LARGE TIME BEHAVIOUR OF A CLASS OF SOLUTIONS OF SECOND ORDER CONSERVATION LAWS

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Abstract. We study the large time behaviour of entropy solutions of the Cauchy problem for a possibly degenerate nonlinear diffusion equation with a nonlinear convection term. The initial function is assumed to have bounded total variation. We prove the convergence of the solution to the entropy solution of a Riemann problem for the corresponding first order conservation law.

1. Introduction. In this paper we consider the problem

\[
\begin{align*}
\begin{cases}
    u_t + f(u)x &= \varphi'(u)xx \\
    u(x,0) &= u_0(x)
\end{cases}
\end{align*}
\]

in \( Q = \mathbb{R} \times \mathbb{R}^+ \)

under the following hypotheses on the data

(H1) \( \varphi, f : \mathbb{R} \rightarrow \mathbb{R} \), \( \varphi \) is nondecreasing and continuous in \( \mathbb{R} \), \( f \) is locally Lipschitz continuous in \( \mathbb{R} \).

(H2) \( u_0 : \mathbb{R} \rightarrow \mathbb{R} \), \( u_0 \in BV(\mathbb{R}) \).

Here \( BV(\mathbb{R}) \) denotes the set of functions of bounded total variation in \( \mathbb{R} \), i.e.

\[
BV(\mathbb{R}) = \{ g \in L^1_{\text{loc}}(\mathbb{R}) : TV(\mathbb{R}) < +\infty \},
\]

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[119]
where

$$TV_R(g) = \sup \left\{ \int_R g\phi' \, dx : \phi \in C_0^1(R), \|\phi\|_{L^\infty(R)} \leq 1 \right\}$$

(see for example [GR]). We shall also consider the function space $BV(I)$, where $I \subset R$ is an open interval and for which the definition is similar. Let us remark that $u_0 \in BV(R)$ implies that $u_0(-\infty)$ and $u_0(+\infty)$ exist in a sense of ess-limits, and that $\|u_0\|_{L^\infty(R)} < \infty$. We define $a = \text{ess lim } u_0(x) \quad \text{and} \quad b = \text{ess lim } u_0(x)$.

The form of the partial differential equation in Problem (P) with nonlinear convection without any convexity assumption and possibly degenerate nonlinear diffusion is natural in view of many applications. A typical example is nonlinear filtration in porous media [GM].

Problem (P) may have no classical solutions. If for example $\varphi(s) = |s|^{m-1}s$ with $m > 1$ one usually considers weak solutions of Problem (P) which are continuous in $Q$. If $\varphi$ is not strictly increasing then the differential equation in (P) reduces to the first order conservation law

$$u_t + f(u)x = 0$$

in regions where $\varphi(u)$ is constant; in this case Problem (P) admits discontinuous solutions. We define solutions of Problem (P) as follows.

**Definition 1.1.** A function $u \in L^\infty(Q)$ is an *entropy solution* of Problem (P) if $u \in L^\infty((0, \infty); BV(R)) \cap C([0, \infty); L^2_{loc}(R)), \varphi(u) \in L^2_{loc}([0, \infty); H^1_{loc}(R))$ and if $u$ satisfies the inequality

$$\frac{\partial}{\partial t} |u - k| + \frac{\partial}{\partial x} (\text{sign}(u - k)(f(u) - f(k))) \leq \frac{\partial^2}{\partial x^2} (\text{sign}(u - k)(\varphi(u) - \varphi(k))) \quad \text{in } D'(Q)$$

for all constants $k \in R$, together with the initial condition $u(0) = u_0$.

This definition extends the notion of entropy solution of equation (1.1) introduced by [K]. Note that if $u$ is an entropy solution of Problem (P), then it satisfies the differential equation

$$u_t + f(u)x = \varphi(u)_{xx} \quad \text{in } D'(Q),$$

which one can check by successively setting $k = \pm \|u\|_{L^\infty(Q)}$ in (1.2).

In order to be able to state the main result of this paper, we consider the Riemann problem

$$\left\{ \begin{array}{ll} u_t + f(u)x = 0 & \text{in } Q \\ u(x,0) = a + (b - a)H(x) = \begin{cases} a & \text{if } x < 0 \\ b & \text{if } x > 0; \end{cases} \end{array} \right.$$

where $H$ is the Heaviside function. It is well known [K], [dB], [MNRR], [Se] that Problem (P$^\infty$) has a unique entropy solution. We remark that the solution $u^\infty$ of Problem (P$^\infty$) can be written using the similarity variable $\eta = x/t$ in the form $u^\infty(x,t) = U(\eta)$, where
$U \in BV(\mathbb{R})$ is a distributional solution of the problem

$$\left\{ \begin{array}{l}
f(U)' = \eta U' \\
U(-\infty) = a, \quad U(+\infty) = b
\end{array} \right. \quad \text{in } \mathbb{R}$$

which satisfies the “entropy” inequality

$$(\text{sign}(U-k)(f(U) - f(k)))' \leq \eta |U-k|' \quad \text{in } D'(\mathbb{R})$$

for all $k \in \mathbb{R}$ (see for instance [Se, p. 50]).

