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LARGE TIME BEHAVIOUR OF A CLASS OF SOLUTIONS OF SECOND ORDER CONSERVATION LAWS

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Abstract. We study the large time behaviour of entropy solutions of the Cauchy problem for a possibly degenerate nonlinear diffusion equation with a nonlinear convection term. The initial function is assumed to have bounded total variation. We prove the convergence of the solution to the entropy solution of a Riemann problem for the corresponding first order conservation law.

1. Introduction. In this paper we consider the problem

$$\text{(P)}\begin{cases} u_t + f(u)_x = \varphi(u)_{xx} & \text{in } Q = \mathbf{R} \times \mathbf{R}^+ \\ u(x,0) = u_0(x) & \text{for } x \in \mathbf{R} \end{cases}$$

under the following hypotheses on the data

(H1) φ, $f : \mathbf{R} \to \mathbf{R}$, φ is nondecreasing and continuous in $\mathbf{R}$, f is locally Lipschitz continuous in $\mathbf{R}$.

(H2) $u_0 : \mathbf{R} \to \mathbf{R}$, $u_0 \in BV(\mathbf{R})$.

Here $BV(\mathbf{R})$ denotes the set of functions of bounded total variation in $\mathbf{R}$, i.e.

$$BV(\mathbf{R}) = \{g \in L^1_{\mathrm{loc}}(\mathbf{R}) : \mathrm{TV}_{\mathbf{R}}(g) < +\infty\},$$

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where

$$\mathrm{TV}_{\mathbf{R}}(g) = \sup\left\{\int_{\mathbf{R}} g\phi'\,dx : \phi \in C_0^1(\mathbf{R}),\ \|\phi\|_{L^\infty(\mathbf{R})} \le 1\right\}$$

(see for example [GR]). We shall also consider the function space $BV(I)$, where $I \subset \mathbf{R}$ is an open interval and for which the definition is similar. Let us remark that $u_0 \in BV(\mathbf{R})$ implies that $u_0(-\infty)$ and $u_0(+\infty)$ exist in a sense of ess-limits, and that $\|u_0\|_{L^\infty(\mathbf{R})} < \infty$. We define $a = \operatorname*{ess\,lim}_{x\to-\infty} u_0(x)$ and $b = \operatorname*{ess\,lim}_{x\to\infty} u_0(x)$.

The form of the partial differential equation in Problem (P) with nonlinear convection without any convexity assumption and possibly degenerate nonlinear diffusion is natural in view of many applications. A typical example is nonlinear filtration in porous media [GM].

Problem (P) may have no classical solutions. If for example $\varphi(s) = |s|^{m-1}s$ with $m > 1$ one usually considers weak solutions of Problem (P) which are continuous in Q. If φ is not strictly increasing then the differential equation in (P) reduces to the first order conservation law

$$u_t + f(u)_x = 0 \tag{1.1}$$

in regions where $\varphi(u)$ is constant; in this case Problem (P) admits discontinuous solutions. We define solutions of Problem (P) as follows.

Definition 1.1. A function $u \in L^\infty(Q)$ is an *entropy solution* of Problem (P) if $u \in L^\infty((0,\infty); BV(\mathbf{R})) \cap C([0,\infty); L^2_{\mathrm{loc}}(\mathbf{R}))$, $\varphi(u) \in L^2_{\mathrm{loc}}([0,\infty); H^1_{\mathrm{loc}}(\mathbf{R}))$ and if u satisfies the inequality

$$\begin{aligned}\frac{\partial}{\partial t}|u-k| &+ \frac{\partial}{\partial x}(\operatorname{sign}(u-k)(f(u)-f(k)))\\ &\le \frac{\partial^2}{\partial x^2}(\operatorname{sign}(u-k)(\varphi(u)-\varphi(k))) \quad \text{in } \mathcal{D}'(Q)\end{aligned} \tag{1.2}$$

for all constants $k \in R$, together with the initial condition $u(0) = u_0$.

This definition extends the notion of entropy solution of equation (1.1) introduced by [K]. Note that if u is an entropy solution of Problem (P), then it satisfies the differential equation

$$u_t + f(u)_x = \varphi(u)_{xx} \quad \text{in } \mathcal{D}'(Q),$$

which one can check by successively setting $k = \pm\|u\|_{L^\infty(Q)}$ in (1.2).

In order to be able to state the main result of this paper, we consider the Riemann problem

$$(\mathrm{P}^\infty)\quad \begin{cases} u_t + f(u)_x = 0 & \text{in } Q\\ u(x,0) = a + (b-a)H(x) = \begin{cases} a & \text{if } x<0\\ b & \text{if } x>0;\end{cases}\end{cases}$$

where H is the Heaviside function. It is well known [K], [dB], [MNRR], [Se] that Problem (P^∞) has a unique entropy solution. We remark that the solution u^∞ of Problem (P^∞) can be written using the similarity variable $\eta = x/t$ in the form $u^\infty(x,t) = \mathcal{U}(\eta)$, where

$\mathcal{U} \in BV(\mathbf{R})$ is a distributional solution of the problem

$$\begin{cases} f(\mathcal{U})' = \eta\, \mathcal{U}' & \text{in } \mathbf{R} \\ \mathcal{U}(-\infty) = a, \quad \mathcal{U}(+\infty) = b \end{cases}$$

which satisfies the "entropy" inequality

$$(\operatorname{sign}(\mathcal{U} - k)(f(\mathcal{U}) - f(k)))' \le \eta\, |\mathcal{U} - k|' \quad \text{in } \mathcal{D}'(\mathbf{R})$$

for all $k \in \mathbf{R}$ (see for instance [Se, p. 50]).

