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LONG-TIME ASYMPTOTICS OF SOLUTIONS TO SOME NONLINEAR WAVE EQUATIONS

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Abstract. In this paper, we survey some recent results on the asymptotic behavior, as time tends to infinity, of solutions to the Cauchy problems for the generalized Korteweg-de Vries-Burgers equation and the generalized Benjamin-Bona-Mahony-Burgers equation. The main results give higher-order terms of the asymptotic expansion of solutions.

1. Introduction. This paper is intended as a survey of recent results on the large-time behavior of solutions to the Cauchy problem for two classes of nonlinear dissipative-dispersive wave equations

$$u_t - \nu u_{xx} + u_{xxx} + f(u)_x = 0 \tag{1.1}$$

and

$$u_t - \nu u_{xx} - u_{xxt} + u_x + f(u)_x = 0, \tag{1.2}$$

both supplemented by the initial condition

$$u(x,0) = u_0(x). \tag{1.3}$$

Here $u = u(x,t)$ is a real-valued function of two real variables: $x \in \mathbb{R}$ (called the spatial variable) and the time $t > 0$. We shall consider $f \in C^2(\mathbb{R})$. The letter ν denotes a positive constant.

Such models appear when one attempts to describe the propagation of small-amplitude long waves in nonlinear dispersive media taking into account dissipative mechanisms. Hence, they arise when dissipation, dispersion, and the effect of nonlinearity are appended to the basic model $u_t + u_x = 0$ for the unidirectional wave propagation.

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The damping is represented here, for simplicity of exposition, by the simple model term $-u_{xx}$, typical for the Burgers equation. However, recall that more general dissipation can be considered (cf. e.g. [19, 30, 12, 2]).

The dispersion term in (1.1) has the form u_{xxx} which appears in the well-known Korteweg-de Vries equation $u_t + u_{xxx} + uu_x = 0$, derived originally as a model for waves propagating on the surface of a canal. It is worth emphasizing that all the results concerning (1.1) that we are going to present below may be rewritten directly for the case when the third derivative in (1.1) is replaced by a pseudo-differential operator with a purely imaginary symbol homogeneous of degree $\alpha > 2$ (cf. [28], for more details).

Equation (1.2) generalizes the model $u_t - u_{xxt} + u_x + uu_x = 0$ put forward in [5] in order to have an alternative model for the Korteweg-de Vries equation. Note that it is called either the regularized long-wave equation or the Benjamin-Bona-Mahony equation. Let us also mention here that equation (1.2) has a counterpart for $x \in \mathbb{R}^n$ (then $\partial^2/\partial x^2$ is replaced by Δ and $\partial/\partial x$ by div). In this case, the physical background of this equation, its well-posedness, as well as some asymptotic properties of solutions are studied in [26].

The typical non-linear term occurring in hydrodynamics in the one-dimensional case has the form $uu_x = (u^2/2)_x$. The most obvious generalization of this nonlinearity consists in replacing the square by $f(u)$ for sufficiently regular f. Indeed, this was done in several papers (cf. [2, 6, 9, 10, 19, 26]). Here we intend to observe more subtle interaction of the nonlinearity with dissipation and dispersion, thus we consider $f(u) = u^q$ with a continuous range of the parameter q. But then, the problem appears with the definition of u^q for negative u and for non-integer q. Hence, in this paper, u^q should be interpreted either as

$$|u|^q \quad \text{or} \quad |u|^{q-1}u. \tag{1.4}$$

To shorten notation in this report, we continue writing u^q in order to denote nonlinearities having the form above. However, note that, in fact, the following property of the nonlinearity will only be important throughout this work: the nonlinear term in (1.1) and (1.2) has the form $f(u)_x$ where the C^1-function f satisfies $|f(u)| \le C|u|^q$, $|f'(u)| \le C|u|^{q-1}$ for every $u \in \mathbb{R}$, $q \ge 2$, and a constant C. Moreover, the limits $\lim_{u\to 0^-} f(u)/|u|^q$, $\lim_{u\to 0^+} f(u)/|u|^q$ exist and the both are different from 0.

Notation. The notation to be used is mostly standard. For $1 \le p \le \infty$ the $L^p(\mathbb{R}^n)$-norm of a Lebesgue measurable real-valued function defined on $\mathbb{R}^n$ is denoted by $\|f\|_p$. We will denote by $\|\cdot\|_{\mathcal{X}}$ the norm of any other Banach space $\mathcal{X}$ used in this paper. If ℓ is a nonnegative integer, $W^{\ell,p}(\mathbb{R}^n)$ will be the Sobolev space consisting of functions in $L^p(\mathbb{R}^n)$ whose generalized derivatives up to order ℓ belong also to $L^p(\mathbb{R}^n)$. The case $p = 2$ deserves the special notation: $W^{\ell,2}(\mathbb{R}^n) = H^\ell(\mathbb{R}^n)$ with the norm $\|u\|_{H^\ell} \equiv \left(\int_{\mathbb{R}^n}(1+|\xi|^2)^\ell|\widehat{u}(\xi)|^2\,d\xi\right)^{1/2}$. The Fourier transform of v is given by $\mathcal{F}v(\xi) = \widehat{v}(\xi) \equiv \int_{\mathbb{R}^n} e^{-ix\xi}v(x)\,dx$. The heat kernel (the fundamental solution of the heat equation) is denoted by $G(x,t) = (4\pi t)^{-n/2}\exp(-|x|^2/(4t))$. Moreover, $M = \int_{\mathbb{R}^n} u_0(x)\,dx$ is called the mass of a solution (one may show that $M = \int_{\mathbb{R}} u(x,t)\,dx$ is conserved for any $t > 0$), and $m = \int_{\mathbb{R}} xu_0(x)\,dx$ is the first moment of the initial data. The letter C will denote generic positive constants which may vary from line to line.