We also consider a sequence of related uniformly parabolic problems, namely

$$(P_{\lambda \varepsilon}) \left\{ \begin{array}{l}
\frac{d}{d\tau} + f_{\varepsilon}(u) = \frac{1}{\varepsilon} \varphi_{\varepsilon}(u) \quad \text{in } Q \\
u(x,0) = u_{0\varepsilon}(\lambda x)
\end{array} \right. \quad \text{for } x \in \mathbb{R}$$

where $0 < \varepsilon \leq 1$, $\lambda > 0$ and the functions $u_{0\varepsilon}$, $\varphi_{\varepsilon}$ and $f_{\varepsilon}$ satisfy the Hypotheses $(H_{\varepsilon})$:

$$(H_{\varepsilon}) \left\{ \begin{array}{l}
(i) \quad u_{0\varepsilon}, \varphi_{\varepsilon}, f_{\varepsilon} \in C^\infty(\mathbb{R}); \\
(ii) \quad \varphi_{\varepsilon} \to \varphi, f_{\varepsilon} \to f \text{ as } \varepsilon \downarrow 0 \text{ uniformly on compact subsets of } \mathbb{R}; \\
(iii) \quad \varepsilon \leq \varphi_{\varepsilon} \leq \frac{1}{\varepsilon} \text{ in } \mathbb{R}; \\
(iv) \quad \text{for all } R > 0 \text{ there exists } L = L(R) \text{ such that } |f_{\varepsilon}'| \leq L(R) \text{ on } (-R,R); \\
(v) \quad u_{0\varepsilon} \to u_0 \text{ in } L^1_{\text{loc}}(\mathbb{R}) \text{ as } \varepsilon \to 0; \\
(vi) \quad \text{ess inf } u_0 \leq u_{0\varepsilon} \leq \text{ess sup } u_0 \text{ in } \mathbb{R}; \\
(vii) \quad \int_{\mathbb{R}} |u_{0\varepsilon}'(x)| \, dx \leq TV(u_0); \\
(viii) \quad u_{0\varepsilon}(x) = a \text{ for } x < -\frac{1}{\varepsilon} \text{ and } u_{0\varepsilon}(x) = b \text{ for } x > \frac{1}{\varepsilon}.
\right.$$
In order to prove Theorem 1.3 we use a scaling technique. For all \( \lambda > 0 \) we set
\[
(1.5) \quad u^\lambda(x, t) = u(\lambda x, \lambda t),
\]
where \( u \) is the limit entropy solution of Problem (P). Then \( u^\lambda \) is a limit entropy solution of Problem \((P^\lambda)\),
\[
(P^\lambda) \begin{cases}
  u_t + f(u)x = \frac{1}{\lambda} \phi(u)xx & \text{in } Q \\
  u(x, 0) = u_0^\lambda(x) = u_0(\lambda x) & \text{for } x \in \mathbb{R},
\end{cases}
\]
where a limit entropy solution \( u^\lambda \) of Problem \((P^\lambda)\) is defined in a similar way as in Definition 1.2. Theorem 1.3 is the consequence of the following convergence result.

**Theorem 1.5.** Let \( \{u^\lambda\}_{\lambda \geq 1} \) be limit entropy solutions of Problem \((P^\lambda)\). Then, for any \( T > 0 \),
\[
\int_{-R}^{R} |u^\lambda(y, 1) - \tilde{U}(y)|^2 \, dy = \int_{-R}^{R} |\tilde{u}(\eta, \lambda) - \tilde{U}(\eta)|^2 \, d\eta \to 0 \quad \text{as } \lambda \to \infty,
\]
which is precisely the result stated in Theorem 1.3.

Indeed it follows from (1.4), (1.5) and Theorem 1.5 that for all \( R > 0 \)
\[
\int_{-R}^{R} |u^\lambda(y, 1) - \tilde{U}(y)|^2 \, dy = \int_{-R}^{R} |\tilde{u}(\eta, \lambda) - \tilde{U}(\eta)|^2 \, d\eta \to 0 \quad \text{as } \lambda \to \infty,
\]
which is precisely the result stated in Theorem 1.3.

The large time behaviour of solutions of Problem (P) has been studied for a long time under various assumptions on \( f, \phi \) and \( u_0 \). We refer to [IO2] and [W] for a historical review and an extensive list of references contained therein. Results related to presented here were obtained by Il’in and Oleinik [IO1], [IO2] in the case that \( \phi(u) = \varepsilon u \), with \( \varepsilon > 0 \) and \( f'' > 0 \) and by Weinberger [W] with the hypotheses that the differential equation in Problem (P) is uniformly parabolic and that \( f'' \) is continuous and only has isolated zeros. Van Duijn and de Graaf [vDdG] also examined a similar problem for a degenerate parabolic equation in the case of power type nonlinearities for the functions \( \phi \) and \( f \). Most of the methods of proof used in those papers are based on maximum principle arguments; here we present an approach based on a scaling method together with energy type estimates. This approach enables us to obtain a unified description of the limiting profile as \( t \to \infty \) of solutions of Problem (P), without standard distinguishing between convexity and concavity of the convection function \( f \). We also refer to [BGH] for a short note about these results. In a forthcoming article we will extend the results that we present here to the case of higher space dimension.

The organization of this paper is as follows. In Section 2 we prove a priori estimates for the solutions \((P^\lambda)\). In Section 3 we deduce from these estimates both the existence of an entropy solution \( u^\lambda \) of Problem \((P^\lambda)\) and the convergence of \( u^\lambda \) to the function \( u^\infty \) as \( \lambda \to \infty \).

