D, to stand for a compound operation (in fact, what is known today as the Laplacean operator). For example, starting with the equation

$$\frac{d^2v}{dt^2} = \frac{d^2v}{dx^2} + \frac{d^2v}{dy^2}, \tag{3.5}$$

he substituted

$$\frac{d^2v}{dx^2} + \frac{d^2v}{dy^2} = Dv, \tag{3.6}$$

so that equation (3.5) became

$$\frac{d^2v}{dt^2} = Dv. \tag{3.7}$$

Naturally, Fourier solved (3.7) as

$$v = \cos(t\sqrt{-D})\varphi(x, y), \tag{3.8}$$

where $\cos(t\sqrt{-D})$ is to be expanded in powers of tD, with

$$D^n = \left(\frac{d}{dx} + \frac{d}{dy}\right)^n.$$

In his collection [1827] of mathematical exercises, Cauchy included three long articles on symbolic methods. He referred to two papers of Brisson (dated 1821, 1823, but unpublished) as his source, and gave a rough outline of their contents.

Cauchy pointed out that the analogy between powers and indices of differentiation led to the idea of representing a linear expression involving a function $u(x, y, z, \ldots)$ and its successive derivatives in the form

$$f(\alpha, \beta, \gamma)u, \tag{3.9}$$

where $f(\alpha, \beta, \gamma)$ is a polynomial of degree m. According to Cauchy, Brisson generalized this by allowing $m = \infty$, so that (3.9) has meaning for a function with a power series expansion. (Similar ideas occur in Arbogast's work.) Cauchy said that Brisson used (3.9) to get solutions of homogeneous and nonhomogeneous partial differential equations. He also considered (3.9) when f can be expanded in descending powers of the variables only, and applied this to find solutions of certain partial differential equations in symbolic form. But he also pointed out the need for care in applying (3.9), it being best to restrict attention to the case where f is a polynomial or a rational function.

Cauchy considered the symbolic operators $F(D)$, $F(\Delta)$, and $F(D, \Delta)$, where $F(x)$, $F(\alpha, \beta)$, are polynomials, and $Dy = \frac{dy}{dx}$, $\Delta y = y(x + \Delta x) - y(x)$. He found the following results

$$F(D)[e^{rx}f(x)] = e^{rx}F(r + D)f(x), \tag{3.10}$$

$$F(\Delta)[e^{rx}f(x)] = e^{rx}F(e^{rx}(1 + \Delta) - 1)f(x), \tag{3.11}$$

$$F(D, \Delta)[e^{rx}f(x)] = e^{rx}F(r + D, e^{rx}((1 + \Delta) - 1)f(x). \tag{3.12}$$

Then he used these rules to solve nonhomogeneous linear partial differential equations with constant coefficients.

For example, if

$$(D - r)y = f(x), \tag{3.13}$$

then by (3.10),

$$y = e^{rx} \int e^{-rx} f(x)dx, \tag{3.14}$$

a result Cauchy attributed to Brisson. Then higher order equations are solvable by factoring. Suppose that the polynomial

$$a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1} x + a_n$$

has the (real or complex) roots $r_1, r_2, \ldots, r_n$. Then the differential equation

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_{n-1} \frac{dy}{dx} + a_n y = f(x), \tag{3.15}$$

can be written as

$$(D - r_1)(D - r_2) \cdots (D - r_n) y = f(x)/a_0. \tag{3.16}$$

Then this can be put into the form of a system

$$\begin{cases} D - r_1) y_{n-1} = f(x)/a_0, \\ (D - r_2) y_{n-2} = y_{n-1}, \\ \vdots \\ (D - r_{n-1}) y_1 = y_2, \\ (D - r_n) y = y_1. \end{cases} \tag{3.17}$$

Using the previous example, Cauchy wrote the solution as

$$y = \frac{e^{r_n x}}{a_0} \int e^{(r_{n-1} - r_n)x} \left(\int e^{(r_{n-2} - r_{n-1})x} \left(\cdots \int e^{(r_1 - r_2)x} f(x) dx \right) \right) \cdots dx. \tag{3.18}$$

This method works for simple or multiple roots. Cauchy used (3.11) to solve difference equations. He found analogous formulas for polynomials in several variables and applied them to solve linear partial differential equations.

According to Koppelman [1971–2], Cauchy's work represents the highest development of the symbolic method on the Continent during the first half of the 19th century. It contained many results which were later used by the English as their starting point. But Cauchy did not trust the method, did not carry it further, and did not give a justification of its basic principles.

Regarding the possibility of fractional indices of operation, there are passing references in the works of Arbogast, Laplace, and Fourier. In [1822], Fourier claimed that his Fourier Integral Theorem,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha f(\alpha) \int_{-\infty}^{\infty} dp \cos(px - p\alpha), \tag{3.19}$$

led to

$$\frac{d^m y}{dx^m} = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\alpha f(\alpha) \int_{-\infty}^{\infty} dp\, p^m \cos\left(px - p\alpha + \frac{m\pi}{2} \right), \tag{3.20}$$

where m is any number.

The first attempt at a coherent theory of fractional indices of operation is due to Joseph Liouville [1832]. Assuming that

$$y = \sum A_m e^{mx} \quad (\text{or } y = \int A(m) e^{mx} dm), \tag{3.21}$$

the defined, for any number μ,

$$\frac{d^\mu y}{dx^\mu} = \sum A_m e^{mx} m^\mu \quad \text{(or the corresponding integral).} \tag{3.22}$$

Liouville considered the analogy between operators and the laws of exponents important, but he did not use it as the starting point of his theory.

In a student paper, Riemann [1953] defined the derivative of order v of $z(x)$ to be the coefficient of h^v (times a constant) in the series expansion of $z(x+h)$:

$$z(x+h) = \sum_{v=-\infty}^{\infty} k_v \delta_x^v z(x) h^v \tag{3.23}$$

where the k_v depend only on h_v. If the exponent is an integer, this reduces to Lagrange's definition of derivative.

For Arbogast and Fourier, the method of using symbols of operations as if they were symbols of quantity was an elegant way of discovering, of expressing, or of verifying theorems, but it was not a method of proof.

Français attempted to state general principles to justify the methods of symbolic operations, roughly on the basis that both obey the same laws of combination. But he gave no clear statement of this. Servois' work was an attempt at a firm theoretical basis for calculus; it was also, in part, a response to the criticism of Wroński. However, none of the Continental mathematicians was interested in developing the method of symbolic operations further, and unlike the English, they did not relate it to a theory of algebra.

At this point, we shall turn to a discussion of the English symbolic algebraists who were influenced by the work described above. They not only played an important role in the English work on symbolic operations, but they inspired the evolution of abstract algebra in general. See especially Koppelman [1971–72] for an excellent description of the work of this period.

This English school included Charles Babbage (1792–1871), George Boole (1815–1864), Augustus De Morgan (1806–1871), Charles Graves (1810–1860), Duncan Farquharson Gregory (1813–1844) and George Peacock (1791–1858).

Their studies began with the response of English mathematicians to Continental advances in analysis. The first polemical work on this topic was by Robert Woodhouse (1773–1827). This book, [1803], was devoted to the foundations of calculus, and claimed that the most satisfying one is that of Lagrange and Arbogast, since they linked calculus to algebra. He wrote

> *the differential calculus . . . is to be considered as a brand of common Algebra, or rather as a part of the common symbolical language in which quantity is treated of.*
> [Quoted in Koppelman, 1971–72, page 176]

However, Woodhouse also pointed out difficulties with the symbolic method. He gave the example of $1/(1+x)$. If this symbol represents the series resulting from dividing 1 by $1+x$, then

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots, \tag{3.24}$$

but $1/(x+1)$ is the symbol representing the series resulting from dividing 1 by $x+1$, so that

$$\frac{1}{x+1} = \frac{1}{x} - \frac{1}{x^2} + \frac{1}{x^3} - \cdots. \tag{3.25}$$

As a result, one cannot affirm that

$$\frac{1}{1+x} = \frac{1}{x+1}.$$

(Since no consideration is given to convergence, the magnitude of x plays no role here.)

Woodhouse' work was taken up by a group that called itself the Analytical Society. It consisted of Babbage, John Herschel (1792–1871), and Peacock. According to Koppeleman [1971–2], their "crusading" writings emphasized certain critical ideas in French works connected with symbolic operations.

Babbage learned differential calculus from the book of Woodhouse, but the latter's most influential book was a textbook on trigonometry [1809]. In this book, he defined the trigonometric functions by series expansions (in keeping with the meaning of term algebraic analysis at the time), and used differential notation throughout. Peacock said that this book, more than any other, contributed to revolutionize the study of mathematics in England.

The Analytical Society emphasized the relationship between calculus and algebra, and work of the French in this area illustrated the superiority of Continental work especially well. The Society translated [1816] Lacroix' condensation [1802] of his 3-volume work [1797–1800], which used the Lagrange theory as a foundation. However, in the shortened work [1802], Lacroix used the theory of limits, a change disapproved of by translators Babbage, Herschel, and Peacock. To them the theory of limits was not acceptable because it tends to separate the principles and "departments" of differential calculus from those of Common Algebra. Of course, Peacock made a similar criticism of Newton's fluxions, which introduced extraneous ideas from geometry and mechanics into the study of purely algebraic problems. Also, differential notation is preferable to fluxional, because it is equally convenient for representing both operations and quantities. Although they used the name "Maclaurin series" in this translation (as did Cauchy later), they did not appreciate the value of his *Treatise*. Ironically, they were actually promoting some of Maclaurin's ideas (from his Book II) on algebraic methods in calculus, apparently without realizing it. In the preface, Babbage and Herschel emphasized the relationship between algebra and calculus, exemplified by their preference for the approach of Lagrange and Arbogast to the foundations of differential calculus, the analogy between repeated operations in calculus and exponentiation, and the method of separation of symbols. They stressed the importance of good notation, especially the idea that a general function can be denoted by a single letter. They claimed that the resulting calculus is more general than any known, so they called it the "calculus of functions."

Herschel, a celebrated astronomer, concentrated his mathematical work on the solution of finite difference equations and their application to functional equations. He used the method of separation of symbols of operation from those of quantity, and emphasized the relationship between calculus and ordinary algebra.