2. Existence of solutions. For the well-posedness of the initial-value problem (1.1)-(1.3), we refer the reader to Saut [33], Naumkin *et al.* [30], Abdelouhab [1], Alarcón [2], Karch [27]. It is sufficient to know for the purpose of this paper that it is always possible to construct global-in-time solutions for any initial data $u_0 \in L^1 \cap H^1$ provided either $\|u_0\|_1 + \|u_0\|_{H^1(\mathbb{R})}$ is small or some restrictions on q are imposed. In [28, Sec. 4], we study this problem more carefully.

The first approach to basic questions concerning existence, uniqueness, as well as the regularity of the solutions to the problem (1.2)-(1.3) was made by Benjamin *et al.* [5], and they have considered the case $f(u) = u^2/2$, and $u_0 \in H^s(\mathbb{R})$ with $s \geq 1$. Their results have been extended by Bona *et al.* [13] and Amick *et al.* [4].

In our work [26], we use the theory of semigroups of linear operators in order to prove the existence of solutions to the multidimensional version of this problem

$$u_t - \Delta u_t - \nu\Delta u - (b, \nabla u) = \nabla \cdot F(u) \tag{2.1}$$

$$u(x, 0) = u_0(x), \tag{2.2}$$

where $\nu > 0$ is a fixed constant, $t \geq 0$, $x \in \mathbb{R}^n$, and $n \geq 1$. Here $b \in \mathbb{R}^n$, $F \in C^1(\mathbb{R}, \mathbb{R}^n)$ is a fixed vector field and $\nabla \cdot F(u) = \sum_{i=1}^n \partial_{x_i} F_i(u)$. To describe our approach, let $K(x)$ denote the fundamental solution of the operator $I - \Delta$ on the whole $\mathbb{R}^n$. K is the well-known Bessel potential of order 2, $K, \nabla K \in L^1(\mathbb{R}^n)$, $K(x) \geq 0$, $\int K(x)\, dx = 1$. Consider the equation (2.1) for u at the point (y, t), multiply the result by $K(x-y)$, and then integrate with respect to y over $\mathbb{R}^n$ in order to obtain the following integral equation

$$u_t - \nu K * \Delta u - K * (b, \nabla u) = K * \nabla \cdot F(u). \tag{2.3}$$

Now, the existence of a unique solution $u(t) \in C([0, T), X)$ to the problem (2.1)-(2.2) for some $T > 0$ and a Banach space X can be obtained using an iteration procedure applied to the integral equation (2.3).

THEOREM 2.1. *i)* (Local existence) *Let* $1 \leq p \leq \infty$. *Assume that* $F \in C^1(\mathbb{R}, \mathbb{R}^n)$, $F(0) = 0$. *Then for each initial value* $u_0 \in L^p(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ *there exist* $T > 0$ *and a unique solution* u *of the problem (2.1)-(2.2) on* $[0, T)$.

ii) (Regularity) *Suppose additionally that* $1 < p < \infty$, $u_0 \in W^{k,p}(\mathbb{R}^n)$, *and* $F \in C^{k+1}(\mathbb{R}, \mathbb{R}^n)$ *for a natural number* k. *Then* $u \in C^1([0, T); W^{k,p}(\mathbb{R}^n))$.

Now, if we multiply (2.1) by u and integrate with respect to x over $\mathbb{R}^n$ we obtain that $\|u(\cdot, t)\|_{H^1(\mathbb{R}^n)} \leq \|u_0(\cdot)\|_{H^1(\mathbb{R}^n)}$. This is the basic *a priori* estimate which leads to the following result.

THEOREM 2.2 (Global existence). *Let* $k = 1$ *or* $k = 2$. *Assume that* α *is a number satisfying* $1 \leq \alpha < \infty$ *if* $n = 2$, *and* $2/n \leq \alpha \leq 2/(n-2)$ *if* $n \geq 3$. *Suppose that there exists a positive constant* C *such that* $|F'(s)| \leq C(1 + |s|^\alpha)$ *for all* $s \in \mathbb{R}$. *If* $k = 1$, *let us assume that* $2 < p < \infty$ *for* $n = 2$, *and* $\max\{2n/(n-2), n\} < p < \infty$ *in the case* $n > 2$. *If* $k = 2$, *assume that* $n/2 < p < n$ *for* $n \geq 2$. *For* $n = 1$, *simply suppose that* $F \in C^1(\mathbb{R}; \mathbb{R}^n)$ *and* $1 \leq p \leq \infty$ *for* $k = 1$.

Then for each initial data $u_0 \in W^{k,p}(\mathbb{R}^n) \cap H^1(\mathbb{R}^n)$ *there exists a unique global solution* u *of the problem (2.1)-(2.2).*

The details of the proofs of Theorems 2.1 and 2.2 may be found in [26], where our approach was based on some ideas adapted from [23].