2. A priori estimates. In this section in a series of lemmas we derive a priori estimates for the solutions \( u^\lambda \) of Problems \((P^\lambda)\), with \( \lambda \geq 1 \).
Lemma 2.1.

\[ \text{ess inf } u_0 \leq u_\lambda \leq \text{ess sup } u_0 \quad \text{in } Q. \]

**Proof.** This result follows from Hypothesis \((H_\epsilon)\) (vi) and applying the standard maximum principle. ■

Lemma 2.2. Let \(0 < \epsilon \leq 1\), \(\lambda \geq 1\) and \(T > 0\) be fixed. Then

\[ u_\lambda - a - (b - a)H(x), \quad u_{xx}, \quad u_{xx} = O(e^{-|x|}) \quad \text{as } |x| \to \infty, \]

uniformly in \([0,T]\).

**Proof.** (i) We first prove that

\[ u_\lambda - b = O(e^{-x}) \quad \text{as } x \to +\infty, \]

uniformly in \([0,T]\). Set \(M = \|u_0\|_{\infty}\). Then, by (2.1),

\[ -M \leq u_\lambda \leq M \quad \text{in } Q. \]

We compare \(u_\lambda\) with the function

\[ \varphi(x, t) = b - \gamma e^{-x+Kt} \]

in the set \(S_{A,K} = \{(x,t) : \ x \geq A + Kt, \ t \geq 0\}\) for some \(\gamma, A,K > 0\). If we choose \(\gamma = (b+M)\epsilon A\) then

\[ \varphi(A + Kt, t) = -M \]

for \(t \geq 0\). Furthermore, if \(A = \frac{1}{2}\) then by \((H_\epsilon)(viii)\)

\[ \varphi(x, 0) = b - \gamma e^{-x} \leq u_{0\epsilon}(x) \]

for \(x \in [A, \infty)\). Finally, for \(K = K_\epsilon\) large enough we have

\[ \varphi_t - \varphi'_\epsilon(\varphi)\varphi_{xx} - \varphi''_\epsilon(\varphi)\varphi_x^2 + f'_\epsilon(\varphi)\varphi_x = \gamma e^{-x+Kt}[\gamma - \varphi'_\epsilon(\varphi) - \gamma e^{-x+Kt}\varphi''_\epsilon(\varphi) + f'_\epsilon(\varphi)] \leq 0 \]

in \(S_{A,K}\). Hence, by the maximum principle \(\varphi(x, t) \leq u_\lambda\) in \(S_{A,K}\) so that

\[ -\gamma e^{KT-x} \leq u_\lambda - b \]

for \(x \geq A + Kt\) and \(t \in [0,T]\). Similarly, comparing \(u_\lambda\) with the function of the form

\[ \omega(x, t) = b + \gamma_1 e^{-x+K_1t} \]

in \(S_{A,K_1}\) for some \(\gamma_1, K_1 > 0\) and \(A\) as before leads to

\[ u_\lambda - b \leq \gamma_1 e^{K_1T-x} \]

for \(x \geq A + K_1t\) and \(t \in [0,T]\).

The proof that \(u_\lambda - a = O(e^{-|x|})\) as \(x \to -\infty\) uniformly in \([0,T]\) is similar.

(ii) In order to prove that

\[ u_{xx} = O(e^{-x}) \quad \text{as } x \to +\infty \]

uniformly in \([0,T]\) we observe that \(p = u_{xx}\) satisfies

\[
\begin{align*}
p_t &= \left( \varphi'_\epsilon(u_\lambda)p_x + \varphi''_\epsilon(u_\lambda)p^2 - f'_\epsilon(u_x)p_x \right) \\
&= \varphi'_\epsilon(u_x)p_{xx} + 3\varphi''_\epsilon(u_x)u_{xx}p_x + \varphi''_\epsilon(u_x)(u_{xx})^2p - f'_\epsilon(u_x)p_x - f''_\epsilon(u_x)u_{xx}p_x,
\end{align*}
\]
and moreover \(|p| \leq M_2\) in \(\mathbb{R} \times [0,T]\) and, by \((H_2)\)(viii), \(p(x,0) = 0\) for \(x > \frac{1}{\varepsilon}\). Thus we can compare \(p\) with functions
\[
\omega(x,t) = \pm \gamma e^{-x + Kt}
\]
in \(S_{A,K}\) for \(\gamma, K > 0\) and \(A = \frac{1}{\varepsilon}\).

The proof that \(u_\varepsilon^{\lambda} = O(e^{-|x|})\) as \(x \to -\infty\) uniformly in \([0,T]\) is similar.

(iii) The proof that \(u_\varepsilon^{\lambda xx} = O(e^{-|x|})\) as \(x \to \pm \infty\) uniformly in \([0,T]\) is similar to the proof given in (ii). ”

**Lemma 2.3.** For all \(t \geq 0\),
\[
(2.2) \quad \int_{\mathbb{R}} |u_\varepsilon^{\lambda}(x,t)| dx \leq \int_{\mathbb{R}} |u_0(x)| dx \leq TV(u_0).
\]

**Proof.** For the sake of simplicity we use the notations \(u\) and \(u_0\) instead of \(u_\varepsilon^{\lambda}\) and \(u_\varepsilon^0\) respectively. To begin with we differentiate the the differential equation in Problem \((P_\varepsilon^\lambda)\) with respect to \(x\), multiply the resulting equation by \(\text{sign}(\varphi'^\prime(u)x)\) and integrate over \(Q_{R,T}\) for fixed \(R\) and \(T > 0\). This leads to
\[
(2.3) \quad \int \int_{Q_{R,T}} u_{xt} \text{sign}(u) + \int \int_{Q_{R,T}} f_x(u)_{xx} \text{sign}(u) = \frac{1}{\varepsilon} \int \int_{Q_{R,T}} \varphi_x(u)_{xxx} \text{sign}(u).
\]

