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NONLOCAL ELLIPTIC PROBLEMS

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Abstract. Some conditions for the existence and uniqueness of solutions of the nonlocal elliptic problem $-\Delta\varphi = M\frac{f(\varphi)}{(\int_\Omega f(\varphi))^p}$, $\varphi|_{\partial\Omega} = 0$ are given.

In this paper we study the following nonlocal elliptic problem:

$$-\Delta\varphi = M\frac{f(\varphi)}{(\int_\Omega f(\varphi))^p}, \tag{1}$$

$$\varphi|_{\partial\Omega} = 0. \tag{2}$$

Here $\varphi : \Omega \to I\!R$ is an unknown function from a bounded subdomain Ω of $I\!R^n$ into $I\!R$, $f : I\!R^n \to I\!R^+$ is a given function and $M > 0$, $p > 0$ are real parameters.

The physical motivations for the study of nonlocal elliptic problems come from statistical mechanics [A], [B], [BKN], [BN1], [BN2], [W], [S], theory of electrolytes [BN2], [BHN], and theory of thermistors [C], [L].

If the nonlinearity $f(\varphi)$ has the form $f(\varphi) = e^{\pm\varphi}$ and $p = 1$ then (1) is identical with the Poisson equation $\Delta\varphi = \rho$ with density ρ of the Boltzmann form. In this case (1) is called the Poisson–Boltzmann equation. If $f(\varphi) = e^{\varphi}$ $(f(\varphi) = e^{-\varphi})$ φ can be interpreted as the gravitational (electric) potential of systems of particles in thermodynamical equilibrium interacting via gravitational (Coulomb) potential. In this interpretation, the parameter M, is the total mass (charge) of the particles of the system.

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The Poisson–Boltzmann equation arises also in investigations of phenomena associated with the occurrence of shear bands in metals being deformed under high strain rates [BT], and in modelling of turbulent behavior of flows [CLMP].

The general problem (1), (2) with given $f(\varphi)$ and positive p appears in modelling of stationary temperature φ when an electric current flows through a material with temperature dependent electrical resistivity $f(\varphi)$, subject to a fixed potential difference [L].

The main tool in the proof of the existence of a solution of (1), (2) is the technique of sub- (super-)solutions [BL], [BT], [T], variational methods [CLMP], [GL], and topological methods (Leray–Schauder theory) [BHN], [KN1], [KN2]. The nonexistence results are a consequence of the Pokhožaev identity [BT], [KN2], or construction of some special subsolutions [BL]. The existence and uniqueness of solutions for the Poisson–Boltzmann problem with $f(\varphi) = e^{-\varphi}$ and arbitrary $M > 0$ was proved in [GL] and [KN1]. In the first paper the variational and in the second − topological methods were applied.

When f has the form $f(\varphi) = e^{\varphi}$, the solutions do not exist for large M and generally are not unique. Moreover, the existence and uniqueness depend on the geometry of the domain Ω [KN2], [KN3].

When $f(\varphi) = e^{-\varphi}$ and $p > 0$ the problem (1), (2) was considered in [C] by using the technique of sub- and supersolutions, maximum principle and rearrangement method.

Under the assumption $\int_0^{+\infty} f(\varphi)\,d\varphi < +\infty$, the problem (1), (2) was investigated in [BL], [T].

We consider our problem in a bounded domain in $\mathbb{R}^n$ with the boundary $\partial\Omega$ of class $C^{1+\epsilon}$. This assumption guarantees the existence of the Green function $G(x,y)$ corresponding to $-\Delta$ and zero boundary data, satisfying the estimate

$$G(x,y) \le \frac{K}{|x-y|^{2-n}}, \quad |\nabla_x G(x,y)| \le \frac{K}{|x-y|^{1-n}}, \tag{3}$$

with some constant K depending on the domain Ω [GW].

To prove the existence of a solution of (1), (2) we shall use the Leray–Schauder theorem. To do this we consider the family of integral equations

$$\varphi(x) = \lambda \frac{\int_\Omega G(x,y) f(\varphi(y))\,dy}{(\int_\Omega f(\varphi(y))\,dy)^p}, \tag{4}$$

where λ is a parameter from $[0, M]$. When $\lambda = M$ the problem (4) is equivalent to (1), (2).

It follows from the estimates (3) that the right hand side of (4) is a continuous, compact operator on the space of continuous functions $C^0(\overline{\Omega})$ with the uniform norm $|\varphi|_\infty = \sup_\Omega |\varphi|$. To apply the Leray–Schauder theorem, a uniform for $\lambda \in [0, M]$ *a priori* estimate of solutions φ of (4) is needed.

First of all, replacing M by $M(f(0))^{1-p}$ we may put $f(0) = 1$.

Assume that f is *decreasing*. Then denoting $z := |\varphi|_\infty$ we have

$$z \le \lambda \Gamma(\Omega) |\Omega|^{-p} (f(z))^{-p},$$

where $\Gamma(\Omega) = sup_{x\in\Omega} \int_\Omega G(x,y)\,dy$ and $|\Omega|$ is the volume of Ω.