3. Decay of solutions and the first order term of asymptotics. The long-time behavior of solutions to equations similar to those in (1.1) and (1.2) has been discussed in several recent works. Concerning the decay of solutions to (1.1) or to (1.2) in various norms Amick *et al.* [4] studied them with $f(u) = u^2/2$. Next, Dix [19] developed the theory of equations more general than that in (1.1) for $f(u) = u^q$ with an integer $q > 2$. His results considerably improved the earlier results of Biler [6]. That research was continued by Bona and Luo [9], where assuming that $q \geq 2$ it was shown that if u is a solution corresponding to suitably restricted data, the L^2-norm of u decays at the rate $t^{-1/4}$ as $t \to \infty$, and this rate is sharp for a generic class of initial data. Note that this is the same rate that one obtains via the Fourier analysis of the linearized equation, i.e. the equation obtained by dropping the nonlinear term in (1.1) or (1.2). More results on the decay of solutions either to (1.1) or to (1.2) may be found in [8, 11, 18, 25, 26, 30, 34, 36, 37].

Below, we describe briefly the contribution of the author to this theory concerning the multidimensional version of the Benjamin-Bona-Mahony-Burgers equation (2.1).

First, we deal with the linear equation, i.e. $F \equiv 0$. Using the Fourier transform it is possible to express the solution $v(x,t) \equiv S(t)v_0(x)$ of the equation $v_t - \Delta v_t - \nu\Delta v - (b, \nabla v) = 0$ with the initial condition $v(x,0) = v_0(x)$ by the oscillatory integral

$$v(x,t) = S(t)v_0(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} \exp(t\Phi(\xi) + ix\xi)\widehat{v_0}(\xi)\, d\xi \tag{3.1}$$

with the phase function $\Phi(\xi) = (-\nu|\xi|^2 + i(b,\xi))(1+|\xi|^2)^{-1}$. It is easy to see that for every $v_0 \in \mathcal{S}(\mathbb{R}^n)$ the oscillatory integral (3.1) converges, and it represents a smooth rapidly decreasing function for every t. In fact, (3.1) defines a solution to the linear equation also for less regular initial data v_0 (cf. the next theorem, below). The following theorem was proved first in [25, Prop. 2.1] in the one-dimensional case. Then its multidimensional improved version appeared in [26, Thm. 2.1].

Theorem 3.1 ([25, 26]). *Let $1 \leq q \leq p \leq \infty$ and $n \geq 1$. There exist positive constants C, ε (independent of t and v_0) such that for every $v_0 \in L^p(\mathbb{R}^n) \cap L^q(\mathbb{R}^n)$*

$$\|S(t)v_0\|_p \leq C(1+t)^{n(1/p-1/q)/2}\|v_0\|_q + Ce^{-\varepsilon t}\|v_0\|_p \tag{3.2}$$

for all $t \geq 0$.

Let us compare this result with an analogous estimate for the heat semigroup $e^{\nu t\Delta}$ generated by the Laplace operator $\nu\Delta$:

$$\|e^{\nu t\Delta}v_0\|_p \leq Ct^{n(1/p-1/q)/2}\|v_0\|_q \tag{3.3}$$

for all $t > 0$ and $1 \leq q \leq p \leq \infty$. The reason of the same decay rate as $t \to \infty$ in (3.2) and (3.3) is that the real part of the phase function $\Phi(\xi)$ in the definition of $S(t)$ (cf. (3.1)) behaves like $-\nu|\xi|^2$ for $\xi \to 0$, hence as in the formula $e^{\nu t\Delta}v_0(x) = (2\pi)^{-n}\int \exp(-\nu t|\xi|^2 + ix\xi)\widehat{v_0}(\xi)\, d\xi$. On the other hand, the additional exponentially decaying term on the right-hand side of (3.2) has to appear, because $S(t)$ has no smoothing properties typical for

$e^{\nu t\Delta}$. In fact, for every $t \in \mathbb{R}$ and $1 \le p \le \infty$, $S(t) : L^p(\mathbb{R}^n) \to L^p(\mathbb{R}^n)$ is a bijection (cf. [26, Sec. 4]).

The estimate (3.2) suggests that the natural question to study is the long time behavior of $t^{n(1/q-1/p)/2}S(t)v_0$ in $L^p(\mathbb{R}^n)$. The next proposition says that if $\int v_0(x)\,dx \neq 0$, then the decay rate (3.2) is optimal for $q = 1$. Moreover, we can assert that the asymptotic of $S(t)v_0$ is described by the fundamental solution of the heat equation.

Theorem 3.2 ([26]). *Let $1 \le p \le \infty$, $n \ge 1$, assume that $v_0 \in L^1(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$, and define $M = \int v_0(x)\,dx = \widehat{v_0}(0)$, $G(x,t) = (4\pi t)^{-n/2}\exp\left(-|x|^2/(4t)\right)$, $\widetilde{S}(t)v_0(x) \equiv S(t)v_0(x - tb)$. Then*

$$t^{n(1-1/p)/2}\|\widetilde{S}(t)v_0(\cdot) - MG(\cdot,\nu t)\|_p \to 0 \quad as \quad t \to \infty.$$

Remark 3.1. It is proved in [4, Lemma 4.1] that for $n = 1$ and all $v_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$

$$\lim_{t\to\infty} t^{1/4}\|S(t)v_0\|_2 = (8\nu\pi)^{-1/4}\left|\int v_0(x)\,dx\right|.$$

Of course, this is a particular case of Theorem 3.2.

Remark 3.2. Theorem 3.1 improves estimates of $S(t)$ in the one-dimensional case given in [25, Prop. 4.1]. In that work the decay of the L^p-norms of solutions to the linearized equation was established for $u_0 \in H^1(\mathbb{R}) \cap L^1_2(\mathbb{R})$, where $L^2_1(\mathbb{R}) = \{g \in L^2(\mathbb{R}) : \|g\| \equiv \int |g(x)|^2(1+|x|)^2\,dx < \infty\} \subset L^1(\mathbb{R})$. We can also obtain the same conclusion in [25, Thm. 2.1 and 2.2] under the weaker assumption $u_0 \in H^1(\mathbb{R}) \cap L^1(\mathbb{R})$.