Babbage [1827] stressed symbolic manipulations and the formal approach to algebra. In general, the English symbolic algebraists found the approach to the interpretation of arithmetic and algebra on the continent as unsatisfactory as in England. They were led to look for a new approach; in particular, De Morgan, Gregory, and Peacock worked in close cooperation. Peacock introduced his ideas in his textbook on algebra [1830] and in a report of the British Association for the Advancement of Science [1834]. In the latter, he introduced the Principle of Permanence of Equivalent Forms into algebra. The purpose of this principle is to justify various generalizations, e.g., $(1+n)^n$ beyond natural numbers, the definition of a^0, etc. His statement of the principle was:

> *Whatever form is algebraically equivalent to another form expressed in general symbols, must continue to be equivalent, whatever those symbols denote.*
>
> [1834, p. 198]

Peacock used the Principle of Permanence to give meaning to the operation d^α/dx^α for nonintegral values of α.

The English mathematicians were led to study abstract laws of combination by the development of the calculus of operations, rather than by work on algebra, e.g., on the complex number system. Their desire to explain the principles of the calculus of operations and to extend its applicability was important to Gregory, Boole, De Morgan, and even had some influence on Hamilton.

In England, work on symbolic methods in analysis began in the 1830s, with the contributions of Charles Graves, D. F. Gregory, John Hewitt Jellett (1817–1888), and Robert Murphy, among others.

D. Gregory wrote numerous papers on what the English school called the "calculus of operations". He was familiar with Cauchy's work, and his methods are similar to Cauchy's, except that he treats symbols of operation as if they were symbols of quantity. He credited Brisson's unpublished work of 1821 as being the first in which the method was applied to the solution of differential equations, and cites Cauchy's *Exercices* [1827] as his source of information about Brisson's work.

Gregory stated that a linear partial differential equation with constant coefficients in any number of variables could be treated exactly like an ordinary differential equation, with one variable symbol of operation, treating the remaining ones as constants. He named Fourier as his source of ideas on partial differential equations, except that Fourier did not use the method to find solutions. Gregory also applied the method to systems of ordinary differential equations, using analogy to mimic techniques of elimination for solving systems of linear algebraic equations. He even argued that there was no valid distinction between symbols of differentiation and symbols of differencing. He believed that he had a formula for d^n/dx^n for rational values of n, although he ignored questions of convergence of the expansions in infinite series which occur when $n < 0$ or n is a fraction. (Some of his colleagues called attention to this difficulty.) He stated three basic rules for symbols of operation $f, f_1, \ldots$:

(i) index law $f^m f^n(x) = f^{m+n}(x)$

(ii) commutative law $f\left(f_1(x)\right) = f_1\left(f(x)\right)$

(iii) distributive law $f(x+y) = f(x) + f(y)$

Gregory attempted to popularize the calculus of operations in a textbook [1841], in which he used separation of symbols extensively. He also used the method to develop a theory of differentiation of fractional order. He listed his major sources as Servois and Murphy.

Despite some lapses in rigor, Gregory obtained many interesting results. His efforts led to the acceptance of the calculus of operations by many English mathematicians. Because he assumed commutativity of operations (rule (ii)), he restricted his studies to constant coefficient differential equations.

Gregory wrote no books on algebra, but in papers he declared that symbolic algebra is the initial concept, and the Principle of Permanence is not needed. His principal idea was the separation of the symbols of operations from those of quantities. Algebra is the science of combinations of operations, defined not by their nature, but by the laws of combination.

The next step, the study of noncommutative operations, was taken by Robert Murphy (1806–1843). He was unconventional: unlike the usual notation, his operations acted to the left, not the right, and he called commutative operations "relatively free" and non-commutative ones "relatively fixed". Although Murphy ignored convergence questions, and used infinite series of arbitrary operations, he was very careful about inverses, whose non-uniqueness can cause difficulties. He proved that if L is a distributive (linear) operation, then so is its inverse, L^{-1}, and he verified the formula $(LM)^{-1} = M^{-1}L^{-1}$. His discussion of inverses centered about the "appendage" of a linear operation, which seems to be related to the modern concept of the kernel of a linear transformation. Murphy's work was abstract and very general, but it did not address convergence questions in applications to calculus. This work influenced Boole, who used the approach to extend Gregory's studies of linear differential equations to the case of variable coefficients.

The best known of the English school was George Boole, who built on the work of Murphy and Gregory in his study of the operator D of differentiation [1844]. This was his most important work on the subject and it was very influential on later workers on the calculus of operations. He stressed the use of analogy to get results on operators and applied Gregory's three rules to ordinary and partial differential equations. For the case of constant coefficients, Boole assumed that the characteristic polynomial has distinct roots, but instead of following Gregory by factoring it into linear factors, he used partial fractions, claiming that this method is independent of any properties of the variable except the three shown by Gregory to be common to the symbol d/dx and to the algebraic symbols generally supposed to represent numbers. However, the method applied only to a restricted class of partial differential equations, so it fell into disuse. The rigor of the method was also questionable, which bothered someone as concerned with logic as Boole was.

In [1847] he defined the terms "symbolical equation" and "symbolical solution": these mean that the results are such that their validity does not depend on the significance of the symbols which they involve, but only on the truth of the laws of their combination. He did include symbolic methods in his two celebrated textbooks on differential equations

[1859] and on finite difference calculus [1860]. Incidentally, in [1859], Boole used the term "Laplace's method" which eventually became standard, as the Laplace transform. But by the late 1860s interest in such symbolic methods had died off.

Robert Harely, a friend and biographer of Boole, said that after speculation on the logic of the calculus of operations, Boole was led to his calculus of deductive reasoning. In his book on differential equations [1859], Boole says that the true value of symbolic methods lies only partly in their simplicity and power; rather their true importance lies in their connection with the general relationship between language and thought.

The first systematic extension of Boole's method to functions of several variables was by Robert Carmichael (1828 or 9–1861) in 1851. (Although Boole had stated his results for the case of several variables, he only applied them to the case of one variable.) Taylor's Theorem, in the form

$$e^{h\frac{d}{dx}}f(x) = f(x+h), \tag{3.26}$$

was generalized by Herschel as

$$f(1+\Delta)u_x = f(e^{\Delta x D})u_x, \tag{3.27}$$

which in turn was extended by W. R. Hamilton in 1837. Hamilton's most important work in the calculus of operations was his application of it to the evaluation of certain definite integrals which occur in physics.

The results of this early stage were summarized in a textbook of Robert D. Carmichael, *Treatise on the Calculus of Operations* [1855]. This is the earliest book dealing with studies of English mathematicians on operators and their uses in analysis[3]. (Although Arbogast's earlier calculus of derivations encompassed differential calculus, its main goal was the calculation of coefficients of various power series expansions. He provided copious examples of these calculations.) Carmichael's book is mainly a summary and an extension of the work described above. But he also mentions the book of Jellett [1850] on the calculus of variations in the context of the calculus of operations.

Carmichael says that the usefulness of the calculus of operations is due to the simplification it provides for both the student and the mature mathematician. For the former, it makes learning easier; for the latter, it provides a means of recovering known results with ease and elegance, and aids research into new areas by ensuring certainty and providing means for rapid calculation.

Some of the results are due to the author, and his largest debt is due to Jellet, followed by Graves and Gregory. In particular, the applications to partial differential equations are mostly from the book of Gregory [1841].

On page 1, Carmichael gives his definition of the calculus of operations:

> *The calculus of Operations, in the greatest extension of the phrase, may be regarded as that science which treats of the combinations of symbols of operation, conformably to certain given laws, and of the relations by which these symbols are connected with the subjects on which they operate.*

[3]Professor Przeworska-Rolewicz has noted that the book of Lembert [1815], written in German, contains the basic elements of algebraic analysis.

He also passes along a remark made to him by Boole, that the great difficulty in the calculus of operations in the case of operations with respect to integral calculus consists in the interpretation of the symbolic results at which you arrive. By this time, Boole had been working with such operations for a number of years.

Carmichael lists three principal laws of operations, and states that a consequence of these laws is that every theorem in Algebra which depends on them has an analogue in Analysis. These are the laws introduced by Gregory and stressed by Boole [1844], but Carmichael does not mention either of them in this connection.

Carmichael's book was well known at the time and (at least in England) was influential in spreading the calculus of operations.

The popularity of the calculus of operations was enhanced by the appearance of elementary textbooks, by Gregory and Boole. De Morgan [1842] devoted space to the calculus of operations, discussing the work of Gregory on separation of symbols, (whose explanation of the validity of this was accepted by his contemporaries), theorems on symbols (e.g., of Lagrange and Herschel), and the use of symbolic methods for transformations of divergent series. He also discussed derivatives of fractional order, but considered this subject "unsettled". However, De Morgan's calculus book was the first full-length British book to abandon Lagrange's power series method and to use limits instead. He used only an intuitive approach to limits, avoiding $\varepsilon - \delta$ formulations. The ironic part of this is that De Morgan gave an explicit $\varepsilon - \delta$ formulation of the definition of limit in a paper of 1835. De Morgan also lectured in the calculus of operations in his courses.

In 1871, De Morgan discussed the relationships between the calculus of operations, the geometry of complex numbers, and the development of abstract algebra; these led to the notion of the purely abstract nature of symbols. (Incidentally, Murphy's paper of 1837 has an important place in the history of the algebra of linear transformations.)

Gregory, Boole, De Morgan, and Peacock, the promulgators of the new abstract algebra, all worked on the calculus of operations. They were led to the former by the latter. Koppelman [1971–72] claims that the English work in algebra was a direct response of the English to a specific aspect of the work of Continental analysts which was available to them: the calculus of operations (to use the English term for it). The British used the method extensively, and considered the concept of operation a unifying theme in mathematics, and one of the utmost interest. The strongest proponents of this were Gregory and Boole, and these efforts are related to their innovations in algebra and logic. De Morgan presented a brief outline of his views in [1835, 1849].

For more on the development of algebra, in addition to Koppelman [1971–72], see Novy [1973].