We also consider a sequence of related uniformly parabolic problems, namely

$$(\mathrm{P}_\varepsilon^\lambda) \begin{cases} u_t + f_\varepsilon(u)_x = \frac{1}{\lambda}\varphi_\varepsilon(u)_{xx} & \text{in } Q \\ u(x,0) = u_{0\varepsilon}(\lambda x) & \text{for } x \in \mathbf{R} \end{cases}$$

where $0 < \varepsilon \le 1$, $\lambda > 0$ and the functions $u_{0\varepsilon}$, φ_ε and f_ε satisfy the Hypotheses (H_ε):

$$(\mathrm{H}_\varepsilon) \begin{cases} \text{(i)} & u_{0\varepsilon},\ \varphi_\varepsilon,\ f_\varepsilon \in C^\infty(\mathbf{R}); \\ \text{(ii)} & \varphi_\varepsilon \to \varphi,\ f_\varepsilon \to f \text{ as } \varepsilon \downarrow 0 \text{ uniformly on compact subsets of } \mathbf{R}; \\ \text{(iii)} & \varepsilon \le \varphi_\varepsilon' \le \frac{1}{\varepsilon} \text{ in } \mathbf{R}; \\ \text{(iv)} & \text{for all } R > 0 \text{ there exists } L = L(R) \text{ such that } |f_\varepsilon'| \le L(R) \text{ on } (-R,R); \\ \text{(v)} & u_{0\varepsilon} \to u_0 \text{ in } L^1_{\mathrm{loc}}(\mathbf{R}) \text{ as } \varepsilon \to 0; \\ \text{(vi)} & \operatorname{ess\,inf} u_0 \le u_{0\varepsilon} \le \operatorname{ess\,sup} u_0 \text{ in } \mathbf{R}; \\ \text{(vii)} & \displaystyle\int_{\mathbf{R}} |u_{0\varepsilon}'(x)|\, dx \le \mathrm{TV}(u_0); \\ \text{(viii)} & u_{0\varepsilon}(x) = a \text{ for } x < -\frac{1}{\varepsilon} \text{ and } u_{0\varepsilon}(x) = b \text{ for } x > \frac{1}{\varepsilon}. \end{cases}$$

The existence of functions $u_{0\varepsilon}$, φ_ε and f_ε follows from hypotheses (H1)–(H2) by a standard mollifying argument. It follows from [LSU, Chapter V, Theorem 8.1] that for any $0 < \varepsilon \le 1$, $\lambda > 0$ Problem $(\mathrm{P}_\varepsilon^\lambda)$ has a unique classical solution u_ε^λ.

Next we introduce a notion of limit entropy solution of Problem (P).

Definition 1.2. We say that an entropy solution u of Problem (P) is a *limit entropy solution* if it is the limit of a sequence of solutions $\{u_{\varepsilon_n}\}$ of the problems $(\mathrm{P}^1_{\varepsilon_n})$ such that

$$u_{\varepsilon_n} \to u \quad \text{in } C([0,T]; L^2_{\mathrm{loc}}(\mathbf{R})) \quad \text{as } \varepsilon_n \to 0.$$

We refer to Benilan and Touré [BT], Maliki and Touré [MT] and Marcati [M] for a study of semigroup solutions, entropy solutions and limit solutions of Problem (P).

The main result of this paper is the following.

Theorem 1.3. *Let u be the limit entropy solution of Problem (P). Set*

$$\widetilde{u}(\eta, t) = u(x,t). \tag{1.4}$$

Then for all $R > 0$

$$\lim_{t\to\infty} \|\widetilde{u}(\cdot,t) - \mathcal{U}\|_{L^2(-R,R)} \;=\; 0.$$

Remark 1.4. In the (x,t) variables this convergence result reads as

$$\lim_{t\to\infty} \frac{1}{2Rt} \int_{-Rt}^{Rt} |u(x,t) - \mathcal{U}(x/t)|^2\, dx \;=\; 0$$

for all $R > 0$.

In order to prove Theorem 1.3 we use a scaling technique. For all $\lambda > 0$ we set

$$u^\lambda(x,t) = u(\lambda x, \lambda t), \tag{1.5}$$

where u is the limit entropy solution of Problem (P). Then u^λ is a limit entropy solution of Problem (P^λ),

$$(\mathrm{P}^\lambda)\begin{cases} u_t + f(u)_x = \frac{1}{\lambda}\varphi(u)_{xx} & \text{in } Q \\ u(x,0) = u_0^\lambda(x) = u_0(\lambda x) & \text{for } x \in \mathbf{R}, \end{cases}$$

where a limit entropy solution u^λ of Problem (P^λ) is defined in a similar way as in Definition 1.2. Theorem 1.3 is the consequence of the following convergence result.

THEOREM 1.5. *Let $\{u^\lambda\}_{\lambda\ge 1}$ be limit entropy solutions of Problem (P^λ). Then, for any $T > 0$,*

$$u^\lambda \to u^\infty \quad \text{in } C([0,T]; L^2_{\mathrm{loc}}(\mathbf{R}))$$

as $\lambda \to \infty$, where u^∞ is the entropy solution of Problem (P^∞).

Indeed it follows from (1.4), (1.5) and Theorem 1.5 that for all $R > 0$

$$\int_{-R}^{R} |u^\lambda(y,1) - \mathcal{U}(y)|^2\,dy = \int_{-R}^{R} |\widetilde{u}(\eta,\lambda) - \mathcal{U}(\eta)|^2\,d\eta \to 0 \quad \text{as } \lambda \to \infty,$$

which is precisely the result stated in Theorem 1.3.