An important tool in proving the decay of solutions to nonlinear parabolic conservation laws is the property that a nonnegative initial data u_0 produces a nonnegative solution $u(x,t)$ for all $t \ge 0$. A direct consequence of this is: for $u_0 \in L^1(\mathbb{R}^n)$ we have $\|u(\cdot,t)\|_1 \le \|u_0\|_1$ and this estimate gives the decay of other L^p-norms of u, cf. e.g. [22]. In the case of the equations (1.1) and (2.1), the maximum principle mentioned above usually fails, hence it is necessary to use different techniques in order to get boundedness of the norm $\|u(\cdot,t)\|_1$ for $t \ge 0$. This bound seems to be crucial in the proof of the decay of solutions to (1.1) and (1.2), as it was observed in [4, Lemma 5.1]. That result says that if $f(u) = -u^2/2$ and $u_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$, then the estimates $\sup_{t>0}\|u(\cdot,t)\|_1 < \infty$ and $\sup_{t>0} t^{1/2}\|u(\cdot,t)\|_2^2 < \infty$ of solutions to (1.1) or to (1.2) are equivalent. Similar considerations for other norms are used in [4, Cor. 5.2] and in [9, Cor. 5.2]. We extend those results for a general F and other L^p-norms of u. Our first theorem says that for general nonlinearities F the L^p-decay properties of solutions to (2.1) for $n = 1$ and each p are equivalent. This *extrapolation principle* improves results cited above.

Theorem 3.3 ([25]). *Let $u(x,t)$ be a solution of (2.1) with $n = 1$, $F \in C^2(\mathbb{R})$, and $\nu > 0$, corresponding to the initial data $u_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$. Assume that for some $p_0 \in [1,\infty)$ and a constant $C > 0$*

$$\|u(\cdot,t)\|_{p_0} \le C(1+t)^{(1/p_0-1)/2} \tag{3.4}$$

for all $t \ge 0$. Then for every $p \in [1,\infty]$ there exists a constant $C = C(p, \|u_0\|_1, \|u_0\|_{H^1})$ such that the inequality (3.4) holds for p_0 replaced by p.

Theorem 3.3 suggests the question, when the assumed estimate (3.4) does hold for some $p_0 \in [1, \infty)$. The next theorem gives some sufficient conditions guaranteeing the validity of (3.4).

THEOREM 3.4 ([25]). *Let u denote the solution of (2.1) for $n = 1$ corresponding to the initial condition $u_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$, $F \in C^2(\mathbb{R})$, and $\nu > 0$. Suppose in addition that one of the following conditions is satisfied*

(i) $\nu > |b|$ *and* $u_0 \in H^2$,

(ii) $|u_0|_1$ *is sufficiently small.*

(iii) $|F'(u)| \le C|u|^2$ *for some* $C > 0$.

Then for every $1 \le p \le \infty$ *there exists* $C_p = C(p, u_0)$ *such that*

$$\|u(\cdot, t)\|_p \le C_p(1 + t)^{(1/p-1)/2}$$

for all $t \ge 0$.

REMARK 3.3. For $\nu > |b|$ the dissipative term dominates dispersive effects and then the maximum principle mentioned above begins to be valid after a finite time and the inequality $\|v(\cdot, t)\|_1 \le \|v(\cdot, T)\|$ holds for some $T \ge 0$ and all $t > T$, cf. [25, Prop. 3.2 and Cor. 3.1.]. Hence we have (3.4) with $p_0 = 1$. The decay result for (2.1) under the assumption (ii) is new, while an analogous fact for (1.1) is known, see [19]. The decay of solutions to (2.1) with F satisfying (iii) was proved in [9] and [37]. In [25], we present a shorter, direct argument.

Recall that any (sufficiently regular) solution of (2.1)-(2.2) satisfies the integral equation

$$u(t) = S(t)u_0 + \int_0^t S(t - \tau)K * \nabla \cdot F(u(\tau))\, d\tau. \tag{3.5}$$

obtained by computing the Fourier transform of (2.1) with respect to the spatial variable x, solving the resulting ordinary equation via the variation of parameters formula, and taking the inverse Fourier transform.

The representation (3.5) is the main tool in the proof of local and global existence of solutions as well as in obtaining time decay of the L^p-norms of solutions from Theorems 3.3 and 3.4. Moreover, in some cases, it allows us to improve the asymptotic results from those theorems.

PROPOSITION 3.5 ([25]). *Let $n = 1$. Suppose that $S(t)u_0(x)$ is defined by (3.1). Let $2 \le p \le \infty$ and $u_0 \in H^2(\mathbb{R})$. Under the assumptions of Theorem 3.3 if $F''(0) = 0$, then*

$$\|v(\cdot, t) - S(t)u_0(\cdot)\|_p = o(t^{(1/p-1)/2}) \ \text{ as } \ t \to \infty.$$

Next we investigate behavior of solutions to the nonlinear equation (2.1) with $n \ge 2$. In order to avoid technical complications we limit ourselves to the case of quadratic nonlinearities.