To conclude, the calculus of operations was imported into England from France and extended by Babbage and Herschel, and was a focal point for mathematical research in Great Britain during the period 1835–1865. De Morgan noted that the Cambridge and Dublin Mathematics Journal was full of symbolic reasoning. Additional work continued in Great Britain and on the Continent through the end of the 19^{th} century and into the 20^{th}. It was made more rigorous, was better understood, and eventually led to the modern theory of Linear Operators, as a part of Functional Analysis. It can still be found in many 20^{th} century textbooks on differential equations, e.g., Piaggio [1952]. However, there was a lull in the subject after Boole's work in the 1860s, until the 1890s; we now turn briefly to the source of renewed interest in symbolic methods, namely, the work of Oliver D. Heaviside (1850–1925) in the 1890s through his work on electrical engineering. His main contribution to science was the development and reformulation of Maxwell's Electrodynamics, and his mathematical ideas arose in that context.

Heaviside began his work with what he called the telegrapher's equation. It is generally believed that most of Heaviside's work was previously known (e.g., Cooper [1952], Lützen [1979], Petrova [1987]) to Cauchy, Gregory, Boole, and others. However, it was Heaviside who popularized the symbolic method, and it is often referred to as the Heaviside calculus. More precisely, Heaviside made three basic contributions:

1) He worked out operational methods systematically and gave numerous applications which were helpful to electrical engineers.
2) He went beyond merely solving equations and attempted to get an explicit representation of solutions which would be useful in describing physical processes.
3) He understood that he could use asymptotic divergent series to get rapid evaluation of a function.

He also popularized the use of the "Heaviside unit function" by his extensive use of it, although a version of this goes back to Fourier in his paper of 1807 and his book [1822]

$$\varphi(x) = \begin{cases} 1, & \text{if } 0 \leq x \leq \pi, \\ 0, & \text{if } \pi < x \leq 2\pi. \end{cases} \tag{3.28}$$

Heaviside considered an impulsive function at the instant $t = x$, which he denoted by $pH(t-x)$, where H is the Heaviside unit function and p is his symbol for the operator D. This is an early version of the delta-function $\delta(t-x)$. He also considered the difficulty previously discussed by Woodhouse, namely, if

$$\frac{1}{1+D} = 1 - D + D^2 - D^3 + \cdots, \tag{3.29}$$

then one is interpreting D^n as $\left(\frac{d}{dx}\right)^n$, whereas if

$$\frac{1}{1+D} = \frac{1}{D} - \frac{1}{D^2} + \frac{1}{D^3} + \cdots, \tag{3.30}$$

then

$$\frac{1}{D} f(x) = \int_a^x f, \tag{3.31}$$

i.e., D is right-invertible. Practical experience led Heaviside to the second form (3.31) wherever possible. For electrical problems, it is also natural to choose $a = 0$. Heaviside mentioned [1893, Vol. III, p. 236)] the use of the Laplace transform in connection with his operational methods, and van der Pol and Bremmer [1950] claim that Heaviside saw this as a rigorous basis for the method. A change in terminology has occurred here, from the earlier calculus of operations to the form still current: operational calculus.

Since the Laplace transform is often used as a justification for the use of operational methods, we shall briefly discuss the development of the Laplace transform as well as the connection between the two.

4. Representational calculus: the Laplace transform and other indirect methods. Cooper [1952] made a useful distinction in the types of operational calculi: a form of operational calculus in which the operator is manipulated directly according to algebraic rules is a *formal calculus*, and one in which it is assumed that a function can be represented in a certain manner, and the operation of differentiation on the function is made to correspond to ordinary algebraic operations on the representing object, is called a *representational calculus*. (The distinction is not absolute.) In these terms, Heaviside made use of a formal calculus. It will be instructive to examine the relation between these two methods of operational calculus.

One of the basic results of Heaviside's calculus is his expansion theorem, which allowed him to use partial fractions to expand rational functions of operators. Although he probably found this independently, it can be found in forms similar to his in work of Cauchy, Lobatto, and Boole. In fact, Heaviside studied Fourier's book [1822] and Boole's book [1859] on differential equations.

A major step in the development of representational calculi was made by Fourier [1822], who introduced two of them: Fourier series and Fourier integrals. But the first systematic operational calculus is in the work of Cauchy over the period 1815–1850. He wrote the Fourier Integral formula for this in exponential form,

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{iu(x-y)} f(y)dy\,du, \tag{4.1}$$

and for functions of the operator D, he got

$$\varphi(D)f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \varphi(iu)e^{iu(x-y)} f(y)dy\,du, \tag{4.2}$$

and corresponding formulas for functions of several variables. This is the essence of the modern form of representational calculus. Cauchy felt that he had a rigorous justification for symbolic methods involving the operators D and Δ when these are applied to rational functions, but not for infinite series (which Brisson accepted) or for taking operators under the sign of integration (as occurred in work of Brisson and of Poisson). Cauchy proposed the natural test for correctness of these procedures: substitute the result into the equation to be solved, and verify the solution. Of course, he required investigating the convergence of any infinite series that arose in the process. He was especially fond of using his residue theory for this purpose.

Cooper lists three main types of operational calculus for the symbol D.

1) Fourier Operational Calculus for periodic functions. Representing a function f by its Fourier series,

$$f(t) = \sum_{-\infty}^{\infty} a_n e^{int}, \tag{4.3}$$

a function of D is represented as

$$\varphi(D)f(t) = \sum_{-\infty}^{\infty} \varphi(in)a_n e^{int}. \tag{4.4}$$

2) Cauchy Operational Calculus for functions defined on the real line. This is given by (4.1) and (4.2). The drawback here was that functions must behave at infinity for the integrals to converge, and various methods have been devised to generalize the Fourier transform (e.g., E. C. Titchmarsh and T. Carleman). Since the introduction of distribution theory, this could be extended to the class of tempered distributions.
3) Heaviside Operational Calculus for functions defined on $[0,\infty)$. This is usually done via the Laplace transform

$$F(p) = (\mathcal{L}f)(p) = \int_0^\infty f(t)e^{-pt}dt, \tag{4.5}$$

where f is of exponential order, and its inversion,

$$f(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} F(p)e^{pt}dp. \tag{4.6}$$

At this point, we shall take a brief detour to discuss the development of the Laplace transform, following Deakin [1981, 1982]. The modern version of the theory of the Laplace transform is correctly attributed to Gustav Doetsch (1892–1977), who presented a systematic treatment in his book [1937]. However, the subject has a long history, going back to Euler, who in 1737, sought solutions of differential equations in the form

$$\int e^{ax}X(x)dx \qquad \text{(no limits of integration)}. \tag{4.7}$$

The Laplace transform is not merely an attempt to make rigorous the calculus of operations, but it has an independent development based on the search for solutions of differential equations in the form of definite integrals of certain types. Euler discussed such methods in a number of papers and in his textbook on integral calculus [1769, Vol. II]. Also, Lagrange (in a paper on the propagation of sound) transformed a partial differential equation (the wave equation) into an ordinary differential equation by a method equivalent to taking Fourier transforms: he applied integration by parts and chose limits of integration which lead to vanishing of integrated terms. This was an anticipation of the technique later used by Laplace and by Petzval. Having done this for the wave equation in one space variable, he repeated it for the case of two space variables later.

It was Pierre-Simon Laplace (1749–1827) who laid the groundwork of the subject. In [1782] he solved a difference equation by looking for solutions of the form

$$\int e^{-sx}\varphi(x)dx \tag{4.8}$$

which, unlike Euler's (4.7), was a definite integral, but whose limits were to be determined. He went into a long discussion on how to obtain the limits, and applied the same method to differential equations. Then he went on to other things, and only returned to the subject much later [1810–11], where he considered integrals of the form

$$\int_0^\infty f(x)e^{-ax}dx. \tag{4.9}$$

He evaluated many of these integrals. Later, he used a version of the Laplace transform to solve a second order partial differential equation. In the process, he got what is essentially the inverse Laplace transform. He returned to these integrals in his work on probability theory [1812]. Although he was anticipated by Euler and by Lagrange, it was Laplace who produced a systematic theory that went far beyond anything the others had produced. The influential calculus of Lacroix [1800, Vol. 3] describes some of Laplace's work on such integrals.

Cauchy made extensive contributions to the mathematics required for the modern version of the Laplace transform, but not to the theory of the transform itself. These include his calculus of residues, (which got his attention away from symbolic methods), his extensive analysis and applications of Fourier transforms, especially their complex form, and studies of symbolic operators.

In the 1820s N. Abel wrote a paper (published in 1839) which was the first attempt at a systematic exploration of the properties of the Laplace transform. But it was basically unknown until 1892. In 1832, Joseph Liouville published three papers on derivatives of fractional order. Along the way, he systematically produced much of the currently known theory of the Laplace transform. This work duplicates and surpasses that of Abel, which was unknown to him, and was very influential on Joseph Petzval (1807–1891).

Inspired by Liouville's papers, Petzval devoted much of the period 1833–1859 to work on integral transforms for solving differential equations. Contrary to the claims of a former student, Simon Spitzer, Petzval's work was done independently of Laplace's. A long controversy ensued, which ended with Boole's use of the term "Laplace's method" in his book on differential equations [1859]. Because of his standing as a mathematician, the term Laplace transform ultimately became generally accepted. Petzval wrote a large book on differential equations [1853, 1859], which was not exclusively concerned with the Laplace transform, but which used it frequently, especially in the first volume. This book was most responsible for bringing the technique to general notice. Its only drawback was that Petzval was not well versed in Cauchy's residue theory.

Petzval's work was the acme of achievement in this subject up to 1860 and beyond; it was not surpassed until Poincaré's work of 1880. Petzval duplicated and then went beyond Laplace and others, and incorporated the contributions of Liouville, and partly those of Cauchy. Among others, he influenced Boole, Mellin, and Bateman.

H. Poincaré submitted a paper for a prize offered in 1880 by the Academy of Science in Paris. This long paper (182 pages) was never published as submitted. Part 2, which was entirely independent of Part 1, was later published, but not the latter (consisting of pages 5–79). Part 1 was concerned with the differential equation

$$\sum_{i=0}^{n} P_i \left(\frac{d}{dx}\right)^i y = 0, \tag{4.10}$$

where the Ps are entire polynomials in x. He used the Laplace transform in the form

$$y = \int v\, e^{zx} dx, \tag{4.11}$$

where v is a function of z, and the contour of integration is to be determined. This is the technique of Petzval, except for the latter's use of real limits of integration. From the manuscript, it seems clear that Poincaré developed his theory of the Laplace transform *ab initio*. He finally succeeded in publishing the contents of Part 1 in [1885], by which time he found out that the Laplace transform was not his invention. With both Boole and Poincaré using Laplace's name for the method, its general acceptance was assured. Also, Poincaré actually introduced the final form of the name: the Laplace transform.