The large time behaviour of solutions of Problem (P) has been studied for a long time under various assumptions on f, φ and u_0. We refer to [IO2] and [W] for a historical review and an extensive list of references contained therein. Results related to presented here were obtained by Il'in and Oleinik [IO1], [IO2] in the case that $\varphi(u) = \varepsilon u$, with $\varepsilon > 0$ and $f'' > 0$ and by Weinberger [W] with the hypotheses that the differential equation in Problem (P) is uniformly parabolic and that f'' is continuous and only has isolated zeros. Van Duijn and de Graaf [vDdG] also examined a similar problem for a degenerate parabolic equation in the case of power type nonlinearities for the functions φ and f. Most of the methods of proof used in those papers are based on maximum principle arguments; here we present an approach based on a scaling method together with energy type estimates. This approach enables us to obtain a unified description of the limiting profile as $t \to \infty$ of solutions of Problem (P), without standard distinguishing between convexity and concavity of the convection function f. We also refer to [BGH] for a short note about these results. In a forthcoming article we will extend the results that we present here to the case of higher space dimension.

The organization of this paper is as follows. In Section 2 we prove a priori estimates for the solutions of Problems ($\mathrm{P}^\lambda_\varepsilon$). In Section 3 we deduce from these estimates both the existence of an entropy solution u^λ of Problem (P^λ) and the convergence of u^λ to the function u^∞ as $\lambda \to \infty$.

2. A priori estimates. In this section in a series of lemmas we derive a priori estimates for the solutions u^λ_ε of Problems ($\mathrm{P}^\lambda_\varepsilon$), with $\lambda \ge 1$.

LEMMA 2.1.

$$\operatorname{ess\,inf} u_0 \le u_\varepsilon^\lambda \le \operatorname{ess\,sup} u_0 \qquad \text{in } Q. \tag{2.1}$$

PROOF. This result follows from Hypothesis (H_ε) (vi) and applying the standard maximum principle. ■

LEMMA 2.2. *Let $0 < \varepsilon \le 1$, $\lambda \ge 1$ and $T > 0$ be fixed. Then*

$$u_\varepsilon^\lambda - a - (b-a)H(x),\ u_{\varepsilon x}^\lambda,\ u_{\varepsilon xx}^\lambda = O(e^{-|x|}) \qquad \text{as } |x| \to \infty,$$

uniformly in $[0,T]$.

PROOF. (i) We first prove that

$$u_\varepsilon^\lambda - b = O(e^{-x}) \quad \text{as } x \to +\infty,$$

uniformly in $[0,T]$. Set $M = \|u_0\|_\infty$. Then, by (2.1),

$$-M \le u_\varepsilon^\lambda \le M \qquad \text{in } Q.$$

We compare u_ε^λ with the function

$$\underline{\omega}(x,t) = b - \gamma e^{-x+Kt}$$

in the set $S_{A,K} = \{(x,t):\ x \ge A + Kt,\ t \ge 0\}$ for some γ, A, $K > 0$. If we choose $\gamma = (b+M)e^A$ then

$$\underline{\omega}(A+Kt,t) = -M$$

for $t \ge 0$. Furthermore, if $A = \frac{1}{\varepsilon}$ then by (H_ε)(viii)

$$\underline{\omega}(x,0) = b - \gamma e^{-x} \le u_{0\varepsilon}(x)$$

for $x \in [A,\infty)$. Finally, for $K = K_\varepsilon$ large enough we have

$$\begin{aligned} &\underline{\omega}_t - \varphi_\varepsilon'(\underline{\omega})\underline{\omega}_{xx} - \varphi_\varepsilon''(\underline{\omega})\underline{\omega}_x^2 + f_\varepsilon'(\underline{\omega})\underline{\omega}_x = \\ &\qquad \gamma e^{-x+Kt}[-K + \varphi_\varepsilon'(\underline{\omega}) - \gamma e^{-x+Kt}\varphi_\varepsilon''(\underline{\omega}) + f_\varepsilon'(\underline{\omega})] \le 0 \end{aligned}$$

in $S_{A,K}$. Hence, by the maximum principle $\underline{\omega}(x,t) \le u_\varepsilon^\lambda$ in $S_{A,K}$ so that

$$-\gamma e^{KT-x} \le u_\varepsilon^\lambda - b$$

for $x \ge A + Kt$ and $t \in [0,T]$. Similarly, comparing u_ε^λ with the function of the form

$$\overline{\omega}(x,t) = b + \gamma_1 e^{-x+K_1 t}$$

in S_{A,K_1} for some γ_1, $K_1 > 0$ and A as before leads to

$$u_\varepsilon^\lambda - b \le \gamma e^{K_1 T - x}$$

for $x \ge A + K_1 t$ and $t \in [0,T]$.

The proof that $u_\varepsilon^\lambda - a = O(e^{-|x|})$ as $x \to -\infty$ uniformly in $[0,T]$ is similar.

(ii) In order to prove that

$$u_{\varepsilon x}^\lambda = O(e^{-x}) \quad \text{as } x \to +\infty$$

uniformly in $[0,T]$ we observe that $p = u_{\varepsilon x}^\lambda$ satisfies

$$\begin{aligned} p_t &= (\varphi_\varepsilon'(u_\varepsilon^\lambda)p_x + \varphi_\varepsilon''(u_\varepsilon^\lambda)p^2 - f_\varepsilon'(u_\varepsilon^\lambda)p)_x \\ &= \varphi_\varepsilon'(u_\varepsilon^\lambda)p_{xx} + 3\varphi_\varepsilon''(u_\varepsilon^\lambda)u_{\varepsilon x}^\lambda p_x + \varphi_\varepsilon'''(u_\varepsilon^\lambda)(u_{\varepsilon x}^\lambda)^2 p - f_\varepsilon'(u_\varepsilon^\lambda)p_x - f_\varepsilon''(u_\varepsilon^\lambda)u_{\varepsilon x}^\lambda p, \end{aligned}$$

and moreover $|p| \le M_\varepsilon$ in $\mathbf{R} \times [0,T]$ and, by (H_ε)(viii), $p(x,0)=0$ for $x > \frac{1}{\varepsilon}$. Thus we can compare p with functions

$$\omega(x,t) = \pm\gamma e^{-x+Kt}$$

in $S_{A,K}$ for γ, $K > 0$ and $A = \frac{1}{\varepsilon}$.