THEOREM 3.6 ([26, Thm. 2.2]). *Let $n \ge 2$. Assume that $u(x, t)$ is a solution of (2.1) with $\nabla \cdot F(u) = u(a, \nabla u)$ for some $a \in \mathbb{R}^n$ corresponding to the initial data $u_0 \in L^1(\mathbb{R}^n) \cap$*

$H^1(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$. *Then there exist constants* $C_p > 0$ *independent of* t *such that*

$$\|u(t) - S(t)u_0\|_p \leq \begin{cases} C_p(1+t)^{n(1/p-1)/2-1/2}(1+\log(1+t)) & \textit{for} \quad 1 \leq p < \frac{n}{n-1}, \\ C_p(1+t)^{n(1/p-1)/2}(1+\log(1+t)) & \textit{for} \quad \frac{n}{n-1} \leq p < \frac{n}{n-2}. \end{cases} \tag{3.6}$$

REMARK 3.4. Theorem 3.6 implies that the long-time behavior of some L^p-norms of solutions to (2.1) is given by the heat kernel, i.e.

$$t^{n(1-1/p)/2}\|u(\cdot - tb, t) - MG(\cdot, \nu t)\|_p \to 0 \quad \text{as} \quad t \to \infty$$

for $1 \leq p < n/(n-1)$ (for every $p \in [1, \infty]$, if $n = 1$). This is a direct consequence of Theorem 3.2 and the first inequality in (3.6). If $n/(n-1) \leq p < n/(n-2)$, we get almost optimal decay estimates $\|u(\cdot, t)\|_p \leq C(1+t)^{n(1/p-1)/2}(1+\log(1+t))$ by (3.2) and (3.6).

4. Nonlinear asymptotics. In this section, we describe the results from [27]. That paper, where our main goal is to investigate the behavior as $t \to \infty$ of solutions to (1.1) with $f(u) = u^2/2$, was inspired by the work of Amick *et al.* [4]. Using the Hopf-Cole transformation and several subtle estimates, they show that each solution of (1.1) (with $f(u) = u^2/2$) corresponding to the initial data $u_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$ satisfies

$$\|u(\cdot, t)\|_2 \leq Ct^{-1/4} \tag{4.1}$$

for all $t > 0$ and a positive constant C. Next, as a direct consequence of (4.1), they obtain $\|u(\cdot, t)\|_\infty \leq Ct^{-1/2}$ with another $C > 0$, as well as decay rates for other L^p-norms of u and its derivatives (cf. [4, Th. 5.1 and Cor. 5.2]). Moreover, the estimate (4.1) is optimal as long as $\int_{\mathbb{R}} u_0(x)\, dx \neq 0$, because, as it was shown in [4, Th. 5.5],

$$\begin{aligned} &\lim_{t\to\infty} t^{1/2} \int_{\mathbb{R}} u^2(x, t)\, dx \qquad (4.2) \\ &\quad = \frac{4\nu^2(\beta-1)^2}{2\pi\sqrt{\nu}} \int_{\mathbb{R}} \frac{\exp(-2\xi^2)}{\{1 + ((\beta-1)/\sqrt{\pi})\int_\xi^\infty \exp(-s^2)\, ds\}^2}\, d\xi, \end{aligned}$$

where $\beta = \exp\{-(1/2\nu)\int_{\mathbb{R}} u_0(x)\, dx\}$.

In [27], we contribute to a better understanding of the limit (4.2). We show that the large time behavior of solutions to (1.1)-(1.3) satisfying (4.1) is described by a *nonlinear diffusion wave*. In other words, the asymptotic profile of solutions to (1.1) with $f(u) = u^2/2$ is given by a particular solution $U_M = U_M(x, t)$ to the Burgers equation

$$v_t - \nu v_{xx} + v v_x = 0, \tag{4.3}$$

having the properties

$$\int_{\mathbb{R}} U_M(x, t)\, dx = M \quad \text{for all} \quad t > 0 \tag{4.4}$$

and

$$\int_{\mathbb{R}} U_M(x, t)\varphi(x)\, dx \to M\varphi(0) \quad \text{as} \quad t \to 0 \tag{4.5}$$

for all $\varphi \in C^\infty(\mathbb{R})$. Using the terminology from [29] one can say that $U_M(x, t) \to M\delta_0$ (the Dirac delta) as $t \searrow 0$ *narrowly* in $\mathbb{R}$. Such a solution is called a fundamental solution in the linear theory and a *source solution* in a nonlinear case.

We need to recall some properties of solutions to (4.3) before we make precise our results. It is well-known that the Hopf-Cole transformation allows us to solve (4.3) for each initial value $v(0) \equiv v_0 \in L^1(\mathbb{R})$ giving a solution v in an explicit form. This is a unique classical solution belonging to $C([0,\infty);\ L^1(\mathbb{R}))$. Moreover, applying the known result for the linear heat equation and inverting the transformation, one concludes that the large time behavior of v is given by the following explicit solution

$$U_M(x,t) = t^{-1/2} U_M\left(xt^{-1/2}, 1\right) \quad \text{with} \quad U_M(\eta,1) = \frac{\exp(-\eta^2/4)}{C_M + \frac{1}{2}\int_0^\eta \exp(-\xi^2/4)\, d\xi}. \tag{4.6}$$

Here, the constant C_M is uniquely determined as a function of M by the condition $\int_{\mathbb{R}} U_M(\eta,1)\ d\eta = M$. The important point to note here is that for every $M \in \mathbb{R}$ the function U_M is a unique solution to (4.3) in the space $C((0,\infty); L^1(\mathbb{R}))$ which takes the initial data in the narrow sense (4.5) (cf. [20, Thm. 3]). It represent the asymptotic behavior of all solutions to (4.3) with given (conserved in time) M. More precisely, it follows that for all initial data $v_0 \in L^1(\mathbb{R})$ such that $\int_{\mathbb{R}} v_0(x)\ dx = M$, the corresponding solutions of (4.3) satisfy

$$t^{(1-1/p)/2} \|v(\cdot,t) - U_M(\cdot,t)\|_p \to 0 \quad \text{as} \quad t \to \infty \tag{4.7}$$

for each $p \in [1,\infty]$ (see e.g. [17], [22], [24], for details).