In [1887], Salvatore Pincherle (1853–1936) wanted to instill rigor into operational calculus. For an operator E, acting on analytic functions $\varphi(y), E(\varphi)$ is to be an analytic function of x, which is to be distributive, and to satisfy

$$\frac{d}{dx}E(\varphi) = E(y\varphi), \text{ and } xE(\varphi) = -E\left(\frac{d\varphi}{dy}\right). \tag{4.12}$$

He then showed that

$$f(x) = \int_{(\lambda)} e^{xy}\varphi(y)dy \tag{4.13}$$

satisfies these requirements, for a suitable class on contours λ. Pincherle returned to the Laplace transform frequently over a period of forty years, and played a role in ensuring that Laplace's name was attached to the transform.

In 1898 or 1899, Heaviside worked on a version of the Laplace transform, but he had little direct influence on the theory of Laplace transforms although he did influence Bromwich, Carson, and van der Pol to use the transform. Another mathematician who made much use of Laplace tranforms for solving differential equations was Harry Bateman, who encouraged Carson to use them in studies of electricity. As we shall see, the earliest work connecting Laplace transform theory with operational calculus seems to be due to engineers and mathematicians interested in engineering problems.

T. J. Bromwich (1875–1929) gave [1916] an explanation of operational calculus based on function theory. His was one of the first papers to do so, and it was the most influential. Bromwich's claim was that he had provided the final rigorous proof of Heavisde's operational calculus. His method was to use an integral representation to transform a given normal system of differential equations into a system of algebraic equations (or to transform a partial differential equation into a differential equation of one order lower). The differential operator d/dt was transformed into multiplication by the independent variable. Finally, he found a contour integral to represent the solution of the original differential equation. His method was essentially the same as Poincaré's, but he used a standard contour.

Another significant reformulation of operational calculus appeared in series of papers by John R. Carson whose ideas were collected in a book [1926] on the use of operational calculus in electric circuit theory. His work was independent of that of others, such as Bromwich, who were studying the same problem. His method is closer to a strict integral transform approach and he inverted by solving an integral equation rather than using function theory. His transform was

$$F(p) = p\int_0^\infty f(t)e^{-pt}dt, \tag{4.14}$$

which he solved for the function f.

Operational calculus was still not fully integrated into the mainstream of Laplace transform theory in the mid 1920s. Several mathematicians pointed out the relationship between these subjects at that time, e.g., Paul Lévy (1886–1971) in [1926]. The first to use transform methods consistently as a substitute for, and an explanation of, operational methods was Baltezar van der Pol (1889–1959) in a sequence of papers starting in 1927. In his book co-authored with Bremmer [1950], he used the "two-sided" Laplace transform, i.e., integrated over the entire real line $(-\infty, +\infty)$, instead of the usual half-line $(0, +\infty)$.

A synthesis of much of the modern theory of the Laplace transform took place over the period 1917–1937. This is summed up in the book [1937] of Gustav Doetsch (1892–1977), a very rigorous and clear compilation of the theory of Laplace transforms as of 1937. His definition of Laplace transform was

$$\int_0^\infty f(t)e^{-pt}dt. \tag{4.15}$$

(As noted above, some versions of the transform include a factor of p in front of the integral.)

It may seem ironic that in the same year as Doetsch's book appeared, presenting a rigorous representational version of operational calculus via the Laplace transform, there also appeared a book by Eugene Stephens [1937] which made extensive use of direct methods of operational calculus. He referred the reader to Davis [1936] or Poole [1936] for a full justification of his methods. Applications are given to partial as well as to ordinary differential equations.

For instance, Stephens considers

$$F(d_1, d_2) = \sum_{j,k} a_{jk} d_1^j d_2^k, \text{ where } d_1 = \frac{\partial}{\partial x},\ d_2 = \frac{\partial}{\partial y}, \tag{4.16}$$

and the summation may denote an infinite series. For the equation

$$F(d_1, d_2)z = f(x, y) + 0, \tag{4.17}$$

Stephens wrote the solution as the sum of a particular solution and a complementary function

$$z = \frac{1}{F(d_1, d_2)} f(x, y) + \frac{1}{F(d_1, d_2)} 0. \tag{4.18}$$

The advent of distribution theory in the late 1940s brought with it the hope of another justification for operational methods. In his book on distribution theory, L. Schwartz [1950; also 1957, page 8] stated that he had omitted many explicit calculations, but added

> *La plupart de ces calculs seront explicités dans un court fascicule spécial, qui paraîtra ultérieurement dans cette même collection.*

However, this sequel to his book has never appeared.[4]

Doetsch [1974] wrote a textbook on Laplace transforms which is based on the theory of distributions. He did not want to use the Fourier transform, which requires the class of tempered distributions, so he used instead the class of distributions of finite order, which allowed him to get a criterion for the representability of an analytic function as

[4]Professor Przeworska-Rolewicz has informed the author that Schwartz' reason for this was that the formulas turned out to be too cumbersome.

the Laplace transform of a distribution of finite order. For Laplace transforms based on tempered distributions, the condition is only sufficient, but is not necessary.

There is also an approach to operational calculus using distributions by T. P. G. Liverman [1964] whose purpose is to present operational methods using direct manipulation of differential and integral operators in the manner of Heaviside. Liverman uses the sequential theory of distributions, rather than the standard linear functional theory. Only the first of two proposed volumes has appeared, dealing with ordinary differential equations. The second volume was to have covered direct operational methods for partial differential equations.

In [1926], P. Lévy connected the operational calculus with convolution. If

$$(f * g)(t) = \int_0^t f(t-u)g(u)du, \tag{4.19}$$

Lévy corresponded a convolution operator F to a function f by the rule

$$F(g) = f * g. \tag{4.20}$$

Then to the constant function 1 corresponds the integral operator I, where

$$(Ig)(t) = \int_0^t g(u)du, \tag{4.21}$$

and I^n corresponds to the n-fold convolution of 1 with itself. Then he introduced differential operators I^{-n} as left inverses of the I^n, which caused problems due to the lack of commutativity of I^n and I^{-n}. He formed a ring of such operators, but was unable to handle quotients of operators F/G.

Before the Laplace transform became the standard approach to the representational form of the operational calculus, Norbert Wiener (1894–1964) used the Fourier transform as the basis for this calculus [1926]. Although this paper was frequently referred to, its methods were not much used because of their difficulty.

Wiener began with a historical introduction to the concept of operational calculus and stated his goal to provide a rigorous foundation for Heaviside's operational calculus. He mentioned the recent work of Carson, but pointed out that Carson found solutions for particular cases without providing a general theory. He listed a number of other possible approaches: Volterra's permutable kernels, Pincherle's theory of transformations of power series, the Laplace transform in the work of F. Bernstein and G. Doetsch in 1925, and the Fourier transform. But the Laplace transform was only applicable in the case of analytic functions, while the Fourier transform is not limited in that way. However, the Fourier transform presented its own difficulties, e, g., the severe growth restrictions at infinity of the classical Fourier transform, which, he said, are far more severe than those for Heaviside's methods. This led him to a generalization of the Fourier transform. In turn, this generalization led him further, to his celebrated work on Generalized Harmonic Analysis.

Wiener introduced the class of functions f in L^2_{loc} for which there exist numbers A and k such that $f(x) \leq Ax^k$ for all x with $|x|$ sufficiently large. In general, a function of this type has no Fourier transform, so he had to provide an extension of the Fourier transform. In particular, he wanted to have

$$\mathcal{F}\left[F\left(\frac{d}{dt}\right)f\right](\mu) = F(i\mu)\mathcal{F}f(\mu), \tag{4.22}$$

where $\mathcal{F}$ denotes the Fourier transform, for fairly general functions F. This form of the required relation is essentially that used by Calderón and Zygmund in their work on singular integral operators in the 1950s. But Wiener had no Fourier transform for such functions, so he proceeded differently. Wiener made use of a decomposition of the function f which he had used in previous work, into a sum (or a series) of functions, each with harmonics in only a restricted range.

In his review of Wiener's paper, Laurent Schwartz [1979] put the method into the more general framework of distribution theory. If f is a tempered distribution and $F \in O_M$ (i.e., F is of class C^∞ and, with all of its derivatives, is of slow growth at infinity), then $F\left(\frac{1}{i}\frac{d}{dx}\right)f$ is also a tempered distribution, and

$$\mathcal{F}F\left(\frac{1}{i}\frac{d}{dx}\right)f(\xi) = F(\xi)\mathcal{F}f(\xi), \tag{4.23}$$

or

$$F\left(\frac{1}{i}\frac{d}{dx}\right)f = \mathcal{F}^{-1}F * f. \tag{4.24}$$

Wiener's operators $F\left(\frac{d}{dx}\right)$ required special treatment in case F has poles or branch points on $i\mathbb{R}$, and when its behavior at infinity is complicated.

Wiener ended his paper with examples of solutions of partial differential equations obtained by operational methods. In the process, he introduced for the first time the notion of a weak derivative and weak solutions of partial differential equations. Schwartz found it amusing that it was also consideration of generalized solutions of partial differential equations that led him to introduce distributions. (In a later historical review, he called his search a near obsession.) The equivalent of the Sobolev space $W^{2,1}$ also appears in this paper.

Although he did not mention it this paper, Wiener's approach to symbolic methods follows Cauchy in basing the method on Fourier integrals. The method did not become popular; the book of V. Bush [1929], which deals with operational circuit analysis, contains an appendix by Wiener outlining his approach. This appears to be the only attempt at popularizing Wiener's method of Fourier integrals.

Before we leave Wiener's paper [1926], it should be noted that Schwartz [1979] pointed out that Wiener's work can be viewed as an early, one-dimensional version of the theory of pseudodifferential operators developed by J. J. Kohn, L. Nirenberg, and L. Hörmander in the 1960s. As Saint Raymond [1991, page 29] stated:

> *The key idea* [of the theory of pseudodifferential operators] *is to replace all the computations on the operators with algebraic calculations on their symbols.*

This is certainly a concise description of the representational form of operational calculus.