The proof that $u^\lambda_{\varepsilon x} = O(e^{-|x|})$ as $x \to -\infty$ uniformly in $[0,T]$ is similar.

(iii) The proof that $u^\lambda_{\varepsilon xx} = O(e^{-|x|})$ as $x \to \pm\infty$ uniformly in $[0,T]$ is similar to the proof given in (ii). ∎

LEMMA 2.3. *For all* $t \ge 0$,

$$\int_{\mathbf{R}} |u^\lambda_{\varepsilon x}(x,t)|dx \le \int_{\mathbf{R}} |u'_{0\varepsilon}(x)|dx \le \mathrm{TV}(u_0). \tag{2.2}$$

PROOF. For the sake of simplicity we use the notations u and u_0 instead of u^λ_ε and $u_{0\varepsilon}$ respectively. To begin with we differentiate the the differential equation in Problem ($\mathrm{P}^\lambda_\varepsilon$) with respect to x, multiply the resulting equation by $\operatorname{sign} u_x = \operatorname{sign}(\varphi'_\varepsilon(u)u_x)$ and integrate over $Q_{R,T}$ for fixed R and $T>0$. This leads to

$$\iint_{Q_{R,T}} u_{xt} \operatorname{sign} u_x + \iint_{Q_{R,T}} f_\varepsilon(u)_{xx} \operatorname{sign} u_x = \frac{1}{\lambda} \iint_{Q_{R,T}} \varphi_\varepsilon(u)_{xxx} \operatorname{sign} u_x. \tag{2.3}$$

We show below that

$$\iint_{Q_{R,T}} u_{xt} \operatorname{sign} u_x \, dxdt = \int_{-R}^{R} |u_x| \Big|_0^T dx, \tag{2.4}$$

$$\iint_{Q_{R,T}} f_\varepsilon(u)_{xx} \operatorname{sign} u_x \, dxdt = \int_0^T f'_\varepsilon(u)|u_x| \Big|_{-R}^{R} dt, \tag{2.5}$$

$$\iint_{Q_{R,T}} \varphi_\varepsilon(u)_{xxx} \operatorname{sign} u_x \, dxdt \le \int_0^T (\varphi'_\varepsilon(u)u_x)_x \operatorname{sign}(\varphi'_\varepsilon(u)u_x) \Big|_{-R}^{R} dt. \tag{2.6}$$

In order to prove (2.4)-(2.6) we use a sequence of smooth approximations $\{S_\delta\}_{\delta>0}$ of the sign function and set $M_\delta(w) = \int_0^w S_\delta(s)\,ds$ for $w \in \mathbf{R}$. Then $M_\delta(w) \to |w|$ as $\delta \to 0$. We have that

$$\iint_{Q_{R,T}} S_\delta(u_x)u_{xt} = \iint_{Q_{R,T}} (M_\delta(u_x))_t = \int_{-R}^{R} M_\delta(u_x) \Big|_0^T dx,$$

where we let $\delta \to 0$ to obtain (2.4).

In order to prove (2.5) we observe that

$$\begin{aligned}(f'_\varepsilon(u)u_x)_x S_\delta(u_x) &= (f'_\varepsilon(u))_x u_x S_\delta(u_x) + f'_\varepsilon(u)(M_\delta(u_x))_x \\ &= (f'_\varepsilon(u)M_\delta(u_x))_x + (f'_\varepsilon(u))_x[u_x S_\delta(u_x) - M_\delta(u_x)],\end{aligned}$$

which implies that

$$\begin{aligned}\iint_{Q_{R,T}} f_\varepsilon(u)_{xx} S_\delta(u_x) &= \iint_{Q_{R,T}} (f'_\varepsilon(u)M_\delta(u_x))_x \quad + \quad J(\delta) \\ &= \int_0^T f'_\varepsilon(u)M_\delta(u_x) \Big|_{-R}^{R} dt \quad + \quad J(\delta),\end{aligned} \tag{2.7}$$

where

$$J(\delta) = \int\int_{Q_{R,T}} (f'_\varepsilon(u))_x [u_x S_\delta(u_x) - M_\delta(u_x)].$$

Since $J(\delta) \to 0$ as $\delta \to 0$ we obtain (2.5) by letting $\delta \to 0$ in (2.7).

Finally we prove (2.6). We have that

$$\begin{aligned}
(2.8) \quad & \int\int_{Q_{R,T}} \varphi_\varepsilon(u)_{xxx} S_\delta(u_x) \\
&= \int\int_{Q_{R,T}} \varphi_\varepsilon(u)_{xxx}[S_\delta(u_x) - S_\delta(\varphi'_\varepsilon(u)u_x)] + \int\int_{Q_{R,T}} (\varphi'_\varepsilon(u)u_x)_{xx} S_\delta(\varphi'_\varepsilon(u)u_x) \\
&= I_1(\delta) + I_2(\delta),
\end{aligned}$$

and remark that since $\varphi'_\varepsilon > 0$ then $I_1(\delta) \to 0$ as $\delta \to 0$. Next we estimate $I_2(\delta)$. We have that

$$\begin{aligned}
(2.9) \quad & I_2(\delta) = \\
& \int_0^T (\varphi'_\varepsilon(u)u_x)_x S_\delta(\varphi'_\varepsilon(u)u_x)\Big|_{-R}^{R} dt - \int\int_{Q_{R,T}} [(\varphi'_\varepsilon(u)u_x)_x]^2 S'_\delta(\varphi'_\varepsilon(u)u_x)\, dxdt \\
& \le \int_0^T (\varphi'_\varepsilon(u)u_x)_x S_\delta(\varphi'_\varepsilon(u)u_x)\Big|_{-R}^{R} dt.
\end{aligned}$$

Substituting (2.9) into (2.8) and letting $\delta \to 0$ we obtain (2.6).