We show below that
\[
(2.4) \quad \int \int_{Q_{R,T}} u_{xt} \text{sign}(u) dx dt = \int_{-R}^{R} \left. |u_x| \right|_0^T dx,
\]
\[
(2.5) \quad \int \int_{Q_{R,T}} f_x(u)_{xx} \text{sign}(u) dx dt = \int_0^T \left. f_x(u)|u_x| \right|_{-R}^{R} dt,
\]
\[
(2.6) \quad \int \int_{Q_{R,T}} \varphi_x(u)_{xxx} \text{sign}(u) dx dt \leq \int_0^T \left. (\varphi_x(u)u_x) \text{sign}(\varphi'^\prime(u)x) \right|_{-R}^{R} dt.
\]

In order to prove (2.4)-(2.6) we use a sequence of smooth approximations \(\{S_\delta\}_{\delta > 0}\) of the sign function and set \(M_\delta(w) = \int_0^w S_\delta(s) ds\) for \(w \in \mathbb{R}\). Then \(M_\delta(w) \to |w|\) as \(\delta \to 0\). We have that
\[
\int \int_{Q_{R,T}} S_\delta(u_x)u_{xt} = \int \int_{Q_{R,T}} (M_\delta(u_x))_{xt} = \int_{-R}^{R} M_\delta(u_x) \left. \right|_0^T dx,
\]
where we let \(\delta \to 0\) to obtain (2.4).

In order to prove (2.5) we observe that
\[
(f_x(u)_{xx} S_\delta(u_x)) = (f_x(u)_{xx} u_x S_\delta(u_x)) + (f_x(u)_{xx} M_\delta(u_x)) = (f_x(u)_{xx} u_x S_\delta(u_x)) + (f_x(u)_{xx} u_x S_\delta(u_x) - M_\delta(u_x)),
\]
which implies that
\[
(2.7) \quad \int \int_{Q_{R,T}} f_x(u)_{xx} S_\delta(u_x) = \int \int_{Q_{R,T}} (f_x(u)_{xx} M_\delta(u_x)) + J(\delta)
= \int_0^T f_x(u)_{xx} M_\delta(u_x) \left. \right|_{-R}^{R} dt + J(\delta),
\]

where
\[ J(\delta) = \int \int_{Q_{T},R} (f_\varepsilon'(u)x_x u_x S_\delta(u_x) - M_\delta(u_x)). \]

Since \( J(\delta) \to 0 \) as \( \delta \to 0 \) we obtain (2.5) by letting \( \delta \to 0 \) in (2.7).

Finally we prove (2.6). We have that
\[
\begin{align*}
(2.8) \quad \int \int_{Q_{T},R} \varphi_\varepsilon'(u)x_x x S_\delta(u_x) \\
= \int \int_{Q_{T},R} \varphi_\varepsilon'(u)x_x x [S_\delta(u_x) - S_\delta(\varphi_\varepsilon'(u)x_x x)] + \int \int_{Q_{T},R} (\varphi_\varepsilon'(u)x_x x S_\delta(\varphi_\varepsilon'(u)x_x x)) \\
= I_1(\delta) + I_2(\delta),
\end{align*}
\]

and remark that since \( \varphi_\varepsilon' > 0 \) then \( I_1(\delta) \to 0 \) as \( \delta \to 0 \). Next we estimate \( I_2(\delta) \). We have that
\[
(2.9) \quad I_2(\delta) = \int_0^T \left( \varphi_\varepsilon'(u)x_x x S_\delta(\varphi_\varepsilon'(u)x_x x) \right) \bigg|_R^{\infty} dt - \int \int_{Q_{T},R} \left[ (\varphi_\varepsilon'(u)x_x x S_\delta(\varphi_\varepsilon'(u)x_x x)) \right] dtdx \\
\leq \int_0^T \left( \varphi_\varepsilon'(u)x_x x S_\delta(\varphi_\varepsilon'(u)x_x x) \right) \bigg|_R^{\infty} dt.
\]

Substituting (2.9) into (2.8) and letting \( \delta \to 0 \) we obtain (2.6).

Now it follows from (2.3)-(2.6) that
\[
\int \int_{Q_{T},R} \varphi_\varepsilon'(u)x_x x S_\delta(\varphi_\varepsilon'(u)x_x x) \bigg|_R^{\infty} dt - \int \int_{Q_{T},R} \left[ (\varphi_\varepsilon'(u)x_x x S_\delta(\varphi_\varepsilon'(u)x_x x)) \right] dtdx \\
\leq \int_0^T \left( \varphi_\varepsilon'(u)x_x x S_\delta(\varphi_\varepsilon'(u)x_x x) \right) \bigg|_R^{\infty} dt.
\]

for all \( R, T > 0 \). Hence, by Lemma 2.2, in the limit as \( R \to \infty \)
\[
\int \int_{R} \left( | u_{x_x x}^\lambda (x, T) | dx - \lambda \int \left| u_{0_x x}^\lambda (\lambda x) \right| dx \right) 
\leq 0,
\]

which yields (2.2) by (H2)(vii).

**Lemma 2.4.** There exists a positive constant \( C = C(R, T) \) such that
\[
(2.10) \quad \| f_\varepsilon(u_x^\lambda) \|_{L^2(0,T;H^{-1}(-R,R))} \leq C.
\]

**Proof.** Here again we omit the lower index \( \varepsilon \) and the upper index \( \lambda \) from the notation.

Let \( R > 0 \) and \( \zeta \in C^\infty_0(-R,R) \). We have that
\[
\langle f_\varepsilon(u_x^\lambda) ; \zeta \rangle = \int_{-R}^{R} f_\varepsilon(u_x^\lambda)(x,t) \zeta(x) \, dx = - \int_{-R}^{R} f_\varepsilon(\zeta)(x,t) \zeta^\lambda(x) \, dx,
\]

which imply that
\[
| \langle f_\varepsilon(u_x^\lambda) ; \zeta \rangle | \leq \left( \int_{-R}^{R} | f_\varepsilon(u_x^\lambda)(x,t) |^2 \, dx \right)^{1/2} \left( \int_{-R}^{R} | \zeta^\lambda(x) |^2 \, dx \right)^{1/2} \\
\leq \left( \int_{-R}^{R} | f_\varepsilon(\zeta)(x,t) |^2 \, dx \right)^{1/2} \| \zeta \|_{H^1_\varepsilon(-R,R)}.
\]
for all $t \in [0, T]$. Hence

\[
\|f_\varepsilon(u)_x(\cdot, t)\|_{H^{-1}(-R, R)} \leq \left( \int_{-R}^R |f_\varepsilon(u)(x, t)|^2 \, dx \right)^{1/2}
\]

for all $t \in [0, T]$ and consequently by (H$_\varepsilon$)(ii) and Lemma 2.1

\[
\int_0^T \|f_\varepsilon(u)_x(\cdot, t)\|_{H^{-1}(-R, R)}^2 \, dt \leq \int \int_{Q_{R,T}} |f_\varepsilon(u)|^2 \leq C
\]

for some positive constant $C = C(R, T)$. ■

**Lemma 2.5.** There exists a positive constant $C = C(R, T)$ such that

\[
(2.11) \quad \|\varphi_\varepsilon(u^\lambda_\varepsilon)_x\|_{L^2((-R, R) \times (0, T))} \leq C\sqrt{\lambda}.
\]

**Proof.** For simplicity we write $u$ and $u_0$ instead of $u^\lambda_\varepsilon$ and $u_{0\varepsilon}$ respectively. Let $R > 0$ and $\psi$ be a smooth function such that

\[
\psi(x) = \left\{ \begin{array}{ll} 1 & \text{if } |x| \leq R \\ 0 & \text{if } |x| \geq R + 1. \end{array} \right.
\]

We multiply the differential equation in Problem (P$_1^\lambda$) by $\varphi_\varepsilon(u)\psi^2$ and write the resulting equality as

\[
\Phi_\varepsilon(u)\psi^2 + \Psi_\varepsilon(u)_x\psi^2 = \frac{1}{\lambda}\varphi_\varepsilon(u)_x\varphi_\varepsilon(u)\psi^2,
\]

where we have set $\Phi_\varepsilon(u) = \int_0^u \varphi_\varepsilon(s) \, ds$ and $\Psi_\varepsilon(u) = \int_0^u f'_\varepsilon(s)\varphi_\varepsilon(s) \, ds$. Integrating by parts on the domain $Q_{R+1,T} = (-R-1, R+1) \times (0, T)$ gives

\[
\int_{-(R+1)}^{R+1} (\Phi_\varepsilon(u(x, T)) - \Phi_\varepsilon(u_0(x)))\psi^2(x) \, dx - \int \int_{Q_{R+1,T}} \Psi_\varepsilon(u)(\psi^2)'
\]

\[
= -\frac{1}{\lambda} \int \int_{Q_{R+1,T}} (\varphi_\varepsilon(u)_x)^2\psi^2 - \frac{2}{\lambda} \int \int_{Q_{R+1,T}} \varphi_\varepsilon(u)_x\varphi_\varepsilon(u)\psi\psi'.
\]

Applying the Cauchy-Schwarz inequality to the second term of the right-hand side of the equality above gives

\[
\int_{-(R+1)}^{R+1} (\Phi_\varepsilon(u(x, T)) - \Phi_\varepsilon(u_0))\psi^2(x) \, dx - \int \int_{Q_{R+1,T}} \Psi_\varepsilon(u)(\psi^2)'
\]

\[
\leq -\frac{1}{2\lambda} \int \int_{Q_{R+1,T}} (\varphi_\varepsilon(u)_x)^2\psi^2 + \frac{2}{\lambda} \int \int_{Q_{R+1,T}} (\varphi_\varepsilon(u))^2(\psi')^2.
\]

Therefore, in view of (H$_\varepsilon$) and Lemma 2.1

\[
\frac{1}{\lambda} \int \int_{Q_{R,T}} (\varphi_\varepsilon(u)_x)^2 \leq C_1
\]

where the positive constant $C_1 = C_1(R, T)$ does not depend on $\varepsilon$ and $\lambda$. ■