At about the same time, Hermann Weyl [1927] gave a formal definition of an operator $a(x, D)$ corresponding to a symbol $a(x, \xi)$, i.e., a function of a certain type. He did this via Fourier transformation and the integral in (4.25) is not convergent, but formally is equivalent (according to Kohn and Nirenberg) to

$$Wf(x) = \frac{1}{(2\pi)^n} \iint e^{i\xi\cdot(x-y)} a\left(\frac{x+y}{2}, \xi\right) f(y)\, dy\, d\xi. \tag{4.25}$$

This also appeared in his book on quantum mechanics [1928, English translation, 1932, p. 274], in the form

$$f(p,q) = \int \int_{-\infty}^{+\infty} e\,[i(\sigma p + \tau q)]\, \xi(\sigma,\tau) d\sigma\ d\tau, \tag{4.26}$$

where $e(x) = e^{ix}$. Weyl's paper of 1927 appeared just a year too late to be useful in the development of quantum mechanics. Moreover, mathematicians did not make much use of Weyl's operational calculus until L. Hörmander [1979] worked out the detailed theory, long after the general theory of pseudodifferential operators had been developed. It is interesting to note that the theory of pseudodifferential operators did not arise from rather natural formal procedures using Fourier transforms applied to linear partial differential equations with variable coefficients, but rather from the theory of singular integral operators of Calderón and Zygmund.

5. Return to formal operational calculus. Since many of the authors in this area refer to the work of Vito Volterra (1860–1940), we shall briefly describe his contributions. In [1913b] Volterra studied a composition of functions of the form

$$(f * g)(x,y) = \int_x^y f(x,\xi) g(\xi,y) d\xi, \tag{5.1}$$

which is a generalization of a matrix product, from the discrete to the continuous case. This work was a sequel to his work [1913a] on integral and integro-differential equations. (Actually, the notation was not $f * g$, but $f^* g^*$.)

In his book with Pérès [1924], Volterra continued this work. His goal in introducing the operation of composition was to solve linear integral equations by forming integral powers of composition kernels of integral equations and showing that these powers are functions permutable among them. He viewed permutable functions and composition as natural extensions to functions of the notions of product of matrices or substitutions.

If matrices M and N satisfy $MN = NM$, he called them *permutable*. The passage to the continuous from the discrete is achieved by replacing integer-valued indices by continuous variables, and sums with respect to indices by integrals with respect to continuous variables. Thus

$$\sum_k a_{ik} b_{kj} \to \int f(x,\xi) g(\xi,y) d\xi.$$

Composition of functions is of two types: the *first type* is that of (5.1) with variable limits of integration, and the *second type* is given by the same integral but with fixed limits. *Permutability* means

$$\int_x^y f(x,\xi) g(\xi,y) d\xi = \int_x^y g(x,\xi) f(\xi,y) d\xi. \tag{5.2}$$

Thus $f * g$ is the resultant of the composition of f and g, and it is associative and distributive, but not commutative in general. The functions are *permutable* if their resultant is commutative.

A simple family of permutable functions consists of those permutable with constants (which may be taken to be identically one).

$$\int_x^y f(x,\xi)d\xi = \int_x^y f(\xi,y)d\xi = \varphi(x,y). \tag{5.3}$$

Volterra and Pérès show that φ depends on a single variable $x-y$. These are called "functions of the closed cycle", and their composition reduces to convolution:

$$\int_0^t f(t-s)g(s)ds. \tag{5.4}$$

Bourbaki [1994] criticizes Volterra's algebraic formalism for not bringing out the connection with the group structure of the real line or of the Fourier transform. However, neither of these seem to be particularly relevant to Volterra's purposes.

The most satisfactory direct version of the operational calculus seems to be that of Jan G. Mikusiński (1913–1987).[5] His first work [1944], written under harsh wartime conditions, was called *Hypernumbers, Part I Algebra.* Only 7 copies of this work, in Polish, were produced. A *hypernumber* (α, f) is an ordered pair consisting of a complex number α and an element f of a complex linear algebra W. Addition of these numbers is defined in the natural way, and Mikusiński wrote $(\alpha, f) = \alpha + f$. A non-commutative multiplication is defined by

$$(\alpha, f)(\beta, g) = (\alpha\beta,\ \beta f + \alpha g + fg). \tag{5.5}$$

The resulting set of hypernumbers is denoted by [W]. His idea was to apply hypernumbers to the theory of integral equations and differential equations by reducing certain analytic problems to purely algebraic ones. Then applications to constant coefficient linear differential equations have an appearance almost identical to Heaviside's calculus. However, unlike the latter, the new version applies to non-zero initial conditions. Mikusiński stated that hypernumbers provide a new algebraic basis for that calculus, one that is conceptually simpler than the Laplace transform. He also considered the permutable functions of Volterra and Pérès. The first part of the work, algebra, presents the theory and certain applications depending on the four elementary arithmetic operations, while the second was to introduce the sum of infinite series of hypernumbers and applications to Volterra integral equations.

For his first example, Mikusiński took W to be the space D of all complex valued functions which are continuous on a given interval $[a, b)$ of the real line, where b may be ∞. Addition in D is the usual one, but multiplication is convolution

$$f(t) * g(t) = \int_a^t f(a + t - \tau)g(\tau)d\tau. \tag{5.6}$$

[5]Since Professor Skórnik presented an excellent sketch of Mikusiński's method, less space will be devoted to it here than its importance requires.

(Mikusiński actually wrote $\{f(t)\}$ for vectors in D.) Then D is a commutative algebra; applications of hypernumbers of $[D]$ to constant coefficient linear differential equations are given. Mikusiński gave another example which had applications to Fredholm integral equations.

After presenting a brief sketch of a sequential theory of distributions, which was carried out later (by G. Temple, by J. Korevaar, and by Mikusiński and R. Sikorski), Mikusiński returned to his operational calculus, but now making use of Titchmarsh' theorem on convolutions, which made the ring structure he used into an integral domain [1947, 1949]. This allowed him to embed the integral domain into a quotient field, and to carry out operations in this field, now known as that of Mikusiński operators. Reviewing the paper of 1949 in Mathematical Reviews in 1951, L. Schwartz said

"This theory gives satisfying formulae for symbolic calculus."

Mikusiński's book [1953] on his operators was well received, and was translated into several languages and has gone into several editions.

Other direct approaches were also being constructed at that time. For example, J. Dalton [1954] had the aim of presenting a elementary but sound presentation of operational calculus. He said that Heaviside's theory failed because for two basic reasons. First, because of an imperfect comprehension of the distinction between $\frac{d}{dx}$ and $(\int_0^t \cdots d\vartheta)^{-1}$, and second because of an unfortunate choice of primary operator on which the whole theory is based. As for the Laplace transform, this is an "alternative" and "essentially different process" which neither explains Heaviside's techniques nor establishes their validity. Dalton's new idea was that the basic operation should be integration over a bounded time-interval. In contrast, the Laplace transform starts with integration over an unbounded time interval, and so is forced to restrict the growth of the functions it is able to handle. This is foreign to Heaviside's method. Dalton's approach does not seem to be in use today.

6. Some further types of algebraic analysis. There are many other examples of studies that could very reasonably be called algebraic analysis. One of these is an approach to distribution theory which essentially re-echoes the old Euler-Lagrange view of algebraic analysis as the study of basic operations via power series. It is sometimes called the axiomatic theory of distributions and was developed, mostly independently, by H. König, J. Sebastião e Silva, and R. Sikorski. At bottom, the basis of their method is the idea of representing a "generalized" function as a generalized derivative of a continuous function. In turn, this idea has its roots in Bochner's work [1932] on the Fourier integral.

In his thesis, König [1953] worked with formal power series

$$\tau = \sum_s f_s z^s, \tag{6.1}$$

where locally, only a finite number of terms are non-zero. The $s = (s_1, s_2, \ldots, s_n)$ are multi-indices and the f_s are functions on an open set $\Omega \subseteq \mathbb{R}^n$. The set of such series forms a vector space B with respect to the natural operations. If e_i denotes the multi-index with 1 in the i^{th} place and zeros elsewhere, a corresponding derivation is introduced in B by

$$\frac{\partial \tau}{\partial x_i} = \sum_s f_s z^{s+e_i}, \tag{6.2}$$

and all of these new series are elements of B. König defines a subspace U of B which is sequentially closed and is closed with respect to all of the derivations (6.2). Finally, he forms the quotient space $F = B/U$, and defines derivations and convergence in F. Distributions are then identified with elements of a subspace L of F which are images in F of elements of B of the form (6.1) with f_s locally integrable on Ω. (L is isomorphic to the space of Schwartz distributions on Ω.) There are derivations and a notion of convergence on L, but convolution, Fourier transforms, and Laplace transforms are missing.

König's theory is presented in more detail in the second edition of Sauer's book [1958] on partial differential equations, where it is applied to generalized solutions of partial differential equations.

The axiomatic versions of Sikorski and of Silva are very similar, so we will give a sketch only of the latter. Sikorski's approach appeared in a 3-page note [1954] and in Vol. I of his book on Real Analysis [1958, pp 425–437].

The most extensive presentation of the axiomatic approach is that of Silva [1954–55]. Silva's early work was a re-working of the operational calculus of L. Fantappiè in the framework of functional analysis. Fantappiè's ideas on analytic functionals can be traced back to Volterra and Pincherle. Silva notes that in Vol. I of Schwartz' book on distributions [1950], "Bochner distributions" are defined, basically as derivatives of continuous functions which don't necessarily have derivatives in the usual sense. (Theorem XXI of Chapter III.) But Schwartz says that it is preferable to have this property as a theorem rather than a definition, because of the non-determination of the order of differentiation of the continuous function. In the work of Silva and of Sikorski, this is easily overcome by an algebraic method analogous to the construction of the rational numbers from the integers (which is also the basis of the construction of Mikusiński's operators.) Silva mentions König's paper [1953] as an example of a direct, purely formal construction of the theory of distributions, but that his own ideas were independent of König's. However, the latter's work did influence him after he became aware of it. Silva criticized König's work as decisive but not definitive, because it does not give a true axiomatic version of distributions, i.e., a categorical system of axioms. In particular, König shows that his system is algebraically isomorphic to the space of distributions, and that his sequential convergence is equivalent to that of Schwartz, but the topological aspects of this correspondence is not considered.