Now it follows from (2.3)-(2.6) that

$$\begin{aligned}
& \int_{-R}^{R} |u^\lambda_{\varepsilon x}(x,T)|\, dx - \lambda \int_{-R}^{R} |u'_{0\varepsilon}(\lambda x)|\, dx \le \\
& \int_0^T (\varphi'_\varepsilon(u_\varepsilon)u^\lambda_{\varepsilon x})_x \operatorname{sign}(\varphi'_\varepsilon(u_\varepsilon)u^\lambda_{\varepsilon x})\Big|_{-R}^{R} dt + \int_0^T f'_\varepsilon(u_\varepsilon)|u^\lambda_{\varepsilon x}|\Big|_{-R}^{R} dt
\end{aligned}$$

for all $R, T > 0$. Hence, by Lemma 2.2, in the limit as $R \to \infty$

$$\int_{\mathbf{R}} |u^\lambda_{\varepsilon x}(x,T)|\, dx - \lambda \int_{\mathbf{R}} |u'_0(\lambda x)|\, dx \le 0,$$

which yields (2.2) by (H_ε)(vii). ∎

Lemma 2.4. *There exists a positive constant $C = C(R,T)$ such that*

$$(2.10) \qquad \|f_\varepsilon(u^\lambda_\varepsilon)_x\|_{L^2((0,T);H^{-1}(-R,R))} \le C.$$

Proof. Here again we omit the lower index ε and the upper index λ from the notation. Let $R > 0$ and $\zeta \in C_0^\infty(-R,R)$. We have that

$$\langle f_\varepsilon(u)_x(\cdot,t), \zeta\rangle = \int_{-R}^{R} f_\varepsilon(u)_x(x,t)\zeta(x)\, dx = -\int_{-R}^{R} f_\varepsilon(u)(x,t)\zeta'(x)\, dx,$$

which imply that

$$\begin{aligned}
|\langle f_\varepsilon(u)_x(\cdot,t), \zeta\rangle| &\le \left(\int_{-R}^{R} |f_\varepsilon(u)(x,t)|^2\, dx\right)^{1/2} \left(\int_{-R}^{R} |\zeta'(x)|^2\, dx\right)^{1/2} \\
&\le \left(\int_{-R}^{R} |f_\varepsilon(u)(x,t)|^2\, dx\right)^{1/2} \|\zeta\|_{H_0^1(-R,R)}
\end{aligned}$$

for all $t \in [0,T]$. Hence

$$\|f_\varepsilon(u)_x(\cdot,t)\|_{H^{-1}(-R,R)} \le \left(\int_{-R}^{R} |f_\varepsilon(u)(x,t)|^2\,dx\right)^{1/2}$$

for all $t \in [0,T]$ and consequently by (H_ε)(ii) and Lemma 2.1

$$\int_0^T \|f_\varepsilon(u)_x(\cdot,t)\|^2_{H^{-1}(-R,R)}\,dt \le \iint_{Q_{R,T}} |f_\varepsilon(u)|^2 \le C$$

for some positive constant $C = C(R,T)$. ■

LEMMA 2.5. *There exists a positive constant $C = C(R,T)$ such that*

$$\|\varphi_\varepsilon(u_\varepsilon^\lambda)_x\|_{L^2((-R,R)\times(0,T))} \le C\sqrt{\lambda}. \tag{2.11}$$

PROOF. For simplicity we write u and u_0 instead of u_ε^λ and $u_{0\varepsilon}$ respectively. Let $R > 0$ and ψ be a smooth function such that

$$\psi(x) = \begin{cases} 1 & \text{if } |x| \le R \\ 0 & \text{if } |x| \ge R+1. \end{cases}$$

We multiply the differential equation in Problem $(\mathrm{P}_\varepsilon^\lambda)$ by $\varphi_\varepsilon(u)\psi^2$ and write the resulting equality as

$$\Phi_\varepsilon(u)_t\psi^2 + \Psi_\varepsilon(u)_x\psi^2 = \frac{1}{\lambda}\varphi_\varepsilon(u)_{xx}\varphi_\varepsilon(u)\psi^2,$$

where we have set $\Phi_\varepsilon(u) = \int_0^u \varphi_\varepsilon(s)\,ds$ and $\Psi_\varepsilon(u) = \int_0^u f'_\varepsilon(s)\varphi_\varepsilon(s)\,ds$. Integrating by parts on the domain $Q_{R+1,T} = (-R-1,R+1)\times(0,T)$ gives

$$\int_{-(R+1)}^{R+1} (\Phi_\varepsilon(u(x,T)) - \Phi_\varepsilon(u_0(x)))\psi^2(x)\,dx - \iint_{Q_{R+1,T}} \Psi_\varepsilon(u)(\psi^2)'$$
$$= -\frac{1}{\lambda}\iint_{Q_{R+1,T}} (\varphi_\varepsilon(u)_x)^2\psi^2 - \frac{2}{\lambda}\iint_{Q_{R+1,T}} \varphi_\varepsilon(u)_x\varphi_\varepsilon(u)\psi\psi'.$$

Applying the Cauchy-Schwarz inequality to the second term of the right-hand side of the equality above gives

$$\int_{-(R+1)}^{R+1} (\Phi_\varepsilon(u(x,T)) - \Phi_\varepsilon(u_0))\psi^2(x)\,dx - \iint_{Q_{R+1,T}} \Psi_\varepsilon(u)(\psi^2)'$$
$$\le -\frac{1}{2\lambda}\iint_{Q_{R+1,T}} (\varphi_\varepsilon(u)_x)^2\psi^2 + \frac{2}{\lambda}\iint_{Q_{R+1,T}} (\varphi_\varepsilon(u))^2(\psi')^2.$$