Therefore z will be uniformly bounded for all $\lambda \in [0, M]$, if

$$\lim_{z\to+\infty} z(f(z))^p > M\Gamma(\Omega)|\Omega|^{-p}. \tag{5}$$

Hence

THEOREM 1. *If f is continuous, positive and decreasing, then (1), (2) has a solution for M, f and p satisfing (5).*

In a similar way, for f positive, *increasing* we have from (4)

$$z \le \lambda\Gamma(\Omega)f(z)|\Omega|^{-p}.$$

Hence we get

THEOREM 2. *If f is continuous, positive and increasing, then (1), (2) has a solution for M, f and p satisfing*

$$\lim_{z\to+\infty} \frac{z}{f(z)} > M\Gamma(\Omega)|\Omega|^{-p}.$$

To get a stronger results, a more subtle estimate of $\int_\Omega f(\varphi)$ is needed.
We start with a simple

LEMMA 1. *Let $f, F : I\!R \to I\!R$ be continuous functions, f positive, F nondecreasing. Then for any continuous function $\varphi : \Omega \to I\!R$*

$$\frac{\int_\Omega f(\varphi)F(f(\varphi))}{\int_\Omega f(\varphi)} \ge \frac{1}{|\Omega|}\int_\Omega F(f(\varphi)). \tag{6}$$

PROOF. The inequality (6) is equivalent to

$$\int_{\Omega\times\Omega} f(\varphi(x))F(f(\varphi(x)))\,dx\,dy - \int_{\Omega\times\Omega} f(\varphi(y))F(f(\varphi(x)))\,dx\,dy \ge 0.$$

The left hand side may be transformed to

$$\frac{1}{2}\int_{\Omega\times\Omega}(F(f(\varphi(x))) - F(f(\varphi(y)))(f(\varphi(x)) - f(\varphi(y)))\,dx\,dy,$$

which is nonnegative.

It is clear that the assumed continuity of all functions may be relaxed, however only in this weak form the Lemma 1 will be used.

THEOREM 3. *If f is a positive, decreasing differentiable function such that $\sup|f'/f| < +\infty$ and $0 < p \le 1$, then the problem (1), (2) has a unique solution for all $M > 0$.*

PROOF. Applying the Jensen inequality we have

$$\exp\left(\frac{1}{|\Omega|}\int_\Omega \ln f(\varphi)\right) \le \frac{1}{|\Omega|}\int_\Omega f(\varphi). \tag{7}$$

Using the Cauchy and Poincaré inequality, we have with some positive constant C

$$\begin{aligned}\left(\int_\Omega \ln f(\varphi)\right)^2 &\le C|\Omega|\int_\Omega (f'(\varphi)/f(\varphi))^2|\nabla\varphi|^2\\ &\le C|\Omega|\sup|f'/f|\int_\Omega |f'(\varphi)/f(\varphi)||\nabla\varphi|^2.\end{aligned}$$

Now we multiply (1) by $\ln f(\varphi)$ and integrate over Ω, what gives

$$0 \ge \int_\Omega (f'/f)|\nabla\varphi|^2 = \frac{\lambda}{(\int_\Omega f(\varphi))^p}\int_\Omega f(\varphi)\ln f(\varphi) \ge \frac{\lambda}{(\int_\Omega f(\varphi))^{p-1}|\Omega|}\int_\Omega \ln f(\varphi).$$

The last inequality follows from Lemma 1. Two inequalities above imply

$$0 \ge \int_\Omega \ln f(\varphi) \ge -\lambda C \sup|f'/f|\Big(\int_\Omega f(\varphi)\Big)^{1-p}. \tag{8}$$

From (8)

$$\int_\Omega \ln f(\varphi) \ge -MC\sup|f'/f||\Omega|^{1-p},$$

hence, using (4) and (7), we get

$$|\varphi|_\infty \le M\Gamma(\Omega)|\Omega|^{-p}\exp(MCp\sup|f'/f||\Omega|^{-p}),$$

which is the desired estimate.

To prove the uniqueness, let φ_i, $i=1,2$ satisfy

$$-\Delta\varphi_i = M\mu_i^p f(\varphi_i), \quad \varphi_i|_{\partial\Omega} = 0, \tag{9}$$

with $\mu_i = (\int_\Omega f(\varphi_i(x))\,dx)^{-1}$. We distinguish two cases: $\mu=\mu_1=\mu_2$ and $\mu_1\ne\mu_2$. In the first case, we take the difference of two equations (9), multiply it by $\varphi_1-\varphi_2$ and integrate over Ω, which gives $\int_\Omega|\nabla(\varphi_1-\varphi_2)|^2 = M\mu^p\int_\Omega(f(\varphi_1)-f(\varphi_2))(\varphi_1-\varphi_2)$. The right hand side of the last equation is nonpositive, hence $\nabla(\varphi_1-\varphi_2)=0$, what implies $\varphi_1=\varphi_2$ in Ω due to $\varphi_1=\varphi_2$ on $\partial\Omega$.