To answer Schwartz' objections to the Bochner method, Silva says that suitable definition of equivalence of formal derivatives is necessary. Thus, if D_i, D_j, $i \neq j$, are two symbols of differentiation, and f, g, are continuous functions, it is possible to choose functions F and G such that

$$D_i f = D_i D_j F, \qquad D_j g = D_j D_i G. \tag{6.3}$$

He follows by analogy the construction of the rational number system, where one uses the fact that two rational numbers can always be represented by fractions with a common denominator. It is notable that no topological structure of the space of distributions is used in the axiomatization; only algebraic constructions are used. However, Silva's paper is long and intricate, due to a very general and abstract introduction of the algebraic projective limit of groups in order to define the order of a distribution (finite or infinite).

In this, Silva followed a suggestion of Schwartz to use the Patching Principle to obtain distributions of infinite order from those of finite order.

In generalizing his approach from scalar-valued to vector-valued distributions, Silva [1960] used more direct methods. He acknowledged that his previous formalism was too abstract and that it masked the elementary and easy character of this construction. A bit later, Silva [1962, 1964] simplified his presentation by restricting attention to distributions of finite order, since those of infinite order are mostly of theoretical rather than practical interest. He repeated [1962] the idea that the passage from the notion of function to that of distribution is of the same character as the successive extensions of numbers, from integers to rationals to reals to complex numbers. To emphasize his point, he quoted Schwartz to the effect: in Analysis, who uses the fact that a real number is a cut or a class of sequences? The formal properties of operations and order relations are entirely sufficient for users of the theory. The method used is elegant from the mathematical viewpoint because it solves an algebraic problem by strictly algebraic means. (Topology plays an artificial role here.) Moreover, a considerable part of the theory of distributions, including Laplace transforms, Fourier series, and Fourier transforms, and their practical applications, may be described more simply without topology.

For simplicity, consider the one-variable case. Let $I \subseteq \mathbb{R}$ be an interval and $C = C(I)$ denote the set of complex-valued continuous functions on I. For an arbitrary point c of I, let

$$\mathfrak{I} f(x) = \int_c^x f, \quad \text{for } f \in C. \tag{6.4}$$

Primitive notions here are continuous function and derivative. The axioms are:

Axiom 1. Every complex-valued function f, defined and continuous on I, is a distribution on I.

Axiom 2. To each distribution T on I corresponds a distribution DT on I, called the *derivative* of T, such that if T is a function with a continuous derivative in the usual sense, then DT coincides with this derivative.

The derivative of order r of a distribution T is defined by induction.

Axiom 3. For each distribution T on I, there is a non-negative integer r and a function f in C such that $T = D^r f$.

Axiom 4. If r is a non-negative integer and $f, g \in C$ satisfy $D^r f = D^r g$, then $f - g$ is a polynomial of degree less than r.

(Denote the set of all polynomials of degree $< r$ by P_r.)

This system of axioms is shown to be consistent and categorical. Silva's model (the same as Sikorski's) is the collection of ordered pairs (r, f), where r is a non-negative integer, and $f \in C$, with equivalence of two such pairs (r, f) and (s, g) defined by $(r, f) \sim (s, g)$ if and only if there is some natural number $m \geq r, s$, such that

$$\mathfrak{I}^{m-r} f - \mathfrak{I}^{m-s} g \in P_m. \tag{6.5}$$

This is an equivalence relation and the equivalence class of (r, f) is denoted by $[r, f]$. The set of all equivalence classes is denoted by $\tilde{C}$. The mapping $f \mapsto [0, f]$ of C to $\tilde{C}$ is $1-1$, so f can be identified with $[0, f]$. A derivation is defined on $\tilde{C}$ by

$$D[r, f] = [r+1, f]. \tag{6.6}$$

Then if f has a usual derivative on I,

$$Df = D[0, f] = [1, f] = [0, f'] = f', \tag{6.7}$$

since $f - \mathfrak{I}f' =$ constant, is in P_1. Also,

$$[r, f] = D[r-1, f] = \cdots = D^r[0, f] = D^r f, \tag{6.8}$$

so Axiom 3 is satisfied. Finally,

$$D^r f = D^r g \Rightarrow [r, f] = [r, g] \Rightarrow f - g \in P_r. \tag{6.9}$$

The extension to higher dimensions is simple.

In [1964], Silva presented a more extended version of this theory, which can be thought of as working with the coefficients of a (generalized) power series, without using the power series itself. He also presented there a direct, elementary theory of integration of distributions, from which he obtained a general theory of convolutions and Fourier transforms for tempered distributions. In 1966, Silva was writing a book about his algebraic version of distribution theory, but he died before completing it. Recently, Campos Ferreira [1997] published a textbook on distribution theory using Silva's approach.

As for interest in presenting distributions as generalized derivatives of continuous functions, this method is used by David Kammler in a forthcoming textbook on Fourier Analysis for undergraduates in mathematics and engineering. Distributions are introduced early in this book and are used often.

Poincaré stated that the study of the history of mathematics is not only interesting in itself, but also serves to point the way to future research. With that in mind, we end this survey with a glance at some recent work.

In the algebraic analysis view of distribution theory sketched above, the axiom system is intended to be categorical. Our final example is an algebraic approach to a much wider class of generalized functions than Schwartz distributions, which is not meant to be categorical. This is the theory of E. Rosinger [1990].

Rosinger is interested in generalized solutions of nonlinear partial differential equations. In particular, he believes the most relevant and useful approach to this subject is to first consider the algebraic problems involved in the interaction between differentiation, discontinuity, and multiplication.

A simple example shows the conflicts which arise when dealing with this trio. Start with the Heaviside function $H : \mathbb{R} \to \mathbb{R}$, defined by

$$H(x) = \begin{cases} 0, & x \leq 0 \\ 1, & x > 0. \end{cases} \tag{6.10}$$

When dealing with generalized solutions of nonlinear partial differential equations, the discontinuous function H may be involved with differentiation and multiplication. So consider a ring A of real-valued functions on $\mathbb{R}$, with a derivative operator $D : A \to A$ (so that D is linear and satisfies Leibniz' rule). Then we have some inconvenient consequences, as follows. Since A is associative and commutative,

$$H^m = H \quad \forall\, m \in \mathbb{N}. \tag{6.11}$$

Then

$$mH(DH) = DH,\ \forall\, m \geq 2, \tag{6.12}$$

so that for $p, q \geq 2$ and $p \neq q$,

$$\frac{1}{p}(DH) = \frac{1}{q}(DH) = H(DH), \tag{6.13}$$

$$\left(\frac{1}{p} - \frac{1}{q}\right)(DH) = 0 \in A. \tag{6.14}$$

Thus

$$DH = 0 \in A. \tag{6.15}$$

Now for good reasons, we expect to have

$$DH = \delta, \tag{6.16}$$

which would give $\delta = 0 \in A$, which is not acceptable.

The way out of this dilemma is to relax some of the assumptions; however we do wish to keep (6.16). This leaves a choice for the algebra A and the derivative operator D. First, A need not be an algebra of functions, but may contain more general objects, and multiplication in A need not be closely related to multiplication of functions. In particular, we need not have (6.13). Also, the assumption that $D : A \to A$ implies that $D^m a$ exists $\forall\ a \in A$ and all $m \in \mathbb{N}$. A sufficient condition for this is $A \subseteq C^\infty(\mathbb{R}^n)$. However, $H \in A \backslash C^\infty(\mathbb{R}^n)$. Thus we may wish to have $D : A \to \overline{A}$, where $\overline{A}$ is another algebra of generalized functions. Since we want D to be a derivation, we assume that there is an algebra homomorphism $A \to \overline{A}\ a \mapsto \overline{a}$, such that

$$D(f \cdot g) = (Df) \cdot \overline{g} + \overline{f} \cdot (Dg), \quad f, g \in A. \tag{6.17}$$

By means of examples of generalized solutions of nonlinear partial differential equations, Rosinger shows how the relations on A and on D yield desire results.

The argument above that (6.17) holds doe not use any calculus; it is purely algebraic, using only the algebraic structure of A and the fact that D is a derivation.

Schwartz' linear theory of distributions does not allow within itself for unrestricted use of nonlinear operators, and in particular, multiplication, so we should try to embed distributions into a larger class of generalized functions. The standard extension of $C^m(\Omega)$ is an embedding into a suitable topological vector space E of generalized functions, obtained solely on the basis of approximation. That is, each generalized function $T \in E \backslash C^m(\Omega)$ is assumed to be a limit of classical functions φ_k in $C^m(\Omega)$. This leads to the assumption that topology alone gives the extension E. But Rosinger and J. Colombeau have shown that it is useful to avoid such early and exclusive stress on the approximation interpretation, which involves topology first or even topology alone. Rosinger proposes the alternative of algebra first.

Rosinger's claim is that the ring-theoretic type of algebra involved here belongs to a more fundamental kind of mathematics than the usual calculus, functional analysis, or topology methods customary in the study of partial differential equations.

7. Conclusion. From the earliest times, the term algebraic analysis has meant the study of the processes of analysis by means of power series. Leibniz, John Bernoulli, Euler, and Lagrange viewed analysis as a method of applying algebra to the solution of various problems; for them, power series were entirely algebraic. Lagrange carried this furthest, trying to base analysis on algebra, i.e., power series. It has continued throughout the 19^{th} century and even into the 20^{th}. Thus a book by Rey Pastor, Calleja, and Trejo [1960] on Mathematical Analysis is in three volumes, the first of which has the title "Análisis algebrico - Teoria de equaciones, Cálculo infinitesimal de una variable". In addition, there are subjects, such as differential algebra, which could properly be called algebraic analysis. It may be stretching things a bit to include the axiomatic theory of distributions of König, of Sikorski, and of Sebastião e Silva with power series methods, but the idea may be thought of as working directly with the coefficients of power series; in the case of König, this is done explicitly.

But our concern here has been with algebraic analysis as the study of symbolic methods in analysis, both direct and representational. This is an idea very different from power series as algebra. The beginnings of this study go back to Arbogast and to some of his successors in France, most of whom are lesser-known mathematicians. Moreover, Fourier and Cauchy made use of symbolic methods, but for limited purposes. Eventually, they dropped these studies. Also they made no direct connection between symbolic methods and algebra itself.