The main result of [27] consists in showing the analogous asymptotic behavior of solutions to (1.1).

Theorem 4.1. *Let $u = u(x,t)$ be a solution to the problem (1.1)-(1.3) with $f(u) = u^2/2$ corresponding to the initial data $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and satisfying the estimate (4.1). Then for each $p \in [1,\infty]$*

$$t^{(1-1/p)/2} \|u(\cdot,t) - U_M(\cdot,t)\|_p \to 0 \quad as \quad t \to \infty, \tag{4.8}$$

where $U_M = U_M(x,t)$ is the source solution of the Burgers equation (4.3) given by formula (4.6).

Theorem 4.1 may be summarized by saying that dispersion becomes asymptotically negligible (compared to dissipation and nonlinearity) in the leading-order long-time asymptotic form of solutions to (1.1).

Remark 4.1. As we have mentioned above, each solution of (1.1) with $f(u) = u^2/2$ corresponding to $u_0 \in L^1(\mathbb{R}) \cap H^2(\mathbb{R})$ satisfies (4.1) ([4]). One can take also $u_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$ to get (4.1) assuming additionally that u_0 is small in an appropriate sense (cf. [19]). In our case, as long as we know (4.1), we are able to prove (4.8) for solutions corresponding to $u_0 \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Moreover, in [27, Sec. 5] we discuss possible applications of our method to more general equations.

Remark 4.2. By Theorem 4.1, we discover that the number on the right hand side of (4.2) is equal to $\|U_M(\cdot,1)\|_2^2$. To see this, we need the explicit form of C_M in (4.6), which one obtains by straightforward calculations.

Remark 4.3. It is proved in [19] (see also [27, 28]) that the asymptotic profile as $t \to \infty$ of the solution of the linear equation $u_t - \nu u_{xx} + u_{xxx} = 0$ with $\int_{\mathbb{R}} u_0(x)\ dx = M$

is given by $MG(x, \nu t)$. The same result holds for solutions to the nonlinear equation

$$u_t - \nu u_{xx} + u_{xxx} + (u^q)_x = 0 \tag{4.9}$$

for $q > 2$. This is the case of so-called *asymptotically weak nonlinearity*, and we refer to [19] for a treatment of more general equations with such a property. Theorem 4.1 shows now that the large time behavior of solutions to (4.9) for $q = 2$ is different.

REMARK 4.4. By the previous remark and Theorem 4.1, we see that the first order term in the asymptotic expansion of solutions (with integrable data) to (4.9) for $q \geq 2$ is the same as for the convection-diffusion equation with the analogous nonlinearity (cf. [22]). It would be interesting to know what happens with solutions to

$$u_t - \nu u_{xx} + u_{xxx} + (u^q)_x = 0$$

for $1 < q < 2$ as $t \to \infty$. Do such solutions with integrable data look (for large t) like the entropy solution of the purely convective equation

$$u_t + (u^q)_x = 0$$

with initial data $M\delta_0$? One can hope that effects of the dissipative and dispersive terms ($-\nu u_{xx}$ and u_{xxx}, resp.) disappear completely in the limit. This conjecture is based on similar results obtained for solutions to the convection-diffusion equation (cf. [20]).

The proof of Theorem 4.1 uses a standard scaling argument (cf. e.g. [15, 20, 21, 29]). For $\lambda \geq 1$, we consider the rescaled function

$$u_\lambda(x,t) \equiv \lambda u(\lambda x, \lambda^2 t), \tag{4.10}$$

where u is a solution to (1.1). Going back to (1.1) with $f(u) = u^2/2$, one easily checks that u_λ is a solution to

$$\begin{aligned} &(u_\lambda)_t - \nu(u_\lambda)_{xx} + \lambda^{-1}(u_\lambda)_{xxx} + u_\lambda(u_\lambda)_x = 0, \\ &u_\lambda(x,0) = \lambda u_0(\lambda x) \equiv u_{0,\lambda}(x). \end{aligned} \tag{4.11}$$

Now the investigation of the asymptotic behavior of the solution u can be reduced to studying the convergence of the family $\{u_\lambda\}_{\lambda \geq 1}$ as $\lambda \to \infty$, what will allow us to pass to a weak limit in (4.11). Hence, in [27], we prove some preliminary estimates, which are essential to establish the compactness of the above family. We deal with a crucial difficulty which does not appear in the case of the heat equation with either absorbing or convective term. The third order term in (1.1) causes oscillations of solutions, and the maximum principle fails. Consequently, one cannot follow reasonings from [20, 21, 29], and others based on this important tool.