**Corollary 2.6.**

\[
(2.12) \quad \|\varphi_\varepsilon(u^\lambda_\varepsilon)_x\|_{L^2([0, T]; H^{-1}(-R, R))} \leq C\sqrt{\lambda}.
\]
Proof. As in the proof of Lemma 2.5, we omit the lower index \( \varepsilon \) and the upper index \( \lambda \) from the notation. Let \( R > 0 \), \( \zeta \in C_0^\infty(-R, R) \) and \( t \in [0, T] \); we have that
\[
(\varphi_\varepsilon(u)_{xx}(\cdot,t),\zeta) = \int_{-R}^{R} \varphi_\varepsilon(u)_{xx}(x,t)\zeta(x) \, dx = -\int_{-R}^{R} \varphi_\varepsilon(u)_x(x,t)\zeta'(x) \, dx
\]
so that
\[
|\langle \varphi_\varepsilon(u)_{xx}(\cdot,t),\zeta \rangle| \leq \left( \int_{-R}^{R} |\varphi_\varepsilon(u)_x(x,t)|^2 \, dx \right)^{1/2} \left( \int_{-R}^{R} |\zeta'(x)|^2 \, dx \right)^{1/2}
\]
for all \( t \in [0, T] \). Hence
\[
\|\varphi_\varepsilon(u)_{xx}(\cdot,t)\|_{H^{-1}(-R,R)} \leq \left( \int_{-R}^{R} |\varphi_\varepsilon(u)_x(x,t)|^2 \, dx \right)^{1/2}
\]
for all \( t \in [0, T] \). In view of Lemma 2.5 we obtain
\[
\int_{0}^{T} \|\varphi_\varepsilon(u)_{xx}(\cdot,t)\|_{H^{-1}(-R,R)}^2 \, dt \leq \int_{Q_R,T} |\varphi_\varepsilon(u)_x|^2 \leq C \lambda
\]
for some positive constant \( C = C(R,T) \).

We end this section with the following compactness result.

**Lemma 2.7.** Let \( R > 0 \). The set \( \{u^\lambda_{\varepsilon}\}_{\varepsilon>0,\lambda>1} \) is precompact in \( C([0,T];L^2(-R,R)) \).

**Proof.** It follows from (2.2) and (2.11) that
\[
\|(u^\lambda_\varepsilon)\|_{L^\infty((0,T);W^{1,1}(-R,R))} \leq C(R,T),
\]
while by (2.10), (2.12) and the differential equation of \( (P^\lambda_\varepsilon) \),
\[
\|(u^\lambda_\varepsilon)_t\|_{L^2((0,T);H^{-1}(-R,R))} \leq C(R,T)
\]
for some positive constant \( C(R,T) \). The result then follows from the embeddings
\[
W^{1,1}(-R,R) \subset L^2(-R,R) \subset H^{-1}(-R,R),
\]
the compactness of the embedding \( W^{1,1}(-R,R) \subset L^2(-R,R) \), and a compactness result due to Simon [Si] (Corollary 4, p. 85).

3. Existence and asymptotic behaviour of limit entropy solutions of Problem \( (P^\lambda_\varepsilon) \) as \( \lambda \to \infty \)

**Definition 3.1.** We say that a function \( u^\lambda_\varepsilon \) is an *entropy solution* of Problem \( (P^\lambda_\varepsilon) \) if it satisfies Definition 1.1 with \( \varphi \) replaced by \( (1/\lambda)\varphi \). A limit entropy solution of Problem \( (P^\lambda_\varepsilon) \) is then defined as in Definition 1.2.

We begin with the following lemma.

**Lemma 3.2.** Let \( 0 < \varepsilon \leq 1 \) and \( \lambda \geq 1 \) be fixed and let \( u^\lambda_\varepsilon \) be the classical solution of Problem \( (P^\lambda_\varepsilon_\varepsilon) \). Then \( u^\lambda_\varepsilon \) satisfies the inequality
\[(3.1) \quad \frac{\partial}{\partial t}|u_\varepsilon^\lambda - k| + \frac{\partial}{\partial x}(\text{sign}(u_\varepsilon^\lambda - k)(f_\varepsilon(u_\varepsilon^\lambda) - f_\varepsilon(k))) \leq \frac{1}{\lambda} \frac{\partial^2}{\partial x^2}(\text{sign}(u_\varepsilon^\lambda - k)(\varphi_\varepsilon(u_\varepsilon^\lambda) - \varphi_\varepsilon(k)))\]

in \( \mathcal{D}'(Q) \) for all \( k \in \mathbb{R} \).

**Proof.** As in the proofs above we write \( u \) instead of \( u_\varepsilon^\lambda \). Let \( k \in \mathbb{R} \). Multiplying the differential equation in Problem \((P_\varepsilon^\lambda)\) by \( S_\delta(u - k) \) gives

\[(3.2) \quad u_t S_\delta(u - k) + f_\varepsilon(u)_x S_\delta(u - k) = \frac{1}{\lambda} \varphi_\varepsilon(u)_x S_\delta(u - k)\]
in \( \mathcal{Q} \). Set

\[F_\varepsilon^\delta(w) = \int_k^w f_\varepsilon'(s) S_\delta(s - k) \, ds.\]

Then

\[(3.3) \quad u_t S_\delta(u - k) = (M_\delta(u - k))_t,\]

\[(3.4) \quad f_\varepsilon(u)_x S_\delta(u - k) = (F_\varepsilon^\delta(u))_x,\]

and

\[(3.5) \quad \varphi_\varepsilon(u)_x S_\delta(u - k) = (\varphi_\varepsilon(u)_x S_\delta(u - k))_x - (\varphi_\varepsilon(u)_x S_\delta'(u - k))u_x \leq (\varphi_\varepsilon(u)_x S_\delta(u - k))_x,\]
since \((\varphi_\varepsilon(u)_x S_\delta'(u - k))u_x \geq 0\). Set

\[G_\varepsilon^\delta(w) = \int_k^w \varphi_\varepsilon'(s) S_\delta(s - k) \, ds.\]

Then \((G_\varepsilon^\delta(u))_x = (\varphi_\varepsilon(u)_x S_\delta(u - k))_x\) and therefore combining \((3.2)-(3.5)\) we obtain

\[(M_\delta(u - k))_t + (F_\varepsilon^\delta(u))_x \leq (G_\varepsilon^\delta(u))_{xx}.\]

Letting \( \delta \to 0 \) gives

\[\frac{\partial}{\partial t}|u - k| + \frac{\partial}{\partial x}F_\varepsilon(u) \leq \frac{1}{\lambda} \frac{\partial}{\partial x^2}G_\varepsilon(u) \quad \text{in} \ \mathcal{D}'(Q),\]

where we use the notations

\[F_\varepsilon(w) = \int_k^w f_\varepsilon'(s) \text{sign}(s - k) \, ds, \quad G_\varepsilon(w) = \int_k^w \varphi_\varepsilon'(s) \text{sign}(s - k) \, ds.\]

But

\[G_\varepsilon(w) = \begin{cases} 
\varphi_\varepsilon(k) - \varphi_\varepsilon(w) & \text{if } k > w \\
\varphi_\varepsilon(w) - \varphi_\varepsilon(k) & \text{if } k < w \\
0 & \text{if } k = w. 
\end{cases}\]

Thus \( G_\varepsilon(w) = \text{sign}(w - k)(\varphi_\varepsilon(w) - \varphi_\varepsilon(k)) \). Similarly \( F_\varepsilon(w) = \text{sign}(w - k)(f_\varepsilon(w) - f_\varepsilon(k)) \).

Therefore \( u \) satisfies \((3.1)\).

Next we prove the existence of a limit entropy solution of Problem \((P^\lambda)\) with properties which we use later on.

**Theorem 3.3.** Let \( \lambda \geq 1 \) be fixed and let \( \{u_\varepsilon^\lambda\}_{0 < \varepsilon \leq 1} \) be the classical solutions of Problems \((P_\varepsilon^\lambda)\). There exists a sequence \( \{\varepsilon_n\} \) and a function \( u^\lambda \in L^\infty(Q) \) such that

\[u_\varepsilon^\lambda \to u^\lambda \quad \text{in} \ C([0,T];L^2(-R,R)) \quad \text{as} \ \varepsilon_n \to 0,\]
for all $R$ and $T > 0$. The function $u^\lambda$ is an entropy solution of Problem $(P^\lambda)$ and satisfies the following estimates:

(i) $\text{ess inf } u_0 \leq u^\lambda \leq \text{ess sup } u_0$ a.e. in $Q$;

(ii) $\|\varphi(u^\lambda)\|_{L^2((-R,R) \times (0,T))} \leq C\sqrt{\lambda}$;

(iii) $\text{TV}(u^\lambda(\cdot,t)) \leq \text{TV}(u_0)$ for a.e. $t \in (0,\infty)$;

(iv) $\|u^\lambda\|_{L^2((0,T);H^{-1}(-R,R))} \leq C$,

where the positive constant $C$ only depends on $R$ and $T$.

**Proof.** Let $\lambda \geq 1$. We deduce from Lemma 2.7 that there exists a sequence $\varepsilon_n \to 0$ and a function $u^\lambda \in C([0,\infty);L^1_{\text{loc}}(\mathbb{R}))$ such that as $\varepsilon_n \to 0$

$$u^\lambda_{\varepsilon_n} \to u^\lambda \quad \text{in } C([0,T];L^2(-R,R)) \quad \text{and a.e. in } Q,$$

for all $R > 0$ and all $T > 0$. The assertions (i)-(iv) are consequences of (2.1), (2.2), (2.11), and (2.14), and of the lower semicontinuity of total variation ([EG], [GR]). Observe that by (H$_2$) (ii) and (3.6) as $\varepsilon_n \to 0$,

$$\text{sign}(u^\lambda_{\varepsilon_n} - k) \to \text{sign}(u^\lambda - k)$$

a.e. in $Q \cap \{x, t : u^\lambda - k \neq 0\}$ and

$$f_{\varepsilon_n}(u^\lambda_{\varepsilon_n}) - f_{\varepsilon_n}(k) \to f(u^\lambda) - f(k),$$

$$\varphi_{\varepsilon_n}(u^\lambda_{\varepsilon_n}) - \varphi_{\varepsilon_n}(k) \to \varphi(u^\lambda) - \varphi(k)$$

a.e. in $Q$. Then, letting $\varepsilon_n$ tend to zero in an integrated form of inequality (3.1) and using (2.1), (3.7) - (3.9) and Lebesgue’s dominated convergence theorem, one deduces that $u^\lambda$ satisfies the inequality

$$\frac{\partial}{\partial t}|u^\lambda - k| + \frac{\partial}{\partial x}(\text{sign}(u^\lambda - k)(f(u^\lambda) - f(k)))$$

$$\leq \frac{1}{\lambda} \frac{\partial^2}{\partial x^2}(\text{sign}(u^\lambda - k)(\varphi(u^\lambda) - \varphi(k)))$$

in $D'(Q)$ for all constants $k \in R$. Furthermore it follows from (H$_2$)(v) and from (3.6) that $u^\lambda$ satisfies the initial condition $u^\lambda(0) = u_0$. Thus $u^\lambda$ is a limit entropy solution of Problem $(P^\lambda)$.

**Corollary 3.4.** Let $\lambda \geq 1$ and let $u^\lambda$ be a limit entropy solution of Problem $(P^\lambda)$. Then the statements (i) - (iv) of Theorem 3.3 hold for $u^\lambda$.

**Proof.** This is an immediate consequence of the definition of the limit entropy solution of Problem $(P^\lambda)$ and of Theorem 3.3.

Before proving Theorem 1.5 we give the definition of an entropy solution of Problem $(P^\infty)$.

**Definition 3.5.** A function $u \in L^\infty(Q) \cap C([0,\infty);L^1_{\text{loc}}(\mathbb{R}))$ is an entropy solution of Problem $(P^\infty)$ if it satisfies the entropy inequality

$$\frac{\partial}{\partial t}|u - k| + \frac{\partial}{\partial x}(\text{sign}(u - k)(f(u) - f(k))) \leq 0$$

in $D'(Q)$ for all constants $k \in R$, together with the initial condition $u(0) = u_0$. 

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Proof of Theorem 1.5. Let $\lambda > 1$, $R > 0$, $T > 0$ and let $u^\lambda$ be a limit entropy solution of Problem $(P^\lambda)$. We deduce from Corollary 3.4, Theorem 3.3 (iii), (iv), the embeddings $BV(-R,R) \subset L^2(-R,R) \subset H^{-1}(-R,R)$, the compactness of the imbedding $BV(-R,R) \subset L^2(-R,R)$ which we shall prove in the Appendix and Corollary 4 p. 85 of [Si] that the set $\{u^\lambda\}_{\lambda>1}$ is precompact in $C([0,T];L^2(-R,R))$. Hence there exists a sequence $\lambda_n \to \infty$ and a function $u^\infty \in C([0,\infty);L^2_{loc}(\mathbb{R}))$ such that for all $R > 0$ and $T > 0$

\[(3.11)\]

$u^{\lambda_n} \to u^\infty$ in $C([0,T];L^2(-R,R))$ and a.e. in $Q_{R,T}$ as $n \to \infty$. It then follows from Theorem 3.3 and Corollary 3.4 that $u^\infty \in L^\infty(Q) \cap L^\infty((0,\infty);BV(\mathbb{R}))$. Finally, similarly as it has been done in the proof of Theorem 3.3 one can prove that $u^\infty$ satisfies the entropy inequality (3.10). Thus $u^\infty$ is an entropy solution of Problem $(P^\infty)$.

Now as a consequence of (3.11) and the uniqueness of the entropy solution of Problem $(P^\infty)$ ([K], [dB]) we obtain that for all $R > 0$ and $T > 0$

$u^\lambda \to u^\infty$ in $C([0,T];L^2(-R,R))$ as $\lambda \to \infty$.

This completes the proof of Theorem 1.5.  

4. Appendix. We shall prove the following lemma.

**Lemma A.1.** Let $R > 0$. Then for any $p \geq 1$, $BV(-R,R)$ is compactly embedded in $L^p(-R,R)$.

**Proof.** Since this result is well known for $p = 1$ we prove it for $p > 1$. Let $\{g_n\}_{n=1}^\infty \subset BV(-R,R)$ be such that

\[(A.1)\]

$\|g_n\|_{BV(-R,R)} = \|g_n\|_{L^p(-R,R)} + TV(-R,R)(g_n) \leq M$

for all $n \geq 1$ and for some constant $M > 0$. We first prove that $\{g_n\}_{n=1}^\infty$ is uniformly bounded in $L^\infty(-R,R)$ (the proof is almost a facsimile of the proof of Claim 3, p. 218 in [EG]). Fix $n \geq 1$ and choose $\{g_{nj}\}_{j=1}^\infty \subset BV(-R,R) \cap C^\infty(-R,R)$ such that as $j \to \infty$, $g_{nj} \to g_n$ in $L^1(-R,R)$ and a.e. in $(-R,R)$ and

\[\int_{-R}^{R} |g'_{nj}| \, dx \to TV(-R,R)(g_n).\]

For each $y, z \in (-R,R)$ we have that

$g_{nj}(z) = g_{nj}(y) + \int_y^{z} g'_{nj}(x) \, dx.$

Averaging with respect to $y \in (-R,R)$ gives

$|g_{nj}(z)| \leq 1/(2R) \int_{-R}^{R} |g_{nj}(y)| \, dy + \int_{-R}^{R} |g'_{nj}(x)| \, dx.$
and hence for \( j \) large enough,
\[
\| g_{n_j} \|_{L^\infty(-R,R)} \leq C \| g_{n_j} \|_{BV(-R,R)},
\]
where the constant \( C \) does not depend on \( n \) and \( j \). Taking the limit \( j \to \infty \) yields
\[
(A.2) \quad \| g_n \|_{L^\infty(-R,R)} \leq CM.
\]
Now, by (A.1), (A.2) and the compactness theorem in [EG] p. 176, there exist a sequence \( n_k \to \infty \) and a function \( g \in L^\infty(-R,R) \) such that as \( k \to \infty \),
\[
g_{n_k} \to g \quad \text{in} \quad L^1(-R,R) \quad \text{and a.e. in} \quad (-R,R).
\]
Since
\[
\int_{-R}^{R} |g_{n_k} - g|^p \, dx \leq \sup_{(-R,R)} |g_{n_k} - g|^{p-1} \int_{-R}^{R} |g_{n_k} - g| \, dx \leq (2CM)^{p-1} \int_{-R}^{R} |g_{n_k} - g| \, dx,
\]
the result follows.

References