It was the English mathematicians who, inspired by the work of the French, pushed symbolic methods much further (in some cases further than could be justified). Some of them, e.g., Gregory and Murphy, produced a mixture of interesting results and occasional nonsense. The most influential members of this group, De Morgan and Boole, helped to popularize it, but also stressed the need for care in its use. Unlike the Continental work on symbolic methods, the English work was intimately connected with the evolution of abstract methods in algebra. In particular, Boole [1844, page 282] said

> *The position which I am most anxious to establish is, that any great advance in the higher analysis must be sought for by an increased attention to the laws of the combinations of symbols.*

The idea of the primacy of the laws of combination of symbols over the exact nature of the symbols themselves, led to the development of abstract methods in algebra.

After Boole, interest in symbolic methods waned until Heaviside revived the method of direct manipulation of operators. This was due to his application of such methods to problems of electrical engineering. Eventually, mainly through the work of van der Pol, the Laplace transform became the standard indirect method of justifying symbolic methods.

An interesting exception to the use of the Laplace transform as the basis for operational calculus (as it was now known), was Wiener's idea of basing it on a generalized form of the Fourier transform. Although the method was too cumbersome to be useful to electrical engineers, it has, in hindsight, other interesting points. His method can be viewed as a precursor of the theory of pseudodifferential operators, developed in the mid-1960s. This is especially strange, when coupled with work of Weyl on quantum mechanics, which appeared at the same time. Weyl's work developed an early version of pseudodifferential operators, but it likewise did not stimulate such studies.

Volterra's work on permutable operators stimulated interest in convolution integrals, and was the source for more direct, algebraic approaches to symbolic methods. This was begun be Lévy and brought to complete fruition by Mikusiński.

There were also several attempts to use Schwartz' distributions as a basis for symbolic methods, but despite some limited success, e.g., Doetsch's use of distributions of finite order to characterize Laplace transforms, such methods have been abandoned.

Algebraic analysis, defined as the study of right-invertible operators, falls into the category of direct manipulation of operators. However, it is a much more precise and more rigorous theory, having the benefit of the extensive and well-developed machinery of linear algebra available as a framework. Much of the work in this field is due to D. Przeworska-Rolewicz, who began her studies of algebraic analysis in the early 1960s.

References for algebraic analysis

L.-F. A. Arbogast [1789], *Essai sur de nouveaux principes de Calcul différentiel et intégral indépendans de la théorie des infiniment-petits et celle de limites*, unpublished manuscript.

— [1800], *Du calcul des dérivations*, Strasbourg, 404 pp.

C. Babbage [1827], *On the influence of signs in mathematical reasoning*, Trans. Cambridge Phil. Soc. Vol. II, Part II, 325–377.

S. Banach [1939], *Über das "loi suprème" von Hoene-Wroński*, Bull. Acad. Polon. Sci. et Lettres Sér. A 1–10.

F. A. Berezin [1987], *Introduction to Superanalysis*, D. Reidel, 424 pp.

R. Bittner [1964], *Algebraic and analytic properties of solutions of abstract differential equations*, Rozprawy Matem. 41, 63 pp.

J.-E. Björk [1979], *Rings of Differential Operators*, North-Holland, 374 pp.

S. Bochner [1932], *Vorlesungen über Fouriersche Integrale*, Leipzig, 229 pp.

G. Boole [1844], *On a general method in analysis*, Phil. Trans. Roy. Soc. London 134, 225–282.

— [1847], *On a certain symbolical equation*, Cambridge & Dublin Math. J. 2, 7–12.

— [1859], *A Treatise on Differential Equations*, Macmillan, London, 485 pp. (1877, 566 pp.)

— [1860], *A Treatise on the Calculus of Finite Differences*, Macmillan, London, 420 pp., 3rd ed., 1880, 336 pp.

U. Bottazzini [1986], *The Higher Calculus*, Springer, 332 pp.

N. Bourbaki [1994], *Elements of the History of Mathematics*, Springer, 301 pp.

T. J. I'A. Bromwich [1916], *Normal coordinates in dynamical systems*, Proc. London Math. Soc. 15, 401–448.

F. Brackx R. Delanghe & F. Sommen [1982], *Clifford Analysis*, Pitman, 308 pp.

B. Brisson [1808], *Mémoire sur l'intégration des équations différentielles partielles*, J. de l'École Polytech. 7, 191–261.

V. Bush [1929], *Operational Circuit Analysis*, John Wiley, 392 pp.

J. Campos Ferreira, [1997], *Introduction to the Theory of Distributions*, Longman, 157 pp.

A. Capelli & Garbieri [1894], *Corso di analisi algebrica*, Napoli.

A. Capelli [1909], *Istituzioni di analisi algebrica*, Napoli, 953 pp.

C. Carathéodory [1963], *Algebraic Theory of Measure and Integration*, Chelsea, 378 pp. (Translation of 1956 German edition.)

R. D. Carmichael [1855], *A Treatise on the Calculus of Operations: Designed to Facilitate the Processes of the Differential and Integral Calculus and the Calculus of Finite Differences*, London, 170 pp.

J. R. Carson [1926], *Circuit Theory and the Operational Calculus*, McGraw-Hill, 197 pp.

A. L. Cauchy [1821], *Cours d'analyse. 1re Partie. Analyse algébrique*, Oeuvres Ser. 2, Vol. 2, 476 pp.

— [1827], *Exercices de mathématiques, Seconde année*, Oeuvres, Ser. 2, Vol. 7.

E. Cesàro [1904], *Elementares Lehrbuch der Algebraischen Analysis und der Infinitesimal Rechnung*, Leipzig, 894 pp. Translation and revision of *Corso di Analysi Algebrica*, Torino, 1894, 499 pp.

G. Chrystal G. [1904 ?], *Algebra. An Elementary Textbook*, 6th ed. Part II, 616 pp.

R. M. Cohn [1965], *Difference Algebra*, Interscience Press, 335 pp.

J. L. B. Cooper [1952], *Heaviside and the operational calculus*, The Math. Gazette 36, 5–19.

J. P. Dalton [1954], *Symbolic Operators*, Johannesburg, 194 pp.

H. T. Davis [1936], *The Theory of Linear Operators*, Bloomington, IN, 628 pp.

M. A. B. Deakin [1981, 1982], *The development of the Laplace transform*, Arch. Hist. Exact Sci. I. *Euler to Spitzer*, *1737–1880*, 25, 343–390; II. *Poincaré to Doetsch*, *1800–1937*, 26, 351–381.

A. De Morgan [1835], *The Elements of Algebra. Preliminary to the Differential Calculus and Fit for the Higher Classes of Schools in Which the Principles of Arithmetic are Taught*, London, 248 pp.

— [1842], *The Differential and Integral Calculus*, London, 785 pp.

— [1849], *Trigonometry and Double Algebra*, London, 167 pp.

J. Dianni [1977], *The history of Jan Śniadecki's book "The theory of the algebraic calculus applied to curved lines" in the light of his correspondence* (in Polish), Kwart. Hist. Nauki i Tech. 22, No. 1, 59–71 [MR 58 (1979) #10079].

G. Doetsch [1937], *Theorie und Anwendung der Laplace-Transformation*, Springer, 436 pp.

— [1974], *Introduction to the Theory and Application of the Laplace Transform*, Springer, 326 pp.

L. Euler [1748], *Introductio in Analysis Infinitorum*, Vol. I, 392 pp.; Vol. II, 402 pp.

— [1769], *Institutiones calculi integralis*, Vol. II, 676 pp. Opera Ser. I, Vol. XIII.

J.-B. Fourier [1822], *Théorie Analytique de la Chaleur*, Paris, 561 pp.

J.-F. Français [1812–13], *Mémoire tendant à démontrer la légitimité de la séparation des échelles des fonctions*, Annales de Math. 3, 244–273.

C. Fraser [1987], *Joseph Louis Lagrange's algebraic version of the calculus*, Historia Math. 14, 38–53.

J. Grabiner [1981], *The Origin of Cauchy's Rigorous Calculus*, M.I.T. Press, 252 pp.

— [1990], *The Calculus as Algebra: J.-L. Lagrange, 1736–1813*, Garland Publ. Co., Boston, 247 pp.

— [1997], *Was Newton's calculus a dead end? The continental influence of Maclaurin's Treatise of Fluxions*, Amer. Math. Monthly 104, 393–410.

I. Grattan-Guinness [1994], *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences*, Routledge, Vol. I, 842 pp.

D. F. GREGORY [1841], *Examples of the Processes of the Differential and Integral Calculus*, Cambridge, 2nd ed., 1846, 529 pp.

K. GÜRLEBECK and W. SPRÖSSIG [1997], *Quaternionic and Clifford Calculus for Physicists and Engineers*, Wiley, 371 pp.

H. HANKEL [1867], *Theorie der Complexen Zahlensystemes*, Leipzig, 196 pp.

O. D. HEAVISIDE [1893, 1899, 1912], *Electromagnetic Theory*, London, Vol. I, 466 pp.; Vol. II, 542 pp.; Vol. III, 519 pp.

J. M. HOENE-WROŃSKI [1811], *Introduction à la Philosophie des Mathématiques*, Paris, 270 pp.

— [1812], *Réfutation de la Théorie des Fonctions Analytiques de Lagrange*, Paris, 136 pp.

L. HÖRMANDER [1979], *The Weyl calculus of pseudo-differential operators*, Comm. Pure Appl. Math. 32, 359–443.

H. N. JAHNKE [1993], *Algebraic analysis in Germany: 1780–1840*, Historia Math. 20, 265–284.

J. H. JELLETT [1858], *An Elementary Treatise on the Calculus of Variations*, Dublin, 377 pp.

I. KAPLANSKY [1957], *Introduction to Differential Algebra*, Hermann, 62 pp., 2nd ed., 1976.

M. KASHIWARA & T. KAWAI (eds.) [1988], *Algebraic Analysis: Papers Dedicated to Professor Miko Sato on the Occasion of His Sixtieth Birthday*, Academic Press, Vol. I, 472 pp.; Vol. II, pp. 473–950.