Thus another argument is needed. Observe that the decay (4.1) of solutions to (1.1) gives an estimate of the family $\{u_\lambda\}_{\lambda \geq 1}$ uniform with respect to λ. Indeed, it follows immediately by a simple change of variables that

$$\|u_\lambda(\cdot,t)\|_2 = \lambda^{1/2}\|u(\cdot,\lambda^2 t)\|_2 \leq C\lambda^{1/2}(\lambda^2 t)^{-1/4} = Ct^{-1/4} \tag{4.12}$$

with C independent of λ. We obtain analogous bounds for derivatives of u_λ using (4.12) and an integral representation of solutions to (4.11). Next, classical compactness results allow us to prove Theorem 4.1 for $p = 2$, first. We are able to handle other $p \in [1, \infty]$

in (4.8) by using (4.7) and an integral representation of solutions of (4.3) (analogous to that in (3.5), with $S(t)$ replaced by $G(t)$).

5. Higher order asymptotics. Bona and Luo proved in [10] that if the initial data has the property that its Fourier transform vanishes at the origin like $|y|^\beta$, $0 \le \beta \le 1$, as $y \to 0$, then for $f(u) = u^q$ with $q \ge 2$ the decay rate of the corresponding solutions of (1.1) or (1.2) will increase by $\beta/2$. Moreover, for $\beta = 1$ they computed explicitly $\lim_{t\to\infty} t^{3/2} \int u^2(x,t)\,dx$ which appears to depend on the first moment of the initial data u_0 and on the double integral

$$\int_0^t \int_{\mathbb{R}} u^q(x,t)\,dxdt. \tag{5.1}$$

In addition, in this special case, the L^2-norm of the difference between solutions of (1.1) or (1.2) and the corresponding linear equation also has the large-time asymptotic form $Ct^{-3/4}$, where C depends on the double integral in (5.1) only (some preliminary results in this direction can be also found in [19]).

To develop the theory described above, we find in [28] the higher-order term of the asymptotic expansion in L^p as $t \to \infty$ of solutions to (1.1) and (1.2) with $f(u) = u^q$. To put this more precisely, assume that $w = w(x,t)$ is a solution to the linearized equation. We compute explicitly a function $Q = Q(x,t)$ such that $u(x,t) - w(x,t) - Q(x,t)$ tends to 0 in $L^p(\mathbb{R})$ ($p \in [1,\infty]$) as t goes to infinity, faster than $u(x,t) - w(x,t)$. Moreover, the form of $Q(x,t)$ differs for $2 < q < 3$, $q = 3$, and $q > 3$. We complete these results by deriving asymptotic formulas as $t \to \infty$ of solutions to the linearized equation.

To describe our results more precisely, let us recall that any solution of (1.1)-(1.3) satisfies the integral equation (the counterpart of (3.5))

$$u(t) = S(t) * u_0 - \int_0^t S_x(t-\tau) * u^q(\tau)\,d\tau. \tag{5.2}$$

Here $S(x,t)$ is the fundamental solution of the equation $u_t - \nu u_{xx} + u_{xxx} = 0$ given by the formula

$$S(x,t) \equiv \frac{1}{2\pi} \int_{\mathbb{R}} e^{(-\nu\xi^2 + i\xi^3)t + ix\xi} d\xi.$$

All the results proved in [28] concern solutions to the integral equation (5.2). However, one can show repeating the reasoning from [27, Thm. 2] that these solutions are, in fact, classical solutions to (1.1)-(1.3) provided q is either an integer or sufficiently large. Hence, without loss of generality, by solutions to (1.1)-(1.3), we always mean solutions to (5.2).

In [28], we impose the following basic assumptions.

1. We consider the asymptotically weak nonlinearity, i.e.
$$f(u) = u^q \quad \text{for} \quad q > 2 \tag{5.3}$$
(cf. the comments following (1.4)).
2. The function $u = u(x,t)$ is a solution of (1.1)-(1.3) satisfying the following decay estimates:
$$\|u(\cdot,t)\|_2 \le Ct^{-1/4} \quad \text{and} \quad \|u(\cdot,t)\|_\infty \le C, \tag{5.4}$$
for all $t > 0$, where the numbers C are independent of t.

REMARK 5.1. The condition in (5.4) is not particularly restrictive, and is imposed for brevity sake. Indeed, the works [19, 30, 6, 9, 8, 11] contain several conditions under which (5.4) holds true. Moreover, we construct in [28] solutions to (1.1)-(1.3) satisfying (5.4) provided u_0 is small in $L^1(\mathbb{R}) \cap H^1(\mathbb{R})$.

We are now in a position to formulate the main theorem from [28].

THEOREM 5.1 ([28]). *Assume $p \in [1, \infty]$. Suppose that u is a solution to (1.1)-(1.3) satisfying the decay estimate (5.4) with $u_0 \in L^1(\mathbb{R}) \cap H^1(\mathbb{R})$.*

i. For $2 < q < 3$ it follows that

$$t^{(q-1/p)/2-1/2}\left\| u(t) - S(t) * u_0 + \int_0^t G_x(t-\tau) * (MG(\tau))^q \, d\tau \right\|_p \to 0 \tag{5.5}$$

as $t \to \infty$.

ii. For $q = 3$ we have

$$\frac{t^{(1-1/p)/2+1/2}}{\log t} \|u(t) - S(t) * u_0 + (\log t) M^3 (4\pi\sqrt{3})^{-1} G_x(t)\|_p \to 0 \tag{5.6}$$

as $t \to \infty$.

iii. For $q > 3$ the following relation

$$t^{(1-1/p)/2+1/2}\left\| u(t) - S(t) * u_0 + \int_0^\infty \int_{\mathbb{R}} u^q(y,\tau)\, dy d\tau \, G_x(t) \right\|_p \to 0 \tag{5.7}$$

holds as $t \to \infty$.