Therefore, in view of (H_ε) and Lemma 2.1

$$\frac{1}{\lambda}\iint_{Q_{R,T}} (\varphi_\varepsilon(u)_x)^2 \le C_1$$

where the positive constant $C_1 = C_1(R,T)$ does not depend on ε and λ. ■

COROLLARY 2.6.

$$\|\varphi_\varepsilon(u_\varepsilon^\lambda)_{xx}\|_{L^2((0,T);H^{-1}(-R,R))} \le C\sqrt{\lambda}. \tag{2.12}$$

PROOF. As in the proof of Lemma 2.5, we omit the lower index ε and the upper index λ from the notation. Let $R > 0$, $\zeta \in C_0^\infty(-R,R)$ and $t \in [0,T]$; we have that

$$\langle \varphi_\varepsilon(u)_{xx}(\cdot,t),\zeta\rangle = \int_{-R}^{R} \varphi_\varepsilon(u)_{xx}(x,t)\zeta(x)\,dx = -\int_{-R}^{R} \varphi_\varepsilon(u)_x(x,t)\zeta'(x)\,dx$$

so that

$$\begin{aligned}|\langle \varphi_\varepsilon(u)_{xx}(\cdot,t),\zeta\rangle| &\le \left(\int_{-R}^{R} |\varphi_\varepsilon(u)_x(x,t)|^2\,dx\right)^{1/2}\left(\int_{-R}^{R} |\zeta'(x)|^2\,dx\right)^{1/2}\\ &\le \left(\int_{-R}^{R} |\varphi_\varepsilon(u)_x(x,t)|^2\,dx\right)^{1/2}\|\zeta\|_{H_0^1(-R,R)}\end{aligned}$$

for all $t \in [0,T]$. Hence

$$\|\varphi_\varepsilon(u)_{xx}(\cdot,t)\|_{H^{-1}(-R,R)} \le \left(\int_{-R}^{R} |\varphi_\varepsilon(u)_x(x,t)|^2\,dx\right)^{1/2}$$

for all $t \in [0,T]$. In view of Lemma 2.5 we obtain

$$\int_0^T \|\varphi_\varepsilon(u)_{xx}(\cdot,t)\|^2_{H^{-1}(-R,R)}\,dt \le \iint_{Q_{R,T}} |\varphi_\varepsilon(u)_x|^2 \le C\lambda$$

for some positive constant $C = C(R,T)$. ■

We end this section with the following compactness result.

LEMMA 2.7. *Let $R > 0$. The set $\{u_\varepsilon^\lambda\}_{\varepsilon>0,\lambda>1}$ is precompact in $C([0,T];L^2(-R,R))$.*

PROOF. It follows from (2.2) and (2.11) that

$$\|(u_\varepsilon^\lambda)\|_{L^\infty((0,T);W^{1,1}(-R,R))} \le C(R,T), \tag{2.13}$$

while by (2.10), (2.12) and the differential equation of $(\mathrm{P}_\varepsilon^\lambda)$,

$$\|(u_\varepsilon^\lambda)_t\|_{L^2((0,T);H^{-1}(-R,R))} \le C(R,T) \tag{2.14}$$

for some positive constant $C(R,T)$. The result then follows from the embeddings

$$W^{1,1}(-R,R) \subset L^2(-R,R) \subset H^{-1}(-R,R),$$

the compactness of the embedding $W^{1,1}(-R,R) \subset L^2(-R,R)$, and a compactness result due to Simon [Si] (Corollary 4, p. 85). ■

3. Existence and asymptotic behaviour of limit entropy solutions of Problem (P^λ) as $\lambda \to \infty$

DEFINITION 3.1. We say that a function u^λ is an *entropy solution* of Problem (P^λ) if it satisfies Definition 1.1 with φ replaced by $(1/\lambda)\varphi$. A limit entropy solution of Problem (P^λ) is then defined as in Definition 1.2.

We begin with the following lemma.

LEMMA 3.2. *Let $0 < \varepsilon \le 1$ and $\lambda \ge 1$ be fixed and let u_ε^λ be the classical solution of Problem (P_ε^λ). Then u_ε^λ satisfies the inequality*

$$\frac{\partial}{\partial t}|u_\varepsilon^\lambda - k| + \frac{\partial}{\partial x}(\operatorname{sign}(u_\varepsilon^\lambda - k)(f_\varepsilon(u_\varepsilon^\lambda) - f_\varepsilon(k)) \le \frac{1}{\lambda}\frac{\partial^2}{\partial x^2}(\operatorname{sign}(u_\varepsilon^\lambda - k)(\varphi_\varepsilon(u_\varepsilon^\lambda) - \varphi_\varepsilon(k))) \tag{3.1}$$

in $\mathcal{D}'(Q)$ *for all* $k \in \mathbf{R}$.

PROOF. As in the proofs above we write u instead of u_ε^λ. Let $k \in \mathbf{R}$. Multiplying the differential equation in Problem $(\mathrm{P}_\varepsilon^\lambda)$ by $S_\delta(u-k)$ gives

$$u_t S_\delta(u-k) + f_\varepsilon(u)_x S_\delta(u-k) = \frac{1}{\lambda}\varphi_\varepsilon(u)_{xx} S_\delta(u-k) \tag{3.2}$$

in Q. Set

$$F_\varepsilon^\delta(w) = \int_k^w f_\varepsilon'(s) S_\delta(s-k)\,ds\,.$$

Then

$$u_t S_\delta(u-k) = (M_\delta(u-k))_t, \tag{3.3}$$

$$f_\varepsilon(u)_x S_\delta(u-k) = (F_\varepsilon^\delta(u))_x, \tag{3.4}$$

and

$$\varphi_\varepsilon(u)_{xx} S_\delta(u-k) = (\varphi_\varepsilon(u)_x S_\delta(u-k))_x - (\varphi_\varepsilon(u)_x S_\delta'(u-k) u_x \le (\varphi_\varepsilon(u)_x S_\delta(u-k))_x, \tag{3.5}$$

since $(\varphi_\varepsilon(u)_x S_\delta'(u-k) u_x \ge 0$. Set

$$G_\varepsilon^\delta(w) = \int_k^w \varphi_\varepsilon'(s) S_\delta(s-k)\,ds.$$

Then $(G_\varepsilon^\delta(u))_{xx} = (\varphi_\varepsilon(u)_x S_\delta(u-k))_x$ and therefore combining (3.2)-(3.5) we obtain

$$(M_\delta(u-k))_t + (F_\varepsilon^\delta(u))_x \le (G_\varepsilon^\delta(u))_{xx}.$$

Letting $\delta \to 0$ gives

$$\frac{\partial}{\partial t}|u-k| + \frac{\partial}{\partial x}F_\varepsilon(u) \le \frac{1}{\lambda}\frac{\partial}{\partial x^2}G_\varepsilon(u) \qquad \text{in } \mathcal{D}'(Q),$$

where we use the notations

$$F_\varepsilon(w) = \int_k^w f_\varepsilon'(s)\operatorname{sign}(s-k)\,ds, \qquad G_\varepsilon(w) = \int_k^w \varphi_\varepsilon'(s)\operatorname{sign}(s-k)\,ds.$$

But

$$G_\varepsilon(w) = \begin{cases} \varphi_\varepsilon(k) - \varphi_\varepsilon(w) & \text{if } k > w \\ \varphi_\varepsilon(w) - \varphi_\varepsilon(k) & \text{if } k < w \\ 0 & \text{if } k = w. \end{cases}$$

Thus $G_\varepsilon(w) = \operatorname{sign}(w-k)(\varphi_\varepsilon(w) - \varphi_\varepsilon(k))$. Similarly $F_\varepsilon(w) = \operatorname{sign}(w-k)(f_\varepsilon(w) - f_\varepsilon(k))$. Therefore u satisfies (3.1). ∎

Next we prove the existence of a limit entropy solution of Problem (P^λ) with properties which we use later on.

THEOREM 3.3. *Let* $\lambda \ge 1$ *be fixed and let* $\{u_\varepsilon^\lambda\}_{0<\varepsilon\le 1}$ *be the classical solutions of Problems* (P_ε^λ)*. There exists a sequence* $\{\varepsilon_n\}$ *and a function* $u^\lambda \in L^\infty(Q)$ *such that*

$$u_{\varepsilon_n}^\lambda \to u^\lambda \qquad \textit{in } C([0,T]; L^2(-R,R)) \textit{ as } \varepsilon_n \to 0,$$

for all R and $T > 0$. The function u^λ is an entropy solution of Problem (P^λ) and satisfies the following estimates:

(i) $\operatorname{ess\,inf} u_0 \le u^\lambda \le \operatorname{ess\,sup} u_0$ *a.e. in Q;*

(ii) $\|\varphi(u^\lambda)_x\|_{L^2((-R,R)\times(0,T))} \le C\sqrt{\lambda}$;

(iii) $\mathrm{TV}(u^\lambda(\cdot,t)) \le \mathrm{TV}(u_0)$ *for a.e. $t \in (0,\infty)$;*

(iv) $\|u_t^\lambda\|_{L^2((0,T);H^{-1}(-R,R))} \le C$,

where the positive constant C only depends on R and T.

Proof. Let $\lambda \ge 1$. We deduce from Lemma 2.7 that there exists a sequence $\varepsilon_n \to 0$ and a function $u^\lambda \in C([0,\infty); L^2_{\mathrm{loc}}(\mathbf{R}))$ such that as $\varepsilon_n \to 0$

$$u^\lambda_{\varepsilon_n} \to u^\lambda \quad \text{in } C([0,T]; L^2(-R,R)) \text{ and a.e. in } Q, \tag{3.6}$$

for all $R > 0$ and all $T > 0$. The assertions (i)-(iv) are consequences of (2.1), (2.2), (2.11), and (2.14), and of the lower semicontinuity of total variation ([EG], [GR]). Observe that by (H_ε) (ii) and (3.6) as $\varepsilon_n \to 0$,

$$\operatorname{sign}(u^\lambda_{\varepsilon_n} - k) \to \operatorname{sign}(u^\lambda - k) \tag{3.7}$$

a.e. in $Q \cap \{(x,t):\ u^\lambda - k \ne 0\}$ and

$$f_{\varepsilon_n}(u^\lambda_{\varepsilon_n}) - f_{\varepsilon_n}(k) \to f(u^\lambda) - f(k), \tag{3.8}$$

$$\varphi_{\varepsilon_n}(u^\lambda_{\varepsilon_n}) - \varphi_{\varepsilon_n}(k) \to \varphi(u^\lambda) - \varphi(k) \tag{3.9}$$

a.e. in Q. Then, letting ε_n tend to zero in an integrated form of inequality (3.1) and using (2.1), (3.7) - (3.9) and Lebesgue's dominated convergence theorem, one deduces that u^λ satisfies the inequality

$$\begin{aligned}\frac{\partial}{\partial t}|u^\lambda - k| &+ \frac{\partial}{\partial x}(\operatorname{sign}(u^\lambda - k)(f(u^\lambda) - f(k))) \\ &\le \frac{1}{\lambda}\frac{\partial^2}{\partial x^2}(\operatorname{sign}(u^\lambda - k)(\varphi(u^\lambda) - \varphi(k))) \quad \text{in } \mathcal{D}'(Q)\end{aligned}$$

for all constants $k \in R$. Furthermore it follows from (H_ε)(v) and from (3.6) that u^λ satisfies the initial condition $u^\lambda(0) = u_0$. Thus u^λ is a limit entropy solution of Problem (P^λ). ∎

Corollary 3.4. *Let $\lambda \ge 1$ and let u^λ be a limit entropy solution of Problem (P^λ). Then the statements (i) - (iv) of Theorem 3.3 hold for u^λ.*

Proof. This is an immediate consequence of the definition of the limit entropy solution of Problem (P^λ) and of Theorem 3.3. ∎

Before proving Theorem 1.5 we give the definition of an entropy solution of Problem (P^∞).