M. KASHIWARA, T. KAWAI & T. KIMURA [1986], *Foundations of Algebraic Analysis*, Princeton University Press, 254 pp.

M. KASHIWARA & P. SCHAPIRA [1990], *Sheaves on Manifolds*, Springer, 512 pp.

E. R. KOLCHIN [1973], *Differential Algebra and Algebraic Groups*, Academic Press, 446 pp.

— [1985], *Differential Algebraic Groups*, Academic Press, 271 pp.

H. KÖNIG [1953], *Neue Begrundung der Theorie der "Distributionen" von L. Schwartz*, Math. Nachr. 9, 129–148.

E. KOPPELMAN [1971–72], *The calculus of operations and the rise of abstract algebra*, Arch. Hist. Exact Sci. 8, 155–242.

S.-F. LACROIX [1797–1800], *Traité du calcul différentiel et du calcul intégral*, 3 Vol., 2nd ed., 1810-1814-1819, Courcier, Paris, Vol. I, 653 pp.; Vol. II, 819 pp.; Vol. III, 776 pp.

— [1802], *Traité élémentaire du calcul différentiel et du calcul intégral*, Duprat, Paris.

— [1816], *An Elementary Treatise on the Differential and Integral Calculus*, Cambridge. Translation of 1802 edition, 720 pp.

J.-L. LAGRANGE [1772], *Sur une nouvelle espèce de calcul relatif à la différentiation et à l'intégration des quantités variables*, Nouvelles mémoires de l'acad. de Berlin, 185–221; Oeuvres, Vol. 3, pp. 439–476.

— [1797], *Théorie des fonctions analytiques*, Duprat, 276 pp.; 2nd ed., 1813, Oeuvres, Vol. 9, 427 pp.

— , [1799], *Leçons sur le calcul des fonctions*, 2nd ed., 1806, 501 pp.; Oeuvres, Vol. 10.

J. LANDEN [1764], *The Residual Analysis, a New Branch of the Algebraic Art*, London, 218 pp.

P. S. LAPLACE [1776], *Mémoire sur l'inclinassion moyenne des orbites comètes sur la figure de la terre et sur les fonctions*, Oeuvres, Vol. 8, 314–321.

— [1782], *Mémoire sur les approximations des formules qui sont fonctions de très grands nombres*, Mém. Acad. Roy. Sci. (Paris), 1–88, Oeuvres 10, 209–229.

— [1810-11], *Mémoire sur les intégrales définies et leur application aux probabilités*, Mém. Acad. Roy. Sci. (Paris) 9, 1-55; Oeuvres 10, 357–412.

— [1812], *Théorie analytique des probabilités*, 3rd ed., 1820, 774 pp.

F. B. A. LEMBERT [1815], *Handbuch der algebraischen Analysis*, Köln.

P. Lévy [1926], *Le calcul symbolique d'Heaviside*, Bull. Sci. Math. 2, 174–192. Also published separately, Gauthier-Villars, 1926.

J. Liouville [1832], *Mémoire sur quelques questions de géométrie et de mécanique, et sur un nouveau genre de calcul pour résoudre ces questions*, J. de L'École Polytech. 13, 1–69.

T. P. G. Liverman [1964], *Generalized Functions and Direct Operational Methods*, Prentice-Hall, Vol. I, Non-Analytic Functions in One Dimension, 338 pp.

J. Lützen [1979], *Heaviside's operational calculus and the attempts to rigorise it*, Arch. Hist. Exact Sci. 21, 161–200.

C. Maclaurin [1742], *A Treatise of Fluxions in Two Books*, Edinburgh, Vol. I, 416 pp.; Vol. II, 388 pp.

P. Mansion [1898], *Mélanges mathématiques*, Gauthier-Villars, 160 pp. + 56 pp.

J. G. Mikusiński [1944], *Hypernumbers*, Kraków, 24 pp.

— [1947], *L'anneau algébrique et ses applications dans l'analyse fonctionnelle*, Ann. Univ. Marie Curie-Skłodowska Sect. A 2, 1–48.

— [1949], *Sur les fondements du calcul opératoire*, Studia Math. 2, 41–70.

— [1953], *Rachunek operatorów*, PWN, 369 pp.

I. Newton [1669], *De Analysi per Aequationes Numero Terminorum Infinitas*, published in 1711.

L. Novy [1973], *Origins of Modern Algebra*, Wolters-Noordhoff, 252 pp.

G. Peacock [1830], *A Treatise on Algebra*, Cambridge; 2nd ed., Vol. I, *Arithmetical Algebra*, 1842, 399 pp.; Vol. II, *On Symbolical Algebra and its Application to the Geometry of Position*, 1846, 355 pp.

— [1834], *Report on the recent progress and present state of certain branches of analysis*, from: Report of the British Association for the Advancement of Science for 1833, London, 185–352.

S. S. Petrova [1987], *Heaviside and the development of the symbolic calculus*, Arch. Hist. Exact Sci. 37, 1–23.

J. Petzval [1853, 1859], *Integration der linearen Differentialgleichungen mit Constanten Coefficienten*, Braumiller, Vienna, Vol. I, 417 pp.; Vol. II, 687 pp.

H. T. H. Piaggio [1952], *An Elementary Treatise on Differential Equations and Their Applications*, rev. ed., 256 pp. (1st ed., 1920).

S. Pincherle [1887], *Della trasformazione di Laplace e di alcune sue applicazioni*, Mem. Accad. Bolgna 4, 125–143.

H. Poincaré [1885], *Sur les équations linéaires aux différentielles ordinaires et aux différences finies*, Amer. J. Math. 7, 1–56; Oeuvres 1, 225–289.

B. Van der Pol & H. Bremmer [1950], *Operational Calculus Based on the Two-Sided Laplace Integral*, Cambridge University Press, 2nd ed., 1955, 415 pp.

E. G. Poole [1936], *Introduction to the Theory of Linear Differential Equations*, Clarendon Press, 202 pp.

D. Przeworska-Rolewicz [1988], *Algebraic Analysis*, D. Reidel & Polish Sci. Publ., 624 pp.

— [1997], *Algebraic Analysis*, M. Hazewinkel (ed.) *Encyclopaedia of Mathematics. Supplement I*, Kluwer, 42–43.

J. Rey Pastor, P. Calleja & C. A. Trejo [1960], *Análisis Matemático*, Vol. I, Análisis algebraico – Teoria de equaciones, Cálcolo infinitesimal de una variable. 5th ed., 836 pp.; Vol. II, Cálcolo infinitesimal de varias variables – Applicaciones, 3rd ed., 1959, 624 pp.; Vol. III, Análisis funcional y applicaciones, 1959, 718 pp.

B. Riemann [1953], *Versuch einer allgemeinen Auffassung der Integration und Differentiation*, Werke, 2nd ed., 353–366, Dover reprint.

J. F. RITT [1932], *Differential Equations From the Algebraic Standpoint*, Amer. Math. Soc., 172 pp.

— [1950], *Differential Algebra*, Amer. Math. Soc., 184 pp.

E. E. ROSINGER [1990], *Nonlinear Partial Differential Equations. An Algebraic View of Generalized Solutions*, North-Holland, 380 pp.

J. RYAN (Ed.) [1996], *Clifford Algebras in Analysis and Related Topics*, CRC Press, 386 pp.

X. SAINT-RAYMOND [1991], *Elementary Introduction to the Theory of Pseudodifferential Operators*, CRC Press, 108 pp.

R. SAUER [1958], *Anfangswertprobleme bei Partiellen Differentialgleichungen*, Springer, 2nd ed, 284 pp.

O. X. SCHLÖMILCH [1845], *Handbuch der Algebraischen Analysis*.

L. SCHWARTZ [1950], *Théorie des distributions*, Hermann, Tome I, 2nd ed., 1957, 162 pp.

— [1979], *Comments on [26c]*, pp. 425–427 of Wiener [1979].

J. SEBASTIÃO E SILVA [1954–55], *Sur une construction axiomatique de la théorie des distributions*, Rev. Fac. Ciências Lisbao 4, 79–186.

— [1960], *Sur la définition et la structure des distributions vectorielles*, Portugaliae Math. 19, 1–80.

— [1962], *Sur l'axiomatique des distributions et ses possibles modèles*, C. I. M. E., Rome, 15 pp.

— [1964], *Integrals and orders of growth of distributions*, in: Theory of Distributions. Proc. Intern. Instit. Lisbon, 1964, Lisbao, 327–390.

F.-J. SERVOIS [1841-15], *Essai sur un nouveau mode d'exposition des principes du calcul différentiel*, Ann. Math. Pures Appl. 5, 93–170. Also published separately under the same title, Nismes, 1814.

R. SIKORSKI [1954], *A definition of the notion of distribution*, Bull. Acad. Polon. Sci. Cl. III 2, 209–211.

— [1958], *Funkcje Rzeczywiste* (Real Functions), PWN, Vol. I, 534 pp.

J. ŚNIADECKI [1783], *Rachunku algebraicznego teoria przystosowana do linii krzywych* (Theory of the Algebraic Calculus Applied to Curved Lines), 2 Vols. 504 pp.

E. STEPHENS [1937], *The Elementary Theory of Operational Mathematics*, McGraw-Hill, 313 pp.

S. TAKAHASHI [1969], *Analysis in Categories*, Queen's University Papers, 131 pp.

V. VOLTERRA [1913a], *Leçons sur les équations intégrales et les équations intégro-différentielles*, Gauthier-Villars, 164 pp.

— [1913b], *Leçons sur les fonctions de lignes*, Gauthier-Villars, 164 pp.

V. VOLTERRA and J. PÉRÈS [1924], *Leçons sur la composition et les fonctions permutables*, Gauthier-Villars, 183 pp.

E. WARING [1776], *Meditationes analyticae*, Cambridge.

H. WEYL [1927], *Quantenmechanik und Gruppentheorie*, Zeit. Physik 46, 1–46.

— [1928], *Gruppentheorie und Quantenmechanik*, Leipzig, 288 pp.; 2nd ed., 1931, 366 pp.; English translation of 2nd edition, 1932, 422 pp.

N. WIENER [1926], *The operational calculus*, Math. Ann., 557–584.

— [1979], *Collected Works With Commentaries*, M.I.T. Press, Vol. II.

R. WOODHOUSE [1803], *The Principles of Analytical Calculation*, Cambridge, 219 pp.

— [1809], *A Treatise on Plane and Spherical Trigonometry*, Cambridge; 3rd ed., 1819; 5th ed., 344 pp.