REMARK 5.2. Since $\int_{\mathbb{R}} |u(x,\tau)|^q \, dx \le C(1+\tau)^{(1-q)/2}$ (cf. [28, Lemma 5.1]), the integral $\int_0^\infty \int u^q(y,\tau)\, dy d\tau$ converges for $q > 3$. Moreover, multiplying (1.1) by x, integrating over $\mathbb{R} \times (0,t)$, and letting $t \to \infty$, we obtain (at least formally) that (cf. [38, Remark 4])

$$\int_0^\infty \int_{\mathbb{R}} u^q(y,\tau)\, dy d\tau = \lim_{t\to\infty} \int_{\mathbb{R}} u(x,t) x \, dx - \int_{\mathbb{R}} u_0(x) x \, dx. \tag{5.8}$$

Another way of deriving (5.8) is based on (5.2), if we observe that easy computation shows that $\int_{\mathbb{R}} xS(t) * u_0(x)\, dx = \int_{\mathbb{R}} xu_0(x)\, dx$ and $\int_{\mathbb{R}} xS_x(t-\tau) * u^q(x,\tau)\, dx = \int_{\mathbb{R}} u^q(x,\tau)\, dx$.

The asymptotic expansion of $S(x,t)$ derived in [28, Sec. 3] allows us to extend Theorem 5.1.

PROPOSITION 5.2. *Assume that $u_0 \in L^1((1+|x|)\, dx)$. For every $p \in [1,\infty]$,*

$$t^{(1-1/p)/2+1/2}\|S(t) * u_0 - MG(t) + mG_x(t) + tMG_{xxx}(t)\|_p \to 0 \quad as \quad t \to \infty,$$

and

$$\frac{t^{(1-1/p)/2+1/2}}{\log t}\|S(t) * u_0 - MG(t) + mG_x(t)\|_p \to 0 \quad as \quad t \to \infty.$$

Moreover, for $2 < q < 3$,

$$t^{(q-1/p)/2-1/2}\|S(t) * u_0 - MG(t)\|_p \to 0 \quad as \quad t \to \infty.$$

REMARK 5.3. Theorem 5.1 and Proposition 5.2 may be summarized by saying that the dispersive effect (i.e. the term $tMG_{xxx}(t)$) appears in the second-order terms of the large-time behavior of solutions to (1.1)-(1.3) for $q > 3$, only. However, when $q \in (2,3]$

the first and the second-order terms of the large time behavior of solutions is the same as that of solutions to the corresponding Cauchy problem for the convection-diffusion equation $u_t - u_{xx} + (u^q)_x = 0$ (cf. [38]).

Our work [28] is closely related to the one by Bona *et al.* [12]. They use a renormalization mapping that was introduced in another context by Bricmont *et al.* [14] in order to find the asymptotic formula, as $t \to \infty$, for solutions to the equation

$$u_t + (u^q)_x - u_{xxx} + \mathcal{M}u = 0, \tag{5.9}$$

where $\widehat{(\mathcal{M}u)}(\xi) = |\xi|^{2\beta}\widehat{u}(\xi)$, $1 < 2\beta \le 2$, $q \ge 2\beta + 1$ (q is an integer). Hence our results overlap exactly with those in [12] for the generalized KdV-Burgers equation when $q \ge 3$ is an integer.

The primary new aspect of our work is that we propose a new approach to study the higher-order asymptotics of solutions to conservation laws with dispersion and dissipation. This approach is based on subtle estimates of the fundamental solution to the linearized equation and on the decay estimates of the L^p-norms of solutions to the nonlinear equations as well as on their integral representation (cf. (3.5), below). This allows us to impose conditions on the initial data (1.3) simpler than those in [12], and we does not require u_0 to be small.

Moreover, we can control the behavior of solutions in $L^p(\mathbb{R})$ for $1 \le p < 2$, which gives more information when $x/\sqrt{t}$ is large. Up to our knowledge, this is the first result of this type for solutions to conservation laws with dissipation and dispersion.

In [28], we also consider other types of dispersion. This extends the generalized KdV-Burgers equation in a different direction than was done in [12]. In fact, here we model the dissipation effects in (1.1) and (1.2) by the term $-u_{xx}$ for simplicity of the exposition only. Note, however, that a more general dissipation can be handled by our method (cf. [7]).

We also consider fractional power nonlinearities, so we replace u^q by (1.4). This enables us to say what happens when $2 < q < 3$.

Finally, another novel aspect of our work is that we also consider the generalized BBM-Burgers equation (1.2), and we find the higher order terms of the asymptotic expansion of solutions in $L^p(\mathbb{R})$ for $p \in [1, \infty]$.

We conclude this section by reviewing analogous results for other equations. Zuazua [38] was the first who found the second order term in the asymptotic expansion as $t \to \infty$ of solutions to the convection-diffusion equation $u_t - \Delta u = \vec{a} \cdot \nabla(|u|^{q-1}u)$ in $\mathbb{R}^n$. He observed that this asymptotics differs for $1 + 1/n < q < 1 + 2/n$, $q = 1 + 1/n$, and $q > 1 + 2/n$. However, the methods used by him are very specific, and can be applied neither to (1.1) nor to (1.2). Next, Carpio [16] derived an analogous result for solutions to the Cauchy problem for the incompressible Navier-Stokes equations in two and three dimensions. In that case, the semigroup of linear operators generated by the Stokes operator reduces to the classical heat semigroup which simplifies considerations. Nonlinear effects in the asymptotic expansion of small solutions to the nonlinear heat equation was observed by Wayne [35] and to a class of nonlinear Schrödinger equations by Pillet and Wayne [32].

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