Definition 3.5. A function $u \in L^\infty(Q) \cap C([0,\infty); L^1_{\mathrm{loc}}(\mathbf{R}))$ is an *entropy solution of Problem (P^∞)* if it satisfies the entropy inequality

$$\frac{\partial}{\partial t}|u - k| + \frac{\partial}{\partial x}(\operatorname{sign}(u - k)(f(u) - f(k))) \le 0 \tag{3.10}$$

in $\mathcal{D}'(Q)$ for all constants $k \in R$, together with the initial condition $u(0) = u_0$.

Proof of Theorem 1.5. Let $\lambda > 1$, $R > 0$, $T > 0$ and let u^λ be a limit entropy solution of Problem (P^λ). We deduce from Corollary 3.4, Theorem 3.3 (iii), (iv), the embeddings

$$BV(-R,R) \subset L^2(-R,R) \subset H^{-1}(-R,R),$$

the compactness of the imbedding

$$BV(-R,R) \subset L^2(-R,R)$$

which we shall prove in the Appendix and Corollary 4 p. 85 of [Si] that the set $\{u^\lambda\}_{\lambda>1}$ is precompact in $C([0,T];L^2(-R,R))$. Hence there exists a sequence $\lambda_n \to \infty$ and a function $u^\infty \in C([0,\infty);L^2_{\mathrm{loc}}(\mathbf{R}))$ such that for all $R > 0$ and $T > 0$

$$u^{\lambda_n} \to u^\infty \tag{3.11}$$

in $C([0,T];L^2(-R,R))$ and a.e. in $Q_{R,T}$ as $n \to \infty$. It then follows from Theorem 3.3 and Corollary 3.4 that $u^\infty \in L^\infty(Q) \cap L^\infty((0,\infty);BV(\mathbf{R}))$. Finally, similarly as it has been done in the proof of Theorem 3.3 one can prove that u^∞ satisfies the entropy inequality (3.10). Thus u^∞ is an entropy solution of Problem (P^∞).

Now as a consequence of (3.11) and the uniqueness of the entropy solution of Problem (P^∞) ([K], [dB]) we obtain that for all $R > 0$ and $T > 0$

$$u^\lambda \to u^\infty \quad \text{in } C([0,T];L^2(-R,R)) \quad \text{as } \lambda \to \infty.$$

This completes the proof of Theorem 1.5. ■

4. Appendix. We shall prove the following lemma.

LEMMA A.1. *Let $R > 0$. Then for any $p \ge 1$, $BV(-R,R)$ is compactly embedded in $L^p(-R,R)$.*

PROOF. Since this result is well known for $p = 1$ we prove it for $p > 1$. Let $\{g_n\}_{n=1}^\infty \subset BV(-R,R)$ be such that

$$\|g_n\|_{BV(-R,R)} = \|g_n\|_{L^1(-R,R)} + \mathrm{TV}_{(-R,R)}(g_n) \le M \tag{A.1}$$

for all $n \ge 1$ and for some constant $M > 0$. We first prove that $\{g_n\}_{n=1}^\infty$ is uniformly bounded in $L^\infty(-R,R)$ (the proof is almost a facsimile of the proof of Claim 3, p. 218 in [EG]). Fix $n \ge 1$ and choose $\{g_{nj}\}_{j=1}^\infty \subset BV(-R,R) \cap C^\infty(-R,R)$ such that as $j \to \infty$,

$$g_{nj} \to g_n \quad \text{in } L^1(-R,R) \text{ and a.e. in } (-R,R)$$

and

$$\int_{-R}^{R} |g'_{nj}|\,dx \to \mathrm{TV}_{(-R,R)}(g_n).$$

For each $y, z \in (-R,R)$ we have that

$$g_{nj}(z) = g_{nj}(y) + \int_y^z g'_{nj}(x)\,dx.$$

Averaging with respect to $y \in (-R,R)$ gives

$$|g_{nj}(z)| \le 1/(2R)\int_{-R}^{R} |g_{nj}(y)|\,dy + \int_{-R}^{R} |g'_{nj}(x)|\,dx$$

and hence for j large enough,

$$\|g_{nj}\|_{L^\infty(-R,R)} \le C\|g_{nj}\|_{BV(-R,R)},$$

where the constant C does not depend on n and j. Taking the limit $j \to \infty$ yields

$$\|g_n\|_{L^\infty(-R,R)} \le CM. \tag{A.2}$$

Now, by (A.1), (A.2) and the compactness theorem in [EG] p. 176, there exist a sequence $n_k \to \infty$ and a function $g \in L^\infty(-R,R)$ such that as $k \to \infty$,

$$g_{n_k} \to g \quad \text{in } L^1(-R,R) \text{ and a.e. in } (-R,R).$$

Since

$$\int_{-R}^{R} |g_{n_k} - g|^p\, dx \le \sup_{(-R,R)} |g_{n_k} - g|^{p-1} \int_{-R}^{R} |g_{n_k} - g|\, dx \le (2CM)^{p-1} \int_{-R}^{R} |g_{n_k} - g|\, dx,$$

the result follows. ∎

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