Now let $\mu_1>\mu_2$. First we show that $\varphi_1>\varphi_2$ in Ω. If not, there exists x_0 such that $\varphi_1(x_0)\le\varphi_2(x_0)$ and $-\Delta(\varphi_1-\varphi_2)(x_0)\le 0$, whereas the difference $M\mu_1^p f(\varphi_1(x_0)) - M\mu_2^p f(\varphi_2(x_0))$ of the right hand side members of (9) at x_0 is positive, a contradiction. We now have $\frac{\partial\varphi_1}{\partial\nu}\le\frac{\partial\varphi_2}{\partial\nu}$ along $\partial\Omega$, which is a consequence of $\varphi_1\ge\varphi_2$. Moreover, integrating (9) over Ω we get $-M\mu_1^{p-1} = \int_{\partial\Omega}\frac{\partial\varphi_1}{\partial\nu} \ge \int_{\partial\Omega}\frac{\partial\varphi_2}{\partial\nu} = -M\mu_2^{p-1}$. These two facts give us $\frac{\partial\varphi_1}{\partial\nu}=\frac{\partial\varphi_2}{\partial\nu}$ on $\partial\Omega$. Thus

$$\frac{\partial}{\partial\nu}(\varphi_1-\varphi_2)=0 \tag{10}$$

at any boundary point.

From Theorem 10. 2, Ch. IV in [LU] applied to (9) which guarantees that $\varphi_i\in C^2(\overline{\Omega})$ and from $\mu_1>\mu_2$ it follows that near the boundary $\Delta(\varphi_1-\varphi_2)<0$. Therefore zero is the minimal value of $\varphi_1-\varphi_2$ there. Due to the Hopf lemma $\frac{\partial}{\partial\nu}(\varphi_1-\varphi_2)<0$ along $\partial\Omega$, contrary to (10).

Consider now the special case when Ω is a ball of radius 1 centered at the origin and φ is radially symmetric. Then the problem (1), (2) reads

$$-(r^{n-1}\varphi')' = Mr^{n-1}\mu^p f(\varphi), \qquad \varphi'(0)=\varphi(1)=0, \qquad \frac{d}{dr} = {}', \tag{11}$$

where $\mu=\mu_\varphi=(\sigma_n\int_0^1 r^{n-1}f(\varphi(r))\,dr)^{-1}$ and σ_n is the area of the unit sphere in $\mathbb{R}^n$.

Theorem 4. *If f is decreasing, $p\ge 1$, and M is such that*

$$\lim_{z\to+\infty} z(f(z))^{p-1} > M\sigma_n^{-p}n^{p-1}, \tag{12}$$

then the problem (11) has a solution.

Proof. Integrating (11) twice and introducing the operator A_λ

$$A_\lambda \varphi(r) = \lambda\mu^p \int_r^1 \left(t^{n-1} \int_0^t s^{n-1} f(\varphi(s))\, ds \right) dt$$

we may replace (11) by

$$\varphi = A_\lambda \varphi, \quad \lambda \in [0, M], \tag{13}$$

if we put $\lambda = M$.

Consider the family of problems (13). The operator A_λ is continuous on the space $C^0[0,1]$. Now

$$(A_\lambda \varphi(r))' = -\lambda r^{1-n} \mu^p \int_0^r s^{n-1} f(\varphi(s))\, ds \tag{15}$$

is uniformly bounded if φ belongs to a bounded subset of $C^0[0,1]$. Hence A_λ is a compact operator. To apply the Leray-Schauder theorem, it is enough to prove a uniform *a priori* estimate of solutions φ of (13) for $\lambda \in [0, M]$.

First we show that $\varphi'' < 0$. In fact, coming back to the form (11) of (13) we see that

$$\begin{aligned} r^{n-1}\varphi''(r) &= -\lambda r^{n-1}\mu^p f(\varphi(r)) + (n-1)r^{-1}\lambda\mu^p \int_0^r s^{n-1} f(\varphi(s))\, ds \\ &< -\lambda r^{n-1}\mu^p f(\varphi(r)) + (n-1)r^{-1}\lambda\mu^p f(\varphi(r)) \int_0^r s^{n-1} ds < 0. \end{aligned}$$

Hence the minimum φ' is attained at $r = 1$. Putting $r = 1$ in (15) we get $\varphi'(1) = -\lambda\sigma_n^{-1}\mu^{p-1}$, so $z = |\varphi|_\infty < M\sigma_n^{-1}\mu^{p-1}$. We have assumed $p \ge 1$, therefore $\mu^{p-1} < (f(z))^{1-p} n^{p-1} \sigma_n^{1-p}$ which implies $z(f(z))^{p-1} \le M\sigma_n^{-p} n^{p-1}$. From (12) and the last inequality the desired *a priori* estimate follows